Fundamentals : Part 1

- Light as an electromagnetic wave
- Diffraction
  - Ray optics
  - Basic Fourier principles
    (convenient mathematics that help develop intuition)
- Telescope parameters
- Quality of optics.

What is light?

-An electromagnetic wave!

Speed $c = \nu\lambda$

(and don't forget wave-particle duality $E = h\nu$)

Harmonic in free space $\ddot{\psi} = -\omega^2 \psi$

solution: $\psi = \psi_0 e^{-i\omega t}$

it varies in time!

* note: Rieke uses $j = \sqrt{-1}$

For a complete description of wave mathematics, see Hecht Chapter 2 (on reserve at the library)

The point: Waves have an amplitude and a phase

* we're going to focus on plane-waves

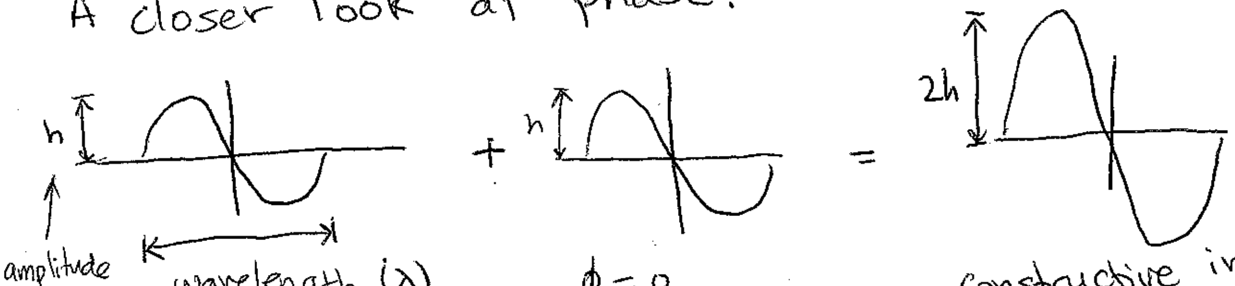
The electric field, for example:

$$E(x,t) = \underbrace{E_0}_{\text{Amplitude}} \underbrace{e^{-i(kx - \omega t)} e^{-i\phi}}_{\text{phase}}$$

important identity:

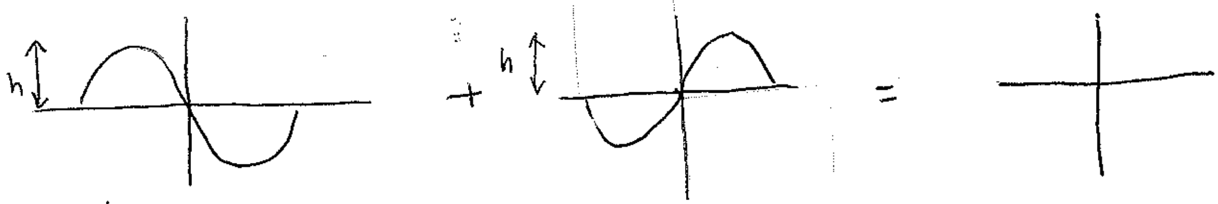
$$e^{ix} = \cos x + i \sin x$$

A closer look at phase:



phase: $\phi = 0$ $\phi = 0$ constructive interference

vs.

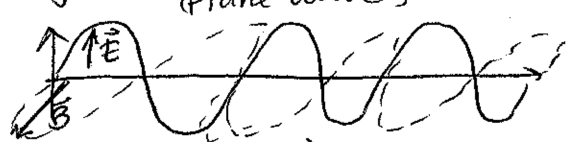


$\phi = 0$ $\phi = \lambda/2$ destructive interference

monochromatic, same wavelength, "coherent"

<u>what is Intensity?</u>

(this is how we, humans, experience light)   (plane wave)



