Discrete homotopy theory and cubical sets

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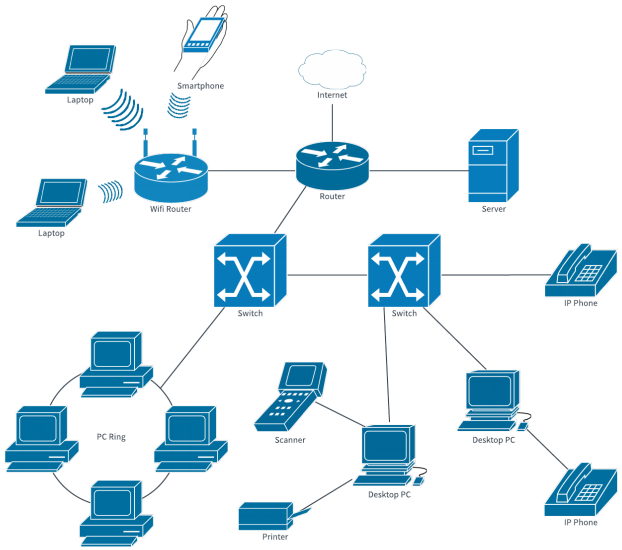
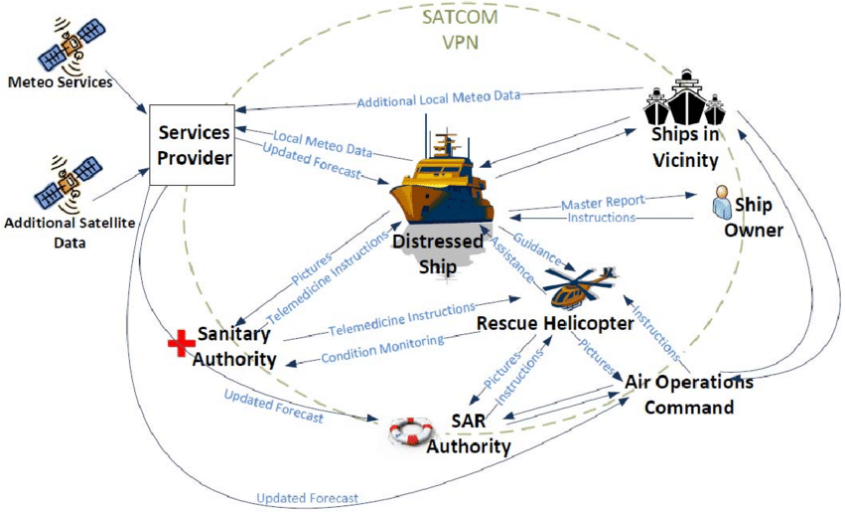
Outline

Original motivation

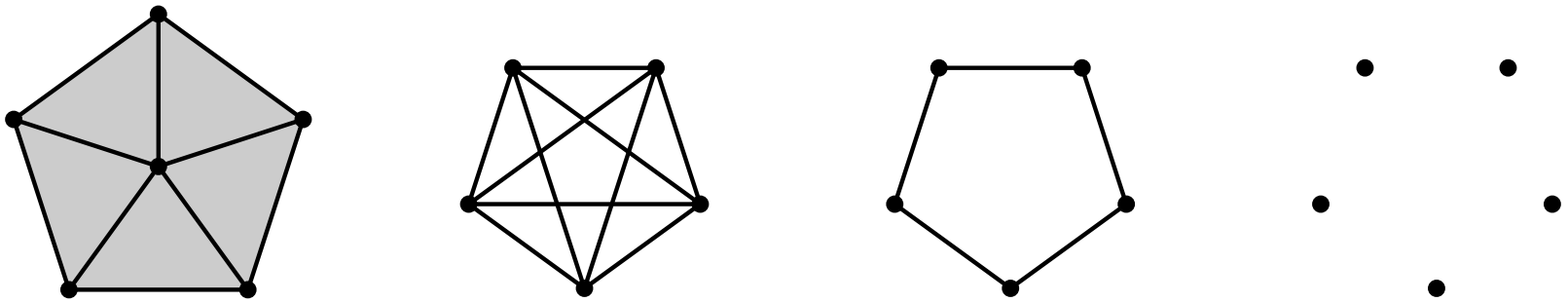
- Represent socio-technical complex systems as simplicial complexes K, possibly with dynamical information attached
- Identify "q-clusters" and "q-holes," i.e. well-connected regions and connectivity gaps in dimension q
- q-holes can represent structural deficiencies in the system
- Method: assign an object to K (for us, a group) measuring **combinatorial** connectedness in each dimension





