Efficiency, sorting, and selection

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Summary

In labor markets in which workers' types are private information, Liu *et al.* (2014) show that stable matchings are positive assortative and efficient within monotonic and supermodular domains. This paper shows that these properties are obtained because within their domains most failures of positive assortativeness and efficiency lead workers to select themselves favorably with respect to the firms, but proves a more general result, because monotonicity can be weakened, that holds true regardless of how firms' values depend on workers' types. The generalization hinges more heavily on firms' higher-order inferences, and the weaker preference domain not only maintains the sorting and efficiency of all stable matchings, but also enlarges the set of transfers that support them.

Keywords: efficiency, incomplete information, selection, stability, sorting.

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1 Introduction

Two landmarks in the theory of two-sided matching markets with transferable utilities are the equivalence between efficiency—surplus maximization—and stability (see, e.g., Shapley & Shubik (1971)) and the positive assortativeness of efficient matchings, when the match surplus is strictly supermodular and strictly increasing in agents' types (see e.g., Becker (1973)). Alas, these results rely on the presence of complete information—i.e., on agents' types being commonly known—so that the scope of their conclusions is somewhat limited.

Interestingly, Liu *et al.* (2014) recently put forward a notion of stability for markets in which workers are privately informed about their types—incomplete-information stability—and show that all incomplete-information stable matchings are positive assortative and efficient, when firms' and workers' values are increasing in both types ([i]), and both match surpluses and workers' values are strictly supermodular ([ii]).

Incomplete-information stability presumes that once an array of workers' types is in place, and all agents commonly observe an individually rational allocation, each firm is informed of the type of its own worker and agrees to participate in a complete-information block—i.e., a blocking opportunity at the given array of workers' types—if and only if the block is profitable at every array of workers' types that accounts for the type of its own worker, the willingness to participate of the blocking worker, and the fact that no other block takes place. Thus, incomplete-information stability captures necessary conditions for stability.

Given that the set of incomplete-information stable matchings is a superset of those under complete information, the search for positive assortativeness in the presence of incomplete information must maintain the strict monotonicity and strict supermodularity of the match surplus. Yet why does strengthening monotonicity from surpluses to values, as required by [i], and adding the strict supermodularity of workers' values to that of match surpluses, as demanded by [ii], deliver assortativeness and efficiency, and to what extent is doing so necessary?

This paper addresses these two questions by means of two related contributions. The first one, Lemma 2, argues that [i] and [ii] imply that some complete-information blocks, including those arising from most failures of positive assortativeness and efficiency, lead the blocking workers to select themselves *favorably* with respect to the blocking firms in the sense that a lower wage may only attract types of workers that generate a lower value for the firms.[1] The second, Proposition 2, refutes the natural, follow-up hypothesis whereby positive assortativeness and efficiency are obtained *only when* workers' selection is favorable by showing that the result in Liu *et al.* (2014) continues to hold under—[ii] and—a weakening of [i] that is silent about how firms' values depend on workers' types.

To understand why the presence of favorable selection—i.e., Lemma 2—drives assortativeness and efficiency under [i] and [ii], consider the simplest failure of positive assortativeness; namely, suppose that an incomplete-information stable allocation were to prescribe that a worker is unmatched despite being of a higher type than a worker who is matched. If the surplus increases with a worker's type, as implied by [i], there would be a blocking opportunity between the unmatched worker and the given firm, regardless of what transfers are at play. Given the incomplete-information stable nature of the allocation, there must be an alternative array of workers' types, consistent with the firm's information, at which every blocking transfer is still profitable for the unmatched worker, but not for the blocking firm. Because [i] demands the worker's value to increase with respect to her own types, however, every blocking transfer at the original array is only profitable to the unmatched worker at alternative arrays at which her type is *higher*. But since firms' values increase with respect to workers' types, also demanded by [i],

[1] The term "favorable" was coined by Jovanovic (1982), but the literature often uses "advantageous" as well (see, e.g., Ali *et al.* (2021)).

the worker's selection would then be favorable to the firm. Hence, the firm would not be able to refuse, and the incomplete-information stability hypothesis would be refuted. Intuitively, [i] implies that the "lowest" transfer the worker is willing to accept *decreases* with her own type. Thus, "lower prices lead to higher quality."

Proposition 2 notices that the incomplete-information stability hypothesis would still be refuted *without* any assumption on how firms' values vary with respect to workers' types, and thus without Lemma 2, because the ability all firms have to draw higher-order inferences would lead the firm in question to infer what it wanted to learn. Thus, neither how firms' values depend on their own types and on those of workers, nor how workers' values depend on firms' types, plays a role that cannot be mimicked by the standard monotonicity of the surpluses and the embedded presence of higher-order inferences. In a nutshell: We can replace [i] with the weaker assumption that workers' values are strictly increasing in their own types and all match surpluses are strictly increasing in both dimensions ([iii]).

To be clear, Lemma 2 implies that, under [i] and [ii], most other failures of positive assortativeness and efficiency also lead to favorable selection (Section 5.1), even though Proposition 2 reveals, once again, that that is not necessary (Section 6). Further, Lemma 2 not only applies to complete-information blocks that arise from the disassortative or inefficient nature of the underlying matching, but also to those with respect to matchings that are both assortative and efficient. Thus, replacing [i] with [iii] not only maintains the positive assortativeness and efficiency of all incomplete-information stable matchings, but also enlarges the set of transfers that supports them (Section 7.1). In fact, the content of Lemma 2, namely that under [i] and [ii] workers select themselves favorably in some complete-information blocks, does not change if one works with the variant of [i] where firms' values are instead decreasing in workers' types (Lemma 3).

By revealing that [i] can be substantially weakened, and thus generalizing the main result in Liu *et al.* (2014), Proposition 2 offers a sharper understanding of *when* assortativeness and efficiency are obtained, pretty much regardless of the stability notion one looks at. However, the real value of Proposition 2 might lie in its conclusion that the presence of favorable selection does *not*, in fact, explain *why* assortativeness and efficiency are obtained. That, indeed, assortaiveness and efficiency are properties of incomplete-information stability in a wider set of environments than those covered by Liu *et al.* (2014), including some that prescribe adverse selection.[2] If so, the reader might be thinking that an explanation for *why* assortativeness and efficiency are obtained, under [ii] and [iii], is still called for. After all, the efficiency of incomplete-information stable outcomes does not seem to conform with how lemon markets work. Here the answer is actually well known: The implicit dynamic nature of stability grants all firms, in the presence of [ii] and [iii], a perfect, sequential screening ability (see Section 7.4).