$|\vec{B}| \rightarrow |\vec{E}|/c$

Energy Flux — Poynting vector:

$$\vec{S} = c^2 \epsilon_0 \vec{E} \times \vec{B}$$
$$= c\epsilon_0 E^2 \quad \text{for a plane-wave}$$

consider a harmonic E-field and "time average":

$$\langle c\epsilon_0 E_0^2 \cos^2 \omega t \rangle = \frac{c\epsilon_0 E_0^2}{2}$$

* Hecht calls this irradiance (page 50)

The point: $\boxed{\text{intensity} \propto E^2}$ no phase information

we see intensity (E^2)
we don't "see" phase

we'll start thinking with complex math, $EE^* = |E|^2$

EE^* is real, non-negative (image property!)

Some vocabulary associated with waves:

- Refraction
- Dispersion
- $\boxed{\text{Diffraction}}$

(Refraction and Dispersion:) material interactions, we'll cover this soon

Quick definitions:

Refraction – relative speed of the wave changes in a medium of refractive index n. $v = c/n$

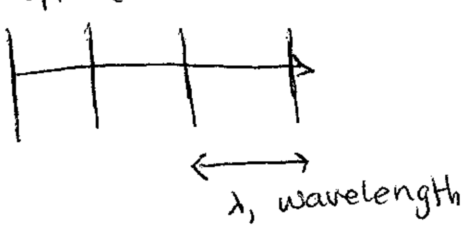
Dispersion – wavelength dependent interaction (e.g. prism)

Diffraction – Think of light scattering off hard/soft edges to produce a characteristic image. But it's deeper than that and requires a full discussion. It is a limit on information

first, a few items for our graphical/mathematical toolbox:

Drawing light rays (ray optics diagrams)

(Plane wave:)

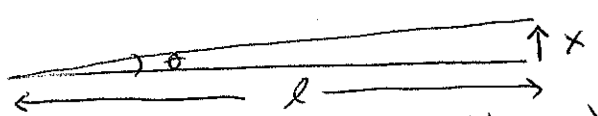


$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

Small angle approximation:

$\sin\theta \approx \theta$



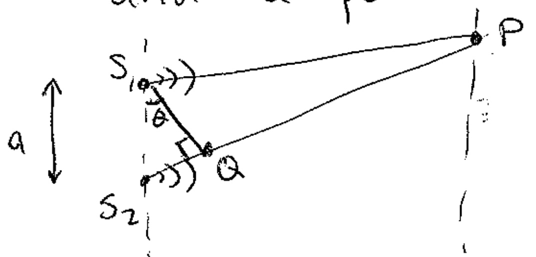
ℓ is very far (e.g. light years)

when $\ell >> x$ $\quad \theta \approx \frac{x}{\ell}$

Question – what is the diffraction limit of a telescope?

Why we care about diffraction – limit on the information we can extract from an image.

Consider two point emitters distance a apart and a point P on a screen



$S_1P = QP$

there is a path length difference between S_2P an S_1P :

$$S_2P - S_1P = S_2Q$$

from the diagram:

$$S_2Q = a \sin\theta$$

depending on the size of S_2Q compared to the wavelength we'll see constructive or destructive interference (or anything in between)

constructive interference when $a\sin\theta$ is an integer multiple of λ

$$a \sin\theta = m\lambda \quad , m = 0, 1, 2 \ldots$$

destructive: $a\sin\theta = (m + \frac{1}{2})\lambda$.

$a \sin\theta \simeq a\theta$

the range of angles between a bright and dark peak is $\theta = \lambda/2a$ Michelson limit for diffraction (for an interferometer)

A moment to reflect:

- how big is Hubble compared to the largest optical ground telescope?
- why put a smaller telescope into space for \$\$\$\$\$?
- If we care about image resolution, we'll need a pixel resolution to match!