Connectivity graphs

- K a simplicial complex
- Let $\Gamma_q(K)$ denote the q-**connectivity graph** of K
  - Vertices: maximal simplices $\sigma \in K$ of dimension $\geq q$
  - Edge between σ and τ if they share a q-face

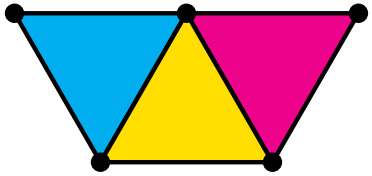


K $\quad$ $\Gamma_0(K)$ $\quad$ $\Gamma_1(K)$ $\quad$ $\Gamma_2(K)$

- q-**holes** are chordless cycles of length ≥ 5 in $\Gamma_q(K)$
- Can detect these combinatorially using homotopical ideas

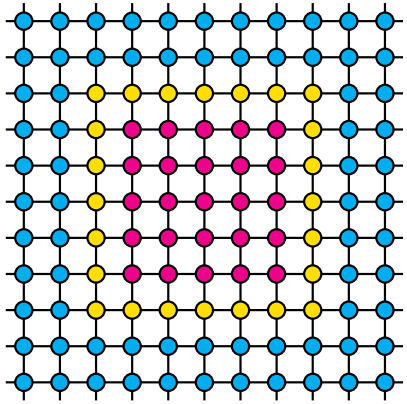
Graph maps and grids

- A **graph map** $f : G \to H$ is a function
  - $f : V(G) \to V(H)$
  - $u \sim v \Rightarrow f(u) \sim f(v)$ or $f(u) = f(v)$
- Let $\mathbb{Z}^n$ denote the infinite n-dimensional grid graph
- We want graph maps $f : \mathbb{Z}^n \to \Gamma_q(K)$ with "finite support" (constant outside finite set)