The rest of the paper is organized as follows. Section 2 introduces the environment and Section 3 the notion of incomplete-information stability introduced by Liu *et al.* (2014). Section 4 introduces the notions of efficiency and positive assortativeness, and Section 5 argues that the main assumptions in Liu *et al.* (2014) lead to favorable selection, which explains why they deliver assortativeness and efficiency. Section 6, on the other hand, contains the main result of the paper, which implies that efficiency and positive assortativeness can be obtained under adverse selection. Finally, Section 7 makes a few final remarks and offers a review of the relevant literature.

2 The environment

There is a finite set of workers, I, and a finite set of firms, J, with $i \in I$ and $j \in J$. There is also a finite set of types of workers, W, and a finite set of types of firms,

[2] Given Lemmas 2 and 3, those environments must feature firms with non-monotonic preferences. The reader is invited to look at Examples 1 and 2, which feature firms with single-peaked and single-dipped preferences, respectively.

F, where $W = \{w^1, w^2, ..., w^K\} \subseteq \mathbb{R}_+$, $F = \{f^1, f^2, ..., f^L\} \subseteq \mathbb{R}_+$, and w^k and f^l are increasing in their indices. Firms' types are commonly known by workers and firms. Thus, a **state** is a vector $\mathbf{w} \in W^{|I|}$ of workers' types. I write $w \in W$ and $f \in F$ for generic elements of W and F, but also use $\mathbf{w}_i$ and $\mathbf{f}_j$ to denote the type of worker i and firm j when the state is $\mathbf{w}$ and the array of firms' types is $\mathbf{f}$.

Value is generated by matches. Following Liu *et al.* (2014), I take as primitive the agents' premuneration values; namely, the aggregate match value each agent receives in the absence of payments. Thus, a match between a worker of type $w \in W$ and a firm of type $f \in F$ gives rise to a premuneration value $\nu_{wf} \in \mathbb{R}$ for the worker and a premuneration value $\phi_{wf} \in \mathbb{R}$ for the firm. The sum of these premuneration values, $S_{wf} := \nu_{wf} + \phi_{wf}$, is the *surplus* of the match. I assume that the premuneration value of unmatched agents is zero and use the notation $f_\emptyset = \emptyset = \omega_\emptyset$, with the convention that $\emptyset < w$ and $\emptyset < f$ for every $\omega \in W$ and every $f \in F$.

Given a state $\mathbf{w}$, a matching between worker i and firm j gives rise to *payoffs*

$$\pi_i^{\mathbf{w}} := \nu_{\mathbf{w}_i \mathbf{f}_j} + p \text{ and } \pi_j^{\mathbf{f}} := \phi_{\mathbf{w}_i \mathbf{f}_j} - p,$$

where $p \in \mathbb{R}$ is the (possibly negative) payment from j to i.

A *matching* is a function $\mu : I \to J \cup \{\emptyset\}$, one-to-one on μ^{-1}, that assigns worker i to $\mu(i)$, where $\mu(i) = \emptyset$ means that i is unmatched. Similarly, $\mu^{-1}(j)$ denotes the assignment of firm j, where $\mu^{-1}(j) = \emptyset$ means that j is unmatched. I will use μ_i and μ_j to denote the (possibly empty) assignments of i and j, respectively.

A *payment scheme* $\mathbf{p}$ associated with a matching μ is a vector that specifies a payment $\mathbf{p}_{i,\mu_i} \in \mathbb{R}$ for each i and a payment $\mathbf{p}_{\mu_j,j} \in \mathbb{R}$ for each j. I assume that $\mathbf{p}_{i,\emptyset} = \mathbf{p}_{\emptyset,j} = 0$.

An *allocation* is a pair $(\mu, \mathbf{p})$, consisting of a matching and a payment scheme, and an *outcome* is a tuple $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f})$.

To capture firms' uncertainty about workers' types, I follow Liu *et al.* (2014)

and assume that the true state is drawn from some distribution with support $\Omega \subseteq W^{|I|}$.

3 Stability

3.1 Individual rationality

Definition 1. *An outcome* $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f})$ *is individually rational if*

$$\nu_{\mathbf{w}_i \mathbf{f}_j} + \mathbf{p}_{i,\mu_i} \geq 0 \text{ for every } i \in I, \text{ and}$$
$$\phi_{\mathbf{w}_i \mathbf{f}_j} - \mathbf{p}_{\mu_j,j} \geq 0 \text{ for every } j \in J.$$

I write $\sum^0$ for the set of individually rational outcomes.

3.2 Complete information

The following definition describes the well-known notion of stability introduced by Shapley & Shubik (1971) for environments with complete information (see also Crawford & Knoer (1981).)

Definition 2. *An outcome* $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f})$ *is* ***complete-information stable*** *if* $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \sum^0$ *and there is no complete-information block; i.e., there is no worker-firm pair* (i, j) *and payment* $p \in \mathbb{R}$ *such that*

$$\nu_{\mathbf{w}_i \mathbf{f}_j} + p > \nu_{\mathbf{w}_i \mathbf{f}_{\mu_i}} + \mathbf{p}_{i,\mu_i} \text{ and } \phi_{\mathbf{w}_i \mathbf{f}_j} - p > \phi_{\mathbf{w}_{\mu_j} \mathbf{f}_j} - \mathbf{p}_{\mu_j,j}.$$

Notice that (i, j) forms a complete-information block at $(\mathbf{w}, \mathbf{f})$ if and only if $S_{\mathbf{w}_i \mathbf{f}_j} > \pi_i^{\mathbf{w}} + \pi_j^{\mathbf{f}}$.

3.3 Incomplete information

The following blocking notion, introduced by Liu *et al.* (2014), extends the notion of complete-information block to markets in which workers' types are private information, but no dissolution is observed.