The Fourier Transform

- in the right approximation (far-field)
- mathematics that describes diffraction

Electric field at the image = Fourier transform [E-field at the aperture]

So:

image formed from a point-source $= \left| \text{F.T.} \{ \text{Aperture / pupil} \} \right|^2$

> $F(\vec{k})$ is the Fourier transform of $f(\vec{x})$ if:
>
> $F(\vec{k}) \rightleftharpoons f(\vec{x})$
>
> $$F(\vec{k}) = \int_{\substack{\text{All}\\\text{space}}} e^{-2\pi i \vec{k}\vec{x}} f(\vec{x})\, dx$$
>
> $\rightleftharpoons$ denotes Fourier transform pair

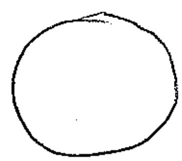
What does it mean? (Think Fourier series)

→ Expressing a (discrete) function as a sum of waves (cosines and sines). What 'waves' is the function made up of?

Optical system apertures (/pupils) we'll think about

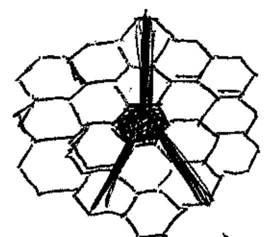
1 

(Like the ray optics example)

2 

common full-aperture approximation

3. others, e.g.

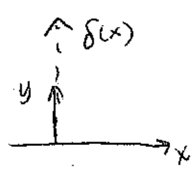


(JWST)

Useful functions and theorems –

The delta function

$$\delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & \text{elsewhere} \end{cases}$$



definition:

$$\int_{-\infty}^{+\infty} \delta(x) = 1$$

$$\delta(x-a) = \begin{cases} \infty, & x = a \\ 0, & \text{elsewhere} \end{cases}$$

"the sampling function"

$\int \delta(x) f(x)\, dx = f(0)$, $\int \delta(x-a) f(x)\, dx = f(a)$

Linearity

$f_1(x) + f_2(x) \rightleftharpoons F_1(k) + F_2(k)$

Shift theorem

linear shift (coordinate) $\rightleftharpoons$ phase shift

$F(k-a) \rightleftharpoons f(x)\, e^{-2\pi i k a}$

proof: $F(k-a) = F(k) * \delta(k-a) \rightleftharpoons f(x) \cdot F.T.\{\delta(k-a)\} = f(x) e^{-2\pi i k a}$

which brings us to...

Convolution theorem

convolution: $g(x) * h(x) = \int g(u-x)\, h(x)\, dx$

$g(x) * \delta(x-a) = g(x-a)$

returns function g at location of the delta function!

the theorem:

$f(x) \rightleftharpoons F(k)$; $g(x) \rightleftharpoons G(k)$

$f(x) * g(x) \rightleftharpoons F(k) \cdot G(k)$

Derivative theorem

$f(x) \rightleftharpoons F(k)$; $\frac{df}{dx} \rightleftharpoons -2\pi i k F(k)$ (homework: prove this)

Power theorem

$\int_{\text{All space}} |F(x)|^2 dx = \int_{\text{All space}} |F(k)|^2 dk$

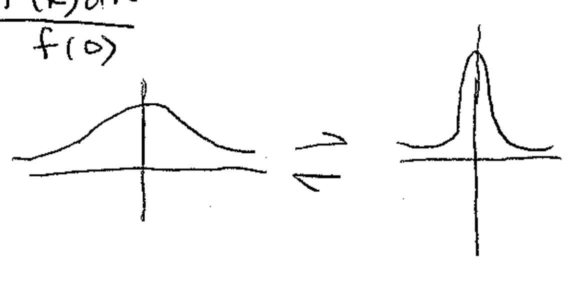
Equivalent width

- $\int_{-\infty}^{+\infty} f(x)\, dx = F(0)$ $\rightarrow$ $\frac{\int f(x) dx}{F(0)} = \frac{\int F(k) dk}{f(0)}$

$\rightarrow$ $\boxed{\frac{\int f(x) dx}{f(0)} = \frac{F(0)}{\int F(k) dk}}$

the area of a function divided by its peak

what it means:



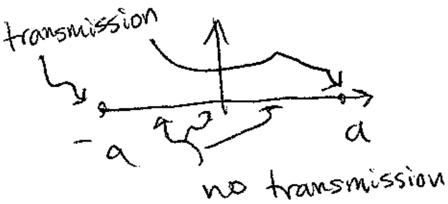
Similarity theorem

$f(x) \rightleftharpoons F(k)$: $f(ax) \rightleftharpoons \frac{1}{a} F\left(\frac{k}{a}\right)$

Proof: $\int e^{-2\pi i k x} f(ax)\, dx = \frac{1}{a} \int e^{-2\pi i \frac{k}{a} ax} f(ax)\, d(ax) = \frac{1}{a} F\left(\frac{k}{a}\right)$

Example #1 – Double slit / two hole interferometer

How can we express this aperture mathematically?



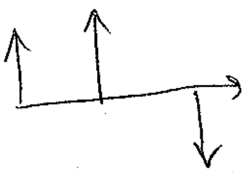
a function that is zero everywhere except $a, -a$ ...

$A(x) = \frac{1}{2} \{ \delta(x+a) + \delta(x-a) \}$

$A(x) \rightleftharpoons a(k)$ $\quad a(k) = \frac{1}{2} \int e^{-2\pi i k x} [\delta(x+a) + \delta(x-a)]\, dx$

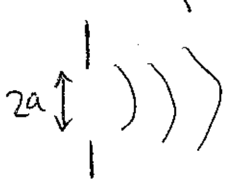
$= \frac{1}{2} \left[e^{-2\pi i k a} + e^{2\pi i k a} \right] = \cos(2\pi k a)$

Answer: brightest fringe at $k=0$ (center)

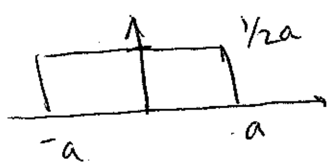
 $\rightleftharpoons$ cosine  $\rightleftharpoons$ sine

question – how can we shift the bright fringe from $k=0$?

Example #2 – single slit aperture



function: "top-hat"

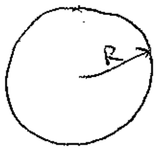


$f(x) = \begin{cases} 1/2a, & |x| < a \\ 0, & \text{elsewhere} \end{cases}$

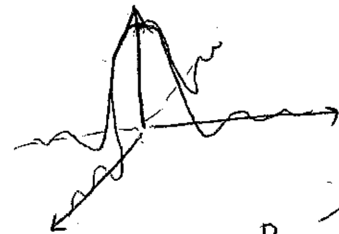
$\frac{1}{2a} \int_{-a}^{a} e^{-2\pi i k x}\, dx = \frac{1}{2a} \frac{-1}{2\pi i k} \left[e^{-2\pi i k a} - e^{2\pi i k a} \right] = \frac{\sin(2\pi k a)}{2\pi k a}$

$\frac{\sin x}{x} = \text{sinc}(x)$ special function! $\qquad \lim_{x \to 0} \frac{\sin x}{x} = 1$

compare to the 2D circular aperture



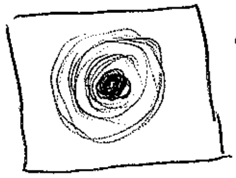
$$f(r) = \begin{cases} 1 & r < R \\ 0 & \text{elsewhere} \end{cases} \rightleftharpoons$$



$\text{Jinc}(k(r)) = \dfrac{J_1(k)}{\pi k}$

Bessel function

Projection on a 2D image:



"Airy rings"

the image of a point-source through a circular aperture is $\text{Jinc}^2(k(r))$ (intensity)

radius to first dark ring? $1.22\,\lambda/D$

→ Rayleigh Criterion of diffraction!

what about an object that isn't a point-source?

$$\text{Object} \times \text{Aperture} \rightleftharpoons \text{F.T.}\{\text{Object}\} * \text{F.T.}\{\text{Aperture}\}$$

* pocket interferometer demo

What is the condition for Fourier transforms?

the math describes Fraunhoffer diffraction

"diffraction in the far-field"

the approximation that we see a flat wavefront.