K

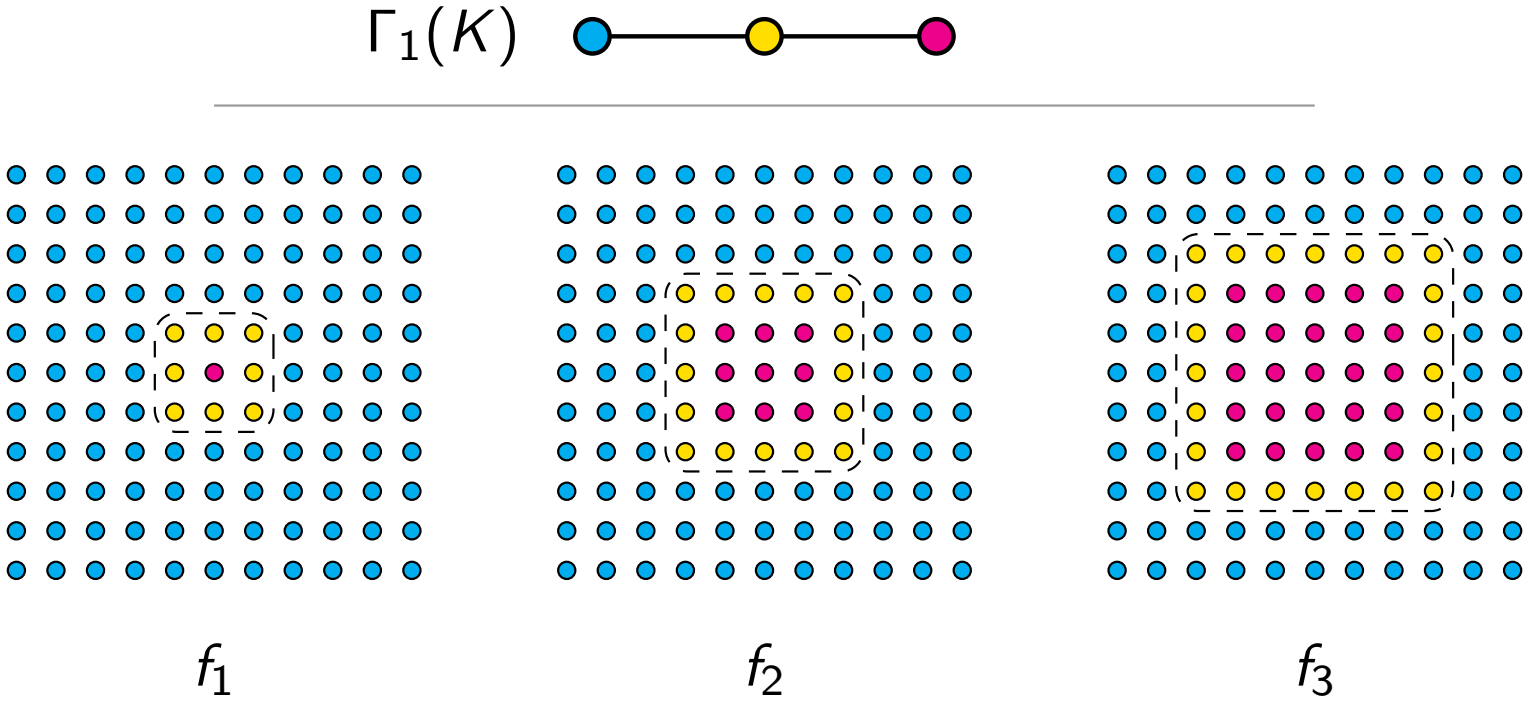
$\Gamma_1(K)$



$f : \mathbb{Z}^2 \to \Gamma_1(K)$

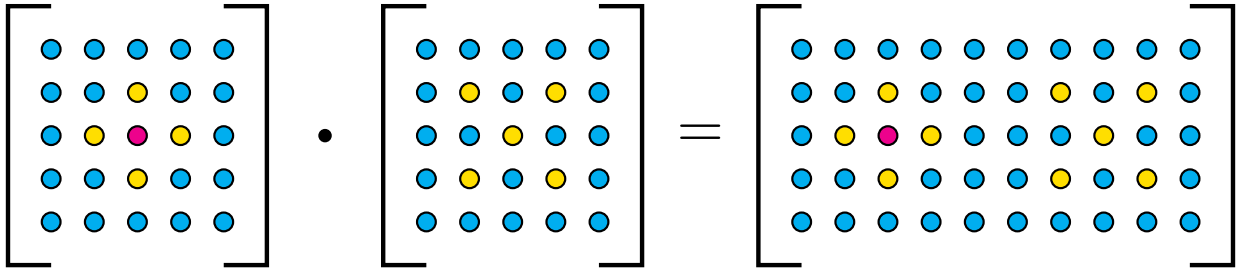
Discrete homotopy

- A **discrete homotopy** consists of
  - Finite sequence of graphs maps $f_i : \mathbb{Z}^n \to \Gamma_q(K)$ with finite support
  - For all i and $v \in V$ we have $f_i(v) \sim f_{i+1}(v)$ or $f_i(v) = f_{i+1}(v)$



Discrete homotopy groups

- Fix a base vertex $\sigma_0 \in \Gamma_q(K)$
- Discrete homotopy defines an equivalence relation on graph maps $f : \mathbb{Z}^n \to \Gamma_q(K)$ based at σ_0 ($f \equiv \sigma_0$ outside finite set)
- Can define a product on discrete homotopy classes:

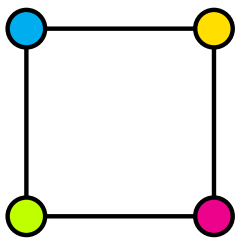


Definition-Theorem (Barcelo–Kramer–Laubenbacher–Weaver 2001)