Definition 3. *Fix any nonempty set* $X \subseteq \sum^0$. *An outcome* $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in X$ *is* X***-blocked*** *if there exists* (i, j) *and* $p \in \mathbb{R}$ *such that*

1. $\nu_{\mathbf{w}_i, \mathbf{f}_j} + p > \nu_{\mathbf{w}_i \mathbf{f}_{\mu_i}} + \mathbf{p}_{i,\mu_i}$, *and*

2. $\phi_{\tilde{\mathbf{w}}_i \mathbf{f}_j} - p > \phi_{\tilde{\mathbf{w}}_{\mu_j} \mathbf{f}_j} - \mathbf{p}_{\mu_j, j}$, *for every* $\tilde{\mathbf{w}} \in \Omega$ *with* $(\mu, \mathbf{p}, \tilde{\mathbf{w}}, \mathbf{f}) \in X$ *such that*

$$\tilde{\mathbf{w}}_{\mu_j} = \mathbf{w}_{\mu_j} \text{ and } \nu_{\tilde{\mathbf{w}}_i, \mathbf{f}_j} + p > \nu_{\tilde{\mathbf{w}}_i \mathbf{f}_{\mu_i}} + \mathbf{p}_{i,\mu_i}.$$

To understand Definition 3, let $X = \sum^0$. An outcome $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \sum^0$ is $\sum^0$-blocked by worker i and firm j if and only if (i, j) forms a complete-information block at $(\mathbf{w}, \mathbf{f})$, *and at every other state* in Ω that is consistent with the signals j receives; namely, with the type of its own worker, the individually rational nature of the allocation, and i's willingness to participate in the block.

Liu *et al.* (2014) assume that firms draw all possible inferences stemming from observing not only that no match is dissolved, but also that no other complete-information block is formed. Thus, they define, for every $k \geq 1$, the sets

$$\sum^k := \{(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) : (\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \sum^{k-1} \text{ and } (\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \text{ is not } \sum^{k-1}\text{-blocked}\}.$$

The set of incomplete-information stable outcomes studied by Liu *et al.* (2014) is given by $\sum =: \bigcap_{k \geq 1} \sum^k$. Notice that the set of incomplete-information stable allocations is nonempty regardless of the arrays of types; i.e., $\sum(\mathbf{w}, \mathbf{f}) \neq \emptyset$ for every $(\mathbf{w}, \mathbf{f})$.

4 Assortativeness and efficiency

Like Liu *et al.* (2014), I will focus on the standard notions of sorting and efficiency.

Definition 4. *A matching* μ *is positive assortative at* $(\mathbf{w}, \mathbf{f})$ *if*

1. *for every* i, i' *with* $\mu_i \neq \emptyset$ *we have* $\mathbf{w}_{i'} > \mathbf{w}_i \Rightarrow \mathbf{f}_{\mu_{i'}} \geq \mathbf{f}_{\mu_i}$, *and*

2. *for every* j, j' *with* $\mu_j \neq \emptyset$ *we have* $\mathbf{f}_{j'} > \mathbf{f}_j \Rightarrow \mathbf{w}_{\mu_{j'}} \geq \mathbf{w}_{\mu_j}$.

I will refer to 1 and 2 in this definition as "worker assortativeness" and "firm assortativeness," respectively. The standard definition of efficiency is as follows:

Definition 5. *A matching* μ *is efficient at* $(\mathbf{w}, \mathbf{f})$ *if* $\sum_{i\in I} S_{\mathbf{w}_i \mathbf{f}_{\mu_i}} = max_{\mu'} \sum_{i\in I} S_{\mathbf{w}_i \mathbf{f}_{\mu'_i}}$.

The main result in Liu *et al.* (2014) is concerned with preference domains that satisfy the following two assumptions:

Assumption 1 (Monotonicity)**.** *Workers' premuneration value* ν_{wf} *and firms' premuneration value* ϕ_{wf} *are increasing in* w *and* f, *with* ν_{wf} *strictly increasing in* w *and* ϕ_{wf} *strictly increasing in* f.

Assumption 2 (Supermodularity)**.** *The worker premuneration value* ν_{wf} *and the match surplus* S_{wf} *are strictly supermodular in* w *and* f.

With these two assumptions, Liu *et al.* (2014) prove the following result:

Proposition 1 (Proposition 3 in Liu *et al.* (2014))**.** *Under Assumptions 1 and 2, every incomplete-information stable outcome is efficient.*

The proof provided by Liu *et al.* (2014) makes use of the following strong link between efficiency and positive assortativeness in environments in which the surplus is strictly increasing and strictly supermodular:[3]

Lemma 1. *Suppose that the surplus is strictly increasing in* w *and* f, *and strictly supermodular. Then, matching* μ *is efficient at* $(\mathbf{w}, \mathbf{f})$ *if and only if* μ *is positive assortative at* $(\mathbf{w}, \mathbf{f})$, $S_{\mathbf{w}_i \mathbf{f}_{\mu_i}} \geq 0$ *for every* i, *and there is no* (i, j) *with* $\mu_i = \mu_j = \emptyset$ *such that* $S_{\mathbf{w}_i \mathbf{f}_j} > 0$.

[3]The proof of Lemma 1 is omitted, but Liu *et al.* (2014) use a similar result, Lemma B.1, which is however stronger than Lemma 1 because it uses Assumption 1, which imposes more than the strict monotonicity of the surplus, and Assumption 2, which demands more than the strict supermodularity of the surplus.

Armed with Lemma 1, the proof of Proposition 1 in Liu *et al.* (2014) first shows that Assumptions 1 and 2 imply that all incomplete-information stable matchings are positive assortative, and then uses that to conclude that all incomplete-information stable matchings must be efficient, because they satisfy $S_{\mathbf{w}_i \mathbf{f}_{\mu_i}} \geq 0$ for every i and $S_{\mathbf{w}_i \mathbf{f}_j} \leq 0$ for every pair (i, j) with $\mu_i = \mu_j = \emptyset$.[4]

5 Favorable selection

This section argues that Assumptions 1 and 2 deliver assortativeness and efficiency in Proposition 1 because they lead workers to select themselves favorably with respect to the firms in a class of complete-information blocks that includes those arising from most failures of assortativeness and efficiency.

The following result says that not every complete-information block is consistent with incomplete-information stability, under Assumptions 1 and 2:

Lemma 2. *Suppose that Assumptions 1 and 2 hold, and fix any* $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \sum^1$. *If* (i, j) *is a complete-information block at* $\mathbf{w}$*, then* $\mu_i \neq \emptyset$ *and* $\mathbf{f}_{\mu_i} \geq \mathbf{f}_j$.