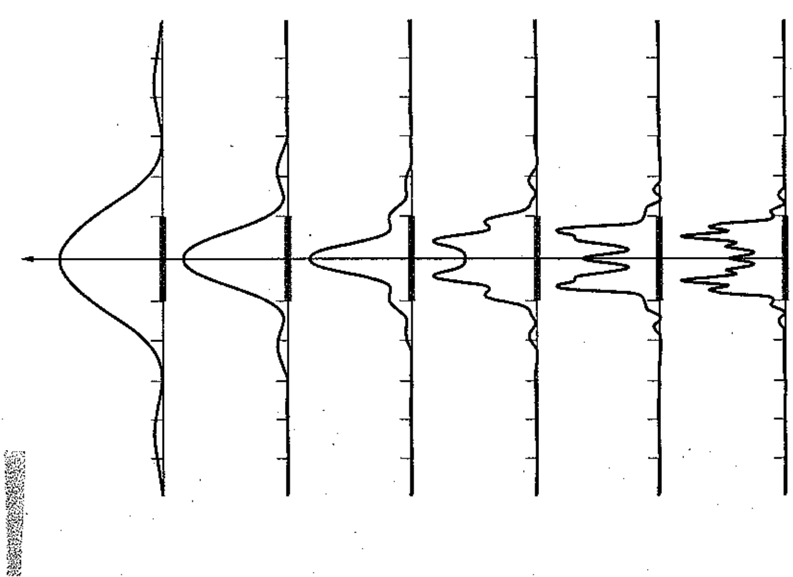
Hecht rule of thumb for far field:

$R > \dfrac{a^2}{\lambda}$

How do we manage?

↳ Lenses!

bring the far field to us!



gure 10.2 A succession of diffraction patterns at increasing distance rm a single slit; Fresnel at the bottom (nearby), going toward aunhofer at the top (faraway). *(Adapted from Fundamentals of Waves and cillations by K. U. Ingard.)*

How do lenses bend light?

"index of refraction" (material property)

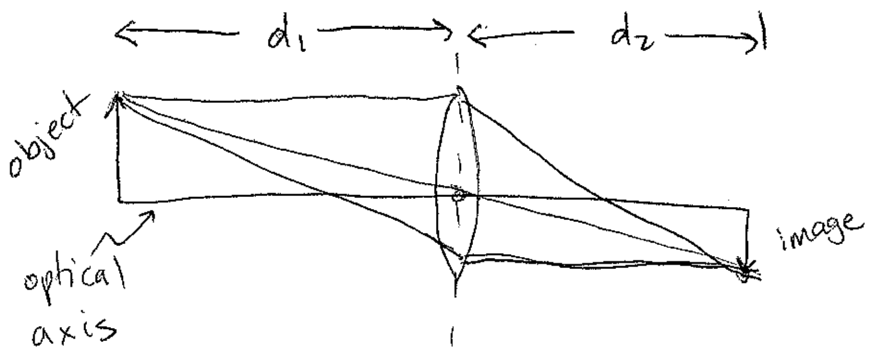
Snell's law: $n \sin\theta = n' \sin\theta'$

Fermat's Principle (from Rieke):

"The optical path from a point on the object through the optical system to the corresponding point on the image must be the same length for all neighboring rays"

This principle guides the way we think about lenses and design an optical system

Consider a converging thin Lens:



Lens Equation: $\frac{1}{f} = \frac{1}{d_1} + \frac{1}{d_2}$

f... focal length

- what if $d_1 = \infty$?

$$\frac{1}{f} = \cancel{\frac{1}{\infty}}^{\,0} + \frac{1}{d_2} \qquad d_2 = f$$

if the object is at ∞ (very far away) the image focus will be a focal length away.