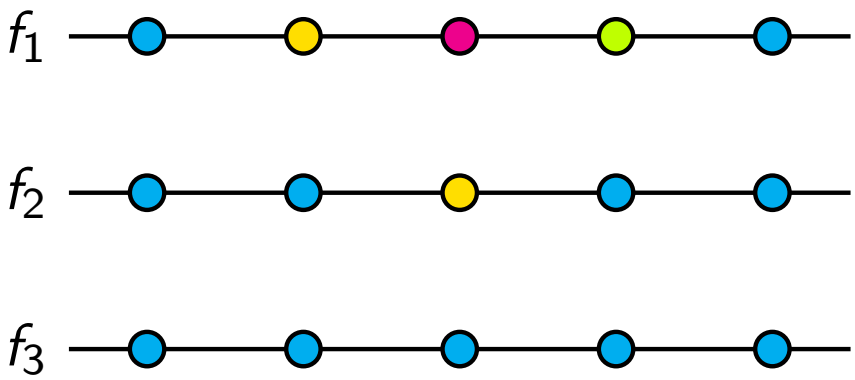
The **discrete homotopy groups** are the groups $A_n^q(K, \sigma_0)$ whose elements are discrete homotopy classes of graph maps $\mathbb{Z}^n \to \Gamma_q(K)$ based at σ_0 and whose products are defined as above.

Contracting the 4-cycle

- While $A_1(K, \sigma_0)$ detects chordless ≥ 5-cycles in $\Gamma_q(K)$, it ignores 3- and 4-cycles
- Highlights the cubical nature of the discrete homotopy groups
- Can contract a discrete loop around the 4-cycle in two steps:

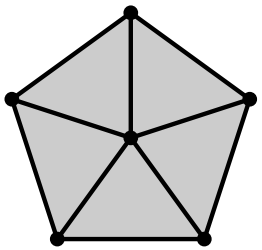


$\Gamma_q(K)$

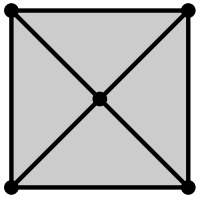


Examples

- If Δ is a simplex, then $A_n^q(\Delta, \sigma_0)$ is trivial for all q, $n > 0$ and σ_0
- If $n > 1$, then $A_n^q(K, \sigma_0)$ is abelian
- $A_1^q(K, \sigma_0)$ detects q-holes of length ≥ 5, but not of length ≤ 4:

$K =$ 

$$A_1^q(K, \sigma_0) \cong \begin{cases} \mathbb{Z} & \text{if } q = 1 \\ 1 & \text{if } q = 0, 2 \end{cases}$$

$L =$ 

$$A_1^q(L, \tau_0) \cong 1 \quad \text{if } q = 0, 1, 2$$

- Suppress the base point σ_0 when $\Gamma_q(K)$ is connected

Remarks

Proposition (Barcelo–Kramer–Laubenbacher–Weaver 2001)

Let $X^q(K)$ be the CW complex obtained by attaching a 2-cell to every 3- and 4-cycle of $\Gamma_q(K)$. Then $A_1^q(K, \sigma_0) \cong \pi_1(X^q(K), \sigma_0)$.

Special case: Graphs

For (connected) graphs $K = G$, we can define discrete homotopy groups $A_n(G)$ directly by using graph maps $\mathbb{Z}^n \to G$ instead of $\mathbb{Z}^n \to \Gamma_0(G)$.

Theorem (L. 2020)

For each n, there is an infinite family of graphs G for which $A_n(G)$ is nontrivial. These are the only known examples of nontrivial higher discrete homotopy groups in the literature.

Fleshing out the theory

Many ideas from classical topology can be meaningfully ported to the discrete setting:

- Discrete Seifert-van Kampen theorem
- Relative discrete homotopy groups and long exact sequences
- Accompanying homology theory for metric spaces, called **discrete singular cubical homology**
  - Satisfies discrete versions of Eilenberg-Steenrod axioms (plays nice with discrete homotopy)
  - Discrete Hurewicz theorem in dimension 1 (first homology group is abelianization of discrete fundamental group)
  - Spectral sequences

Application: Subspace arrangements

- W a finite real reflection group of rank n
- $\Sigma(W)$ the Coxeter complex of type W
- $\mathcal{W}_{n,k}$ the arrangement of fixed subspaces of all rank-$(k-1)$ irreducible parabolic subgroups of W (interesting when $k \geq 3$)
- $\mathcal{W}_{n,k}$ generalizes Coxeter arrangements ($k = 2$) and k-equal arrangements ($W = A_n$)