The proof of Lemma 2 can be found in the Appendix, but it states that incomplete-information stability is *not* consistent with complete-information blocks that involve either an *unmatched* worker or a worker that is matched to a firm with a type that is *lower* than the type of the blocking firm, independently of the number of inferences firms can make.[5] The reason, plainly, is that in all of those complete-information blocks the assumption that firms' values are increasing in workers' types—embedded in Assumption 1—makes the selection of the blocking worker favorable to the blocking firm. Thus, the blocking firm is not able to refuse. To illustrate this, and understand why the presence of favorable selection drives Proposition 1, a sketch of the proof of Lemma 2 might prove helpful.

[4] The reader can look, respectively, at their Lemma B.5, and their argument in page 583.

[5] This result is not stated in Liu *et al.* (2014), but their Lemma 2 contains the same insight.

Suppose that an outcome $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f})$ is complete-information blocked by a pair (i, j) that features $\mu_i = \emptyset$, and consider the transfer

$$p^{\epsilon}_{\mathbf{w}_i} := -\nu_{\mathbf{w}_i \mathbf{f}_j} + \epsilon,$$

where $\epsilon > 0$. When ϵ is small, this is the "smallest" blocking transfer that i is willing to accept. The fact that the outcome is incomplete-information stable implies that there must be some $\mathbf{w}^0$ with $\mathbf{w}^0_i = \mathbf{w}_i$ at which the matching is individually rational and the block involving $p^{\epsilon}_{\mathbf{w}_i}$ is still profitable to i, but not to j, for every $\epsilon > 0$. The latter implies that

$$\phi_{\mathbf{w}_i \mathbf{f}_j} > \phi_{\mathbf{w}^0_i \mathbf{f}_j}, \tag{1}$$

and the former is equivalent to

$$\nu_{\mathbf{w}^0_i \mathbf{f}_j} - \nu_{\mathbf{w}_i \mathbf{f}_j} \geq 0. \tag{2}$$

Intuitively, (1) is necessary for j to say "no," and (2) is necessary for i to say "yes," to $p^{\epsilon}_{\mathbf{w}_i}$, at $\mathbf{w}^0$. Thus, the reader should think of (2) as the selection constraint faced by firm j. Since ν_{wf} is strictly increasing in w, by Assumption 1, (2) delivers $\mathbf{w}_i \leq \mathbf{w}^0_i$, which means that only types that are higher than $\mathbf{w}_i$ would select themselves to say "yes." But since ϕ_{wf} is increasing in w, also required by Assumption 1, it follows that i's selection is favorable to j. Thus, j cannot say "no." Indeed, the assumption that ϕ_{wf} is increasing in w implies that (1) delivers $\mathbf{w}_i > \mathbf{w}^0_i$. Intuitively, lower "prices" drive the "quality" up, not down, because (2) is equivalent to $p^{\epsilon}_{\mathbf{w}_i} > p^{\epsilon}_{\mathbf{w}^0_i}$, so that the "smallest" transfer i is willing to accept *decreases* with her type. As a consequence, j cannot refuse.

A similar argument can be made if $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \sum^0$ is complete-information blocked by a pair (i, j) that features $\mu_i \neq \emptyset$ and $\mathbf{f}_j > \mathbf{f}_{\mu_i}$. To see why, notice that in this case the "smallest" transfer that i is willing to accept at $\mathbf{w}$ is

$$p^{\epsilon}_{\mathbf{w}_i} := \nu_{\mathbf{w}_i \mathbf{f}_{\mu_i}} + \mathbf{p}_{i,\mu_i} - \nu_{\mathbf{w}_i \mathbf{f}_j} + \epsilon.$$

As before, j says "no" only if $\mathbf{w}_i > \mathbf{w}^0_i$ and i says "yes" only if $\mathbf{w}^0_i \geq \mathbf{w}_i$, because i is willing to accept $p^{\epsilon}_{\mathbf{w}_i}$, at $\mathbf{w}^0$, if and only if

$$\nu_{\mathbf{w}^0_i \mathbf{f}_j} - \nu_{\mathbf{w}_i \mathbf{f}_j} + \nu_{\mathbf{w}^0_i \mathbf{f}_{\mu_i}} - \nu_{\mathbf{w}_i \mathbf{f}_{\mu_i}} \geq 0, \tag{3}$$

which implies that $\mathbf{w}^0_i \geq \mathbf{w}_i$, because $\mathbf{f}_j > \mathbf{f}_{\mu_i}$ and ν_{wf} is strictly supermodular, by Assumption 2. Intuitively, i select herself favorably with respect to firm j, because $p^{\epsilon}_{\mathbf{w}_i} > p^{\epsilon}_{\mathbf{w}^0_i}$, so that the smallest transfer i is willing to accept again *decreases* with her type. As a consequence, j cannot say "no." □

In a nutshell: The strict supermodularity and strict monotonicity—with respect to w—of ν_{wf} govern the selection of matched and unmatched workers, respectively, which guarantees that every—and thus the "minimum"—wage a given type of worker is willing to accept is also, *but only*, accepted by higher or lower types, respectively, depending on whether the firm making the offer is better or worse than the worker's firm. If firms' values increase with respect to workers' types, as Assumption 1 requires, the former means that the worker's selection is favorable with respect to the given firm.

5.1 Liu, Mailath, Postlewaite, and Samuelson

It is not hard to see that Lemma 2 is the key driving force in Proposition 1. If worker assortativeness fails, we have i and i', with $\mu_i \neq \emptyset$, such that

$$\mathbf{w}_{i'} > \mathbf{w}_i$$
$$\mathbf{f}_{\mu_{i'}} < \mathbf{f}_{\mu_i}.$$

We can have either $\mu_{i'} = \emptyset$ or $\mu_{i'} \neq \emptyset$. Since Assumption 1 implies that the surplus is strictly increasing in w, the former implies that (i', μ_i) must form a complete-information block, contradicting Lemma 2. If $\mu_{i'} \neq \emptyset$, instead, then the strict supermodularity of the surplus, granted by Assumption 2, implies that either (i', μ_i) or $(i, \mu_{i'})$ must form a complete-information block. The former contradicts Lemma 2, because $\mathbf{f}_{\mu_{i'}} < \mathbf{f}_{\mu_i}$, but a complete-information block by $(i, \mu_{i'})$ does not (see See Section 5.2 below).

Similarly, a failure of firm assortativeness would say that there are two firms, j and j', with $\mu_j \neq \emptyset$, such that

$$\mathbf{f}_{j'} > \mathbf{f}_j$$

$$\mathbf{w}_{\mu_{j'}} < \mathbf{w}_{\mu_j}.$$

As before, we can have either $\mu_{j'} = \emptyset$ or $\mu_{j'} \neq \emptyset$, but the latter would imply a failure of worker assortativeness and can therefore be dealt with using the same arguments described above. If $\mu_{j'} = \emptyset$, then the strict monotonicity of S_{wf} with respect to f embedded in Assumption 1 implies that (μ_j, j') must form a complete-information block at $\mathbf{w}$. This contradicts Lemma 2, because $\mathbf{f}_{j'} > \mathbf{f}_j$.

Given that incomplete-information stability delivers positive assortativeness, Lemma 1 implies that an incomplete-information stable matching μ fails to be efficient if and only if there is some (i, j) with $\mu_i = \mu_j = \emptyset$ such that $S_{\mathbf{w}_i \mathbf{f}_j} > 0$. If that's the case, however, (i, j) must form a complete-information block, contradicting, again, Lemma 2.

5.2 Higher-order inferences

The reason Lemma 2 is not contradicted when $(i, \mu_{i'})$ above forms a complete-information block is that the worker's selection is *adverse* to the firm. Indeed, both (1) and (3) would demand $\mathbf{w}_i^0 < \mathbf{w}_i$, when $\mu_{i'}$ takes the place of j. But then, how does one reach the desired contradiction when dealing with failures of positive assortativeness (and efficiency) that only lead to a complete-information block involving the low-low pair? This is where one invokes the higher-order inferences of the low-type firm, because $\mathbf{w}_{\mu_{i'}}^0 = \mathbf{w}_{\mu_{i'}}$ and $\mathbf{w}_i^0 < \mathbf{w}_i$ imply that the failure of worker assortativeness carries over to $\mathbf{w}^0$. At $\mathbf{w}^0$ the whole argument can be repeated, so that, again, $(i, \mu_{i'})$ is the only complete-information block that does not contradict Lemma 2. The resulting unraveling eventually informs the low-type firm that the true type of i, the low-type worker, must be $\mathbf{w}_i$. The next figure illustrates this:

$$\mathbf{w}_i^n = ... = \mathbf{w}_{i'}^0 = \mathbf{w}_{i'} > \mathbf{w}_i > \mathbf{w}_i^0 > ... > \mathbf{w}_i^n$$
$$\mathbf{f}_{\mu_{i'}} < \mathbf{f}_{\mu_i}.$$

Figure 1: The failure of worker-assortativeness at $\mathbf{w}$ unravels, with $(i, \mu_{i'})$ being the only complete-information block at $\mathbf{w}$ and at $\mathbf{w}^n$, for every $K \geq n \geq 0$.

6 Main result

Consider the following weakening of Assumption 1:

Assumption 3 (Weak Monotonicity)**.** *Workers' premuneration value ν_{wf} is strictly increasing in w, and the match surplus S_{wf} is strictly increasing in w and f.*

The label "Weak Monotonicity" in Assumption 3 intends to emphasize that Assumption 3 is weaker than Assumption 1. Indeed, Assumption 1 implies that the match surplus is strictly increasing in w and f, but Assumption 3 is silent on how firms' values depend on w and f, or how workers' values vary with f. Arguably, Assumption 3 is substantially weaker than Assumption 1.

The following result, which constitutes the main contribution of the paper, generalizes Proposition 1:

Proposition 2. *Under Assumptions 2 and 3, every incomplete-information stable outcome is efficient.*

The proof of Proposition 2 can be found in the Appendix, but Sections 6.1, 6.2, and 6.3 below provide sketches that the interested reader might find helpful.

The key difference between Propositions 1 and 2 is that the latter is *not* driven by Lemma 2, because Lemma 2 does not necessarily hold under Assumptions 2 and 3. Thus, the presence of favorable selection is unnecessary to achieve assortativeness and efficiency.

Plainly, Proposition 2 shows that firms' ability to make higher-order inferences allows one to dispense with any monotonicity assumption on ϕ_{wf}. Intuitively, one can let the incomplete-information stability hypothesis, together with the strict supermodularity and strict monotonicity of ν_{wf} (with respect to w), which govern the selection of workers, dictate whether the alternative array—imagined by any blocking firm—features a higher or lower type for the blocking worker. This is all that we need to ensure that firms' ability to draw higher-order inferences leads them to infer what they need to know about the true array.

Example 1 below illustrates that Proposition 2 does not hinge on Lemma 2 by providing an outcome, $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f})$, that is not positive assortative but belongs to $\sum^1$ even though there is a complete-information block by the high-high pair.

Example 1. *There are two workers and two firms, $I = \{i_1, i_2\}$ and $J = \{j_1, j_2\}$. The set of types are $W = \{2, 3, 4, 5\}$ and $F = \{2, 3\}$. Firms' premuneration values are independent of f, and depend on w as follows:*

$$\phi_w = \begin{cases} 6 & \textit{if } w = 3 \\ 5 & \textit{if } w = 4 \\ 3 & \textit{if } w = 5 \\ 2 & \textit{if } w = 2 \end{cases}$$

Thus, firms' values are neither increasing nor decreasing on w. Workers' premuneration values are described in the following table, where $\bar{\epsilon} > 0$:

	2	3	4	5
2	2	4	$6-\tilde{\epsilon}$	8
3	3	6	$9-\tilde{\epsilon}$	12

Figure 2: Workers' premuneration values.

Notice that workers' values are strictly increasing in their own types, and strictly supermodular, for $\tilde{\epsilon} > 0$ small enough.[6] *The following three tables describe three different outcomes, differing only in their state,* $\mathbf{w}$, $\mathbf{w}'$, *and* $\mathbf{w}''$*:*

[6] Workers' values can be written as

$$\nu_{wf} = \begin{cases} wf - f & \text{if } w \neq 4 \\ wf - f - \tilde{\epsilon} & \text{if } w = 4 \end{cases}$$

where $\tilde{\epsilon} > 0$ should be taken to be "small." Notice that the resulting surplus is strictly supermodular. The reason to include $\tilde{\epsilon}$, instead of simply considering $\nu_{wf} = wf - f$ for every w, is to ensure that the surplus is strictly increasing in workers' types. The following table describes the match surpluses:

	2	3	4	5
2	4	10	$11-\tilde{\epsilon}$	11
3	5	12	$14-\tilde{\epsilon}$	15

Figure 3: Match surpluses.