Theorem (Barcelo–Severs–White 2011)

Let $U(\mathcal{W}_{n,k})$ denote the complement of $\mathcal{W}_{n,k}$. Then

$$\pi_1(U(\mathcal{W}_{n,k})) \cong A_1^{n-k+1}(\Sigma(W)).$$

Real version of a result of Brieskorn

W admits a presentation with generating set S and relations

1. $s^2 = 1$ for all $s \in S$
2. $st = ts$ for all $s, t \in S$ with $m(s,t) = 2$
3. $sts = tst$ for all $s, t \in S$ with $m(s,t) = 3$

$$\vdots$$

Theorem (Rephrasing of Brieskorn 1971)

The fundamental group of the complement of the complexification of $\mathcal{W}_{n,2}$ is given by the above generators and relations, minus relation 1.

Theorem (Rephrasing of Barcelo–Severs–White 2011)

The fundamental group $\pi_1(U(\mathcal{W}_{n,3}))$ is given by the above generators with only the relations 1 and 2.

Application: Group theory

- $A_n(G)$ definition requires vertices of G to be distance ≤ 1 apart; can require only distance $\leq r$ to get generalization $A_{n,r}(G)$
- Let F_S denote free group (finite rank) with normal subgroup N and $\overline{S}$ the image of S in F_S/N
- Can recover N from F_S/N using homotopy of Cayley graph:

$$N \cong \pi_1(\mathrm{Cay}(F_S/N, \overline{S}))$$

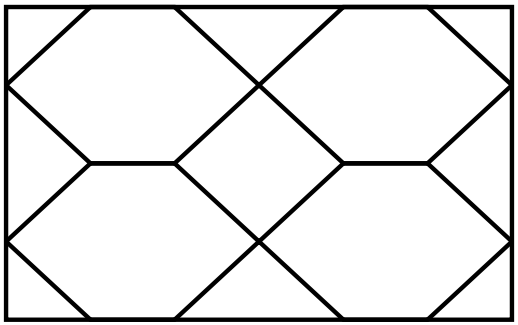
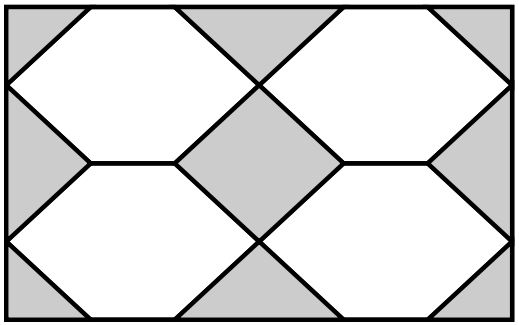
- Discrete homotopy can do the same for any finitely presented group

Theorem (Delabie–Khukhro 2020)

Let $G = \langle S \mid R \rangle$ be a finitely presented group with identity e and normal subgroup N. There is a value of r depending only on S and R such that $N \cong A_{1,r}(\mathrm{Cay}(G/N, \overline{S}), e)$.

What do we want?

- Consider only graphs $K = G$ from now on
- We understand concretely what $A_1(G)$ computes:

A_1  $\cong$ π_1 

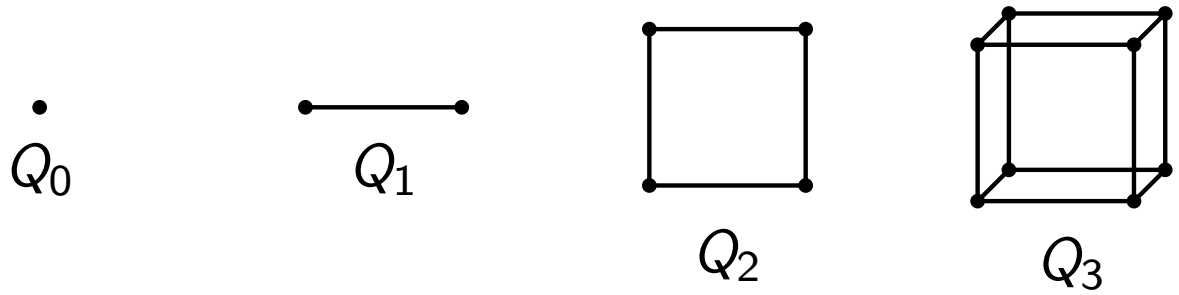
- Can we achieve a similar understanding of higher homotopy groups?