Worker indices:	i_1	i_2
Workers' payoffs, $\pi^{\mathbf{w}}$:	$7-\tilde{\epsilon}$	6
Workers' types, $\mathbf{w}$:	4	3
Transfers, $\mathbf{p}$:	1	0
Firm types, $\mathbf{f}$:	2	3
Firms' payoffs, $\pi^{\mathbf{f}}$:	4	6
Firms' indices:	j_1	j_2

Worker indices:	i_1	i_2
Workers' payoffs, $\pi^{\mathbf{w}'}$:	9	6
Workers' types, $\mathbf{w}'$:	5	3
Transfers, $\mathbf{p}$:	1	0
Firm types, $\mathbf{f}$:	2	3
Firms' payoffs, $\pi^{\mathbf{f}}$:	2	6
Firms' indices:	j_1	j_2

Worker indices:	i_1	i_2
Workers' payoffs, $\pi^{\mathbf{w}''}$:	9	3
Workers' types, $\mathbf{w}''$:	5	2
Transfers, $\mathbf{p}$:	1	0
Firm types, $\mathbf{f}$:	2	3
Firms' payoffs, $\pi^{\mathbf{f}}$:	2	2
Firms' indices:	j_1	j_2

Figure 4: Three different outcomes, with states $\mathbf{w}$, $\mathbf{w}'$, and $\mathbf{w}''$.

Notice that $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f})$ *is complete-information blocked by* (i_1, j_2)*, because* $(9-\tilde{\epsilon})+5 > 13-\tilde{\epsilon}$*, but not by* (i_2, j_1)*, because* $4+6 \leq 10$*. Yet* $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \sum^1$*, because at* $\mathbf{w}'$ *the transfer* $p^{\epsilon}_{\mathbf{w}_{i_1}} = 7-\tilde{\epsilon}-(9-\tilde{\epsilon})+\epsilon = -2+\epsilon$ *is still profitable to* i_1*, but not to* j_2*, for every* $\epsilon > 0$*. Indeed,* $12+(-2+\epsilon) = 10+\epsilon > 9$ *and* $3-(-2+\epsilon) = 5-\epsilon \leq 6$*.*

Similarly, $(\mu, \mathbf{p}, \mathbf{f})$ *is complete-information blocked at* $\mathbf{w}'$ *by* (i_2, j_1)*, because* $4+6 > 8$*, but not by* (i_1, j_2)*, because* $12+3 \leq 15$*. Yet* $(\mu, \mathbf{p}, \mathbf{w}', \mathbf{f}) \in \sum^1$*, because at* $\mathbf{w}''$ *the transfer* $p^{\epsilon}_{\mathbf{w}'_{i_2}} = 6-4+\epsilon = 2+\epsilon$ *is still profitable to* i_2*, but not to* j_1*.Indeed,* $2+(2+\epsilon) = 4+\epsilon > 3$ *and* $2-(2+\epsilon) = \epsilon \leq 2$*.* □

Notice that $(\mu, \mathbf{p}, \mathbf{w}'', \mathbf{f})$ *is complete-information blocked by* (i_1, j_2)*, because* $12+3 > 11$*, but not by* (i_2, j_1)*, because* $2+2 \leq 5$*. However, the highest worker-type is* 5*. Thus,* $(\mu, \mathbf{p}, \mathbf{w}'', \mathbf{f}) \notin \sum^1$*. Hence,* $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \notin \sum$*.* □

The example illustrates that Assumptions 2 and 3 do not necessarily give rise to favorable selection. In fact, the example features adverse selection: The minimum transfer that gives rise to the complete-information block formed at $\mathbf{w}$, namely, $p^{\epsilon}_{\mathbf{w}_{i_1}}$, is accepted by i_1 at $\mathbf{w}'$, when i_1's type is *higher* $(5 > 4)$, but *less* preferred by j_2. Similarly, the minimum transfer that gives rise to the complete-information block formed at $\mathbf{w}'$, namely, $p^{\epsilon}_{\mathbf{w}'_{i_2}}$, is accepted by i_2 at $\mathbf{w}''$, when i_2's type is *smaller* $(2 < 3)$ and *less* preferred by j_1.

It is easy to see that if firms values are instead increasing in w, as required by Assumption 1, incomplete-information stability (in fact, $\sum^1$) is not consistent with the complete-information block by the high-high pair at $\mathbf{w}$, by Lemma 2.

6.1 Worker assortativeness

Suppose that worker assortativeness fails, so that there are two workers, i and i', with $\mu_i \neq \emptyset$, such that

$$\mathbf{w}_{i'} > \mathbf{w}_i$$

$$\mathbf{f}_{\mu_{i'}} < \mathbf{f}_{\mu_i}.$$

6.1.1 i' is unmatched:

If $\mu_{i'} = \emptyset$, then—Assumption 3 implies that—(i', μ_i) must form a complete-information block. Yet this does not contradict incomplete-information stability anymore, because under Assumption 3 it turns out that (1), namely $\phi_{\mathbf{w}_{i'}\mathbf{f}_{\mu_i}} > \phi_{\mathbf{w}^0_{i'}\mathbf{f}_{\mu_i}}$, does not deliver $\mathbf{w}^0_{i'} < \mathbf{w}_{i'}$, but instead the weaker $\mathbf{w}^0_{i'} \neq \mathbf{w}_{i'}$. Since (2), i.e., $\nu_{\mathbf{w}^0_{i'}\mathbf{f}_{\mu_i}} \geq \nu_{\mathbf{w}_{i'}\mathbf{f}_{\mu_i}}$, implies that $\mathbf{w}^0_{i'} \geq \mathbf{w}_{i'}$, we have $\mathbf{w}^0_{i'} > \mathbf{w}_{i'}$, and so the failure of worker assortativeness carries over to $\mathbf{w}^0$. As a consequence, the same argument can be repeated, because the pair (i', μ_i) would form a complete-information block over and over again. The next figure illustrates this:

$$\mathbf{w}_{i'}^n > \ldots > \mathbf{w}_{i'}^0 > \mathbf{w}_{i'} > \mathbf{w}_i = \mathbf{w}_i^0 = \ldots = \mathbf{w}_i^n$$
$$\emptyset < \mathbf{f}_{\mu_i}.$$

Figure 5: The failure of worker-assortativeness at $\mathbf{w}$ unravels, when $\mu_{i'} = \emptyset$.

In the end, the resulting unraveling informs μ_i, given its ability to make higher-order inferences, that the true type of worker i' must indeed be $\mathbf{w}_{i'}$. As a consequence, the complete-information block would have to take place.

Notice that the argument in Liu *et al.* (2014) handles this case without invoking μ_i's higher-order inferences, because the presence of favorable selection—i.e., Lemma 2—makes them unnecessary.

6.1.2 i' is matched:

If $\mu_{i'} \neq \emptyset$, then either (i', μ_i) or $(i, \mu_{i'})$ must form a complete-information block, and now the former does not contradict incomplete-information stability anymore. The reason, again, is that under Assumption 3 we have that (1), i.e., $\phi_{\mathbf{w}_{i'}\mathbf{f}_{\mu_i}} > \phi_{\mathbf{w}_{i'}^0\mathbf{f}_{\mu_i}}$, no longer delivers $\mathbf{w}_{i'}^0 < \mathbf{w}_{i'}$, but instead $\mathbf{w}_{i'}^0 \neq \mathbf{w}_{i'}$. Since (3), i.e.,

$$\nu_{\mathbf{w}_{i'}^0\mathbf{f}_{\mu_i}} + \nu_{\mathbf{w}_{i'}\mathbf{f}_{\mu_{i'}}} - \nu_{\mathbf{w}_{i'}\mathbf{f}_{\mu_i}} - \nu_{\mathbf{w}_{i'}^0\mathbf{f}_{\mu_{i'}}} \geq 0,$$

implies that $\mathbf{w}_{i'}^0 \geq \mathbf{w}_{i'}$, because of Assumption 2, we have $\mathbf{w}_{i'}^0 > \mathbf{w}_{i'}$, and so the failure of worker assortativeness carries over to $\mathbf{w}^0$. Thus, the argument can be repeated, with again either (i', μ_i) or $(i, \mu_{i'})$ forming a complete-information block. The following figure illustrates the case in which the pair that forms a complete-information block alternates:

$$\ldots = \mathbf{w}^2_{i'} > \mathbf{w}^1_{i'} = \mathbf{w}^0_{i'} > \mathbf{w}_{i'} > \mathbf{w}_i = \mathbf{w}^0_i > \mathbf{w}^1_i = \mathbf{w}^2_i > \ldots$$
$$\mathbf{f}_{\mu_{i'}} < \mathbf{f}_{\mu_i}.$$

Figure 6: The failure of worker-assortativeness at $\mathbf{w}$ unravels, when $\mu_{i'} \neq \emptyset$, with only (i', μ_i) forming a complete-information block at $\mathbf{w}$ and at $\mathbf{w}^n$, for odd n, and only $(i, \mu_{i'})$ forming a complete-information block at $\mathbf{w}^n$, for even n.

In the configuration illustrated in Figure 6, the ability of both μ_i and $\mu_{i'}$ to make higher-order inferences implies that the resulting unraveling will inform one of them the true type of worker of the other. As a consequence, one of the complete-information blocks would have to take place. Notice, then, that this case requires firms to be able to make higher-order inferences about (the type of worker of) each other. This is precisely the main feature of Example 1 above. This possibility *cannot* arise in Liu *et al.* (2014), because Lemma 2 implies that in *every* array of *every* sequence like $\{\mathbf{w}, \mathbf{w}^0, \mathbf{w}^1, \ldots\}$ the *only* complete-information block is $(i, \mu_{i'})$.

6.2 Firm assortativeness

If the failure of positive assortativeness is due to a failure of firm assortativeness, there are two firms, j and j', with $\mu_j \neq \emptyset$, such that

$$\mathbf{f}_{j'} > \mathbf{f}_j$$
$$\mathbf{w}_{\mu_{j'}} < \mathbf{w}_{\mu_j}.$$

If $\mu_{j'} \neq \emptyset$, then we have a failure of worker assortativeness, and so can be dealt with using the same arguments described above. If $\mu_{j'} = \emptyset$, then (μ_j, j') forms a complete-information block. However, this no longer contradicts incomplete-information stability. This is so, as before, because under Assumption 3 we have that (1), namely $\phi_{\mathbf{w}_{\mu_j}\mathbf{f}_{j'}} > \phi_{\mathbf{w}^0_{\mu_j}\mathbf{f}_{j'}}$ only implies that $\mathbf{w}_{\mu_j} \neq \mathbf{w}^0_{\mu_j}$. Since (3), i.e.,

$$\nu_{\mathbf{w}^0_{\mu_j}\mathbf{f}_{j'}} + \nu_{\mathbf{w}_{\mu_j}\mathbf{f}_j} - \nu_{\mathbf{w}_{\mu_j}\mathbf{f}_{j'}} - \nu_{\mathbf{w}^0_{\mu_j}\mathbf{f}_j} \geq 0, \tag{4}$$

implies that $\mathbf{w}^0_{\mu_j} \geq \mathbf{w}_{\mu_j}$, because of Assumption 2, it follows that $\mathbf{w}^0_{\mu_j} > \mathbf{w}_{\mu_j}$, so that the failure of firm assortativeness carries over to $\mathbf{w}^0$. The same argument can be repeated, thus leading to the situation illustrated by the following figure:

$$\begin{array}{l} \mathbf{f}_{j'} > \mathbf{f}_j \\ \emptyset < \mathbf{w}_{\mu_j} < \mathbf{w}^0_{\mu_j} < \ldots < \mathbf{w}^n_{\mu_j}. \end{array}$$

Figure 7: The failure of firm-assortativeness at $\mathbf{w}$ unravels, when $\mu_{j'} = \emptyset$.

In the end, the resulting unraveling informs j', given its ability to make higher-order inferences, that the true type of worker μ_j must indeed be $\mathbf{w}_{\mu_j}$. As a consequence, the complete-information block would have to take place. Notice, again, that this argument does not rely on the presence of favorable selection, i.e., on Lemma 2, like the argument in Liu *et al.* (2014), but instead on the ability of firms to make higher-order inferences.