Goal

Construct a topological space X such that $A_n(G) \cong \pi_n(X)$ for all n.

Cube graphs

- Let $Q_n \subset \mathbb{Z}^n$ be induced by all vertices with all coordinates 0 or 1



- Fix a graph G whose discrete homotopy groups we are interested in
- Let $M_n(G) = \mathrm{Hom}(Q_n, G)$ (graph maps from the n-cube to G)
- We will define face and degeneracy maps for $M_\bullet(G)$

The cubical set

- For $i = 1, \ldots, n$ and $\varepsilon = 0, 1$ define

$$\begin{aligned} a_{i,\varepsilon}(n) : Q_{n-1} &\to Q_n \\ (x_1, \ldots, x_{n-1}) &\mapsto (x_1, \ldots, x_{i-1}, \varepsilon, x_{i+1}, \ldots, x_{n-1}) \end{aligned}$$

$$\begin{aligned} b_i(n) : Q_n &\to Q_{n-1} \\ (x_1, \ldots, x_n) &\mapsto (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \end{aligned}$$

- Recall that $M_n(G) = \mathrm{Hom}(Q_n, G)$. There are induced maps $\alpha_{i,\varepsilon}(n) : M_n(G) \to M_{n-1}(G)$ and $\beta_i(n) : M_{n-1}(G) \to M_n(G)$

The cubical set $M_\bullet(G)$

We obtain a cubical set $M_\bullet(G) : \square^{\mathrm{op}} \to \mathrm{Set}$ with face maps $\alpha_{i,\varepsilon}$ and degeneracy maps β_i.

Relating discrete and continuous homotopy groups

Theorem (Babson–Barcelo–de Longueville–Laubenbacher 2006)

Let $X(G)$ denote the geometric realization of $M_\bullet(G)$. If a certain cubical approximation property* holds, then for all n we have

$$A_n(G) \cong \pi_n(X(G)).$$

The asterisk: Proposed cubical approximation theorem

Let X be a cubical set and $f : I^n \to |X|$ a continuous map such that $f|_{\partial I^n}$ is cubical. There exists a cubical subdivision D^n of I^n and a cubical map $f' : D^n \to |X|$ such that $f \simeq f'$ and $f|_{\partial D^n} = f'|_{\partial D^n}$.

- While this statement seems plausible, no one has been able to prove it or find it in the literature!

Big questions (I am not a topologist ☺)

- Does the cubical approximation theorem hold?
- The CW complex $X(G)$ is infinite dimensional in general. Can we find a finite-dimensional deformation retract?
- Can we use the (conditional) fact that $A_n(G) \cong \pi_n(X(G))$ to directly find nontrivial $A_n(G)$ for $n \geq 2$?
- Using the theorem, can the tools of classical homotopy theory be leveraged to prove discrete versions of other famous theorems in topology? (Hurewicz for higher dimensions, Dold–Thom, etc.)

The end

Thank you!

References

E. Babson, H. Barcelo, M. de Longueville and R. Laubenbacher
Homotopy theory of graphs
J. Algebraic Combin., 24(1):31–44, 2006

H. Barcelo, X. Kramer, R. Laubenbacher and C. Weaver
Foundations of a connectivity theory for simplicial complexes
Adv. Appl. Math., 26(2):97–128, 2001

H. Barcelo, C. Severs and J. A. White
k-parabolic subspace arrangements
Trans. Amer. Math. Soc., 363(11):6063–6083, 2011

T. Delabie and A. Khukhro
Coarse fundamental groups and box spaces
Proc. Roy. Soc. Edinburgh Sect. A, 150(3):1139–1154, 2020

References

E. Brieskorn
Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe
Invent. Math., 12:57–61, 1971

B. Lutz
Higher discrete homotopy groups of graphs
arXiv e-prints, arXiv:2003.02390, 2020