6.3 Efficiency

Armed with positive assortativeness, Lemma 1 implies that a matching μ that is incomplete-information stable fails to be efficient if and only if there is some (i, j) with $\mu_i = \mu_j = \emptyset$ such that $S_{\mathbf{w}_i\mathbf{f}_j} > 0$. Thus, (i, j) form a complete-information block. However, this no longer contradicts incomplete-information stability. To see this, notice that under Assumption 3 inequality (1), namely $\phi_{\mathbf{w}_i\mathbf{f}_j} > \phi_{\mathbf{w}^0_i\mathbf{f}_j}$, does not deliver $\mathbf{w}^0_i < \mathbf{w}_i$, but instead the weaker $\mathbf{w}^0_i \neq \mathbf{w}_i$. Since (2), i.e., $\nu_{\mathbf{w}^0_i\mathbf{f}_j} \geq \nu_{\mathbf{w}_i\mathbf{f}_j}$, implies that $\mathbf{w}^0_i \geq \mathbf{w}_i$, we have $\mathbf{w}^0_i > \mathbf{w}_i$, and so the failure of efficiency carries over to $\mathbf{w}^0$. As a consequence, the same argument can be repeated at $\mathbf{w}^0$. The next figure illustrates this:

$$\begin{aligned} \mathbf{w}_i^n > \ldots > \mathbf{w}_i^0 > \mathbf{w}_i > \emptyset \\ \emptyset < \mathbf{f}_j. \end{aligned}$$

Figure 8: The failure of efficiency at $\mathbf{w}$ unravels, when μ is positive assortative.

In the end, the resulting unraveling informs j, given its ability to make higher-order inferences, that the true type of worker i must indeed be $\mathbf{w}_i$. As a consequence, the complete-information block would have to take place.

Notice, once more, the key difference with Liu *et al.* (2014): Under Assumption 3, the selection of i is not necessarily favorable to j, at $\mathbf{w}$, and at $\mathbf{w}^n$, for every $n \geq 0$. Indeed, only types that are higher than $\mathbf{w}_i$ (resp. $\mathbf{w}_i^n$, for every $n \geq 0$) would be willing to accept $p^{\epsilon}_{\mathbf{w}_i}$ (resp. $p^{\epsilon}_{\mathbf{w}_i^n}$, for every $n \geq 0$), but those types might be *less* preferred than $\mathbf{w}_i$ (resp. $\mathbf{w}_i^n$, for every $n \geq 0$), by j.

7 Discussion

7.1 Stable transfers

It is easy to see that Lemma 2 holds true regardless of whether μ is assortative and efficient or not. Thus, Assumptions 1 and 2 might also lead to favorable selection in certain complete-information blocks when the underlying matching is, in fact, assortative and efficient. This is not the case under Assumptions 2 and 3. Thus, weakening Assumption 1 by replacing it with Assumption 3 not only maintains the efficiency and positive assortativeness of all incomplete-information stable matchings, but also enlarges the set of transfers that supports them. These observations are illustrated by the following example.

Example 2. *There are two workers and two firms, $I = \{i_1, i_2\}$ and $J = \{j_1, j_2\}$. The set of types are $W = \{3, 4, 5\}$ and $F = \{2, 3\}$. Workers' premuneration values* *are given by* $\nu_{wf} = wf$.[7] *Suppose, first, that firms' premuneration values are also given by* $\phi_{wf} = wf$*, so that Assumptions 1 and 2 are satisfied, and consider the following outcome:*

Worker indices:	i_1	i_2
Workers' payoffs, $\pi^{\mathbf{w}}$*:*	6	19
Workers' types, $\mathbf{w}$*:*	3	5
Transfers, $\mathbf{p}$*:*	0	4
Firm types, $\mathbf{f}$*:*	2	3
Firms' payoffs, $\pi^{\mathbf{f}}$*:*	6	11
Firms' indices:	j_1	j_2

The underlying matching is both efficient and positive assortative, but not complete-information stable. In particular, (i_1, j_2) *is the only complete-information block. Thus, by Lemma 2, the outcome is not incomplete-information stable.*

Suppose, now, that firms' premuneration values are independent of f*, and depend on* w *as follows:*

$$\phi_w = \begin{cases} 11 & \textit{if } w = 5 \\ 5 - \bar{\epsilon} & \textit{if } w = 3 \\ 3 & \textit{if } w = 4, \end{cases}$$

where $\bar{\epsilon} > 0$*. Thus, firms' values are neither increasing nor decreasing in* w*. Notice, in particular, that Assumptions 2 and 3 are satisfied.*[8] *Thus, the previous outcome becomes:*

[7] These premuneration values satisfy Assumptions 2 and 3, but they also increase with f. The latter plays no role, but simplifies the exposition.

[8] The reason to include $\tilde{\epsilon}$ is to ensure that the surplus is strictly increasing in workers' types. The following table describes the match surpluses:

Worker indices:	i_1	i_2
Workers' payoffs, $\pi^{\mathbf{w}}$*:*	6	19
Workers' types, $\mathbf{w}$*:*	3	5
Transfers, $\mathbf{p}$*:*	0	4
Firm types, $\mathbf{f}$*:*	2	3
Firms' payoffs, $\pi^{\mathbf{f}}$*:*	$5-\bar{\epsilon}$	7
Firms' indices:	j_1	j_2

As before, the outcome is not complete-information stable. In particular, (i_1, j_2) *is the only complete-information block, where* $p^{\epsilon}_{\mathbf{w}_{i_1}} = -3+\epsilon$*, for* $\epsilon > 0$*. Consider the following outcome, describing the same allocation, but another state,* $\mathbf{w}'$*:*

Worker indices:	i_1	i_2
Workers' payoffs, $\pi^{\mathbf{w}'}$*:*	8	19
Workers' types, $\mathbf{w}'$*:*	4	5
Transfers, $\mathbf{p}$*:*	0	4
Firm types, $\mathbf{f}$*:*	2	3
Firms' payoffs, $\pi^{\mathbf{f}}$*:*	3	7
Firms' indices:	j_1	j_2

Notice that $p^{\epsilon}_{\mathbf{w}_{i_1}} = -3+\epsilon$ *is still accepted by* i_1*, but not by* j_2*, because* $12-3+\epsilon > 8$ *and* $3+3-\epsilon = 6-\epsilon \leq 7$*. Thus,* $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \sum^1$*. In fact,* $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \sum$*, because* $(\mu, \mathbf{p}, \mathbf{w}', \mathbf{f})$ *is complete-information stable.* □

	3	4	5
2	$11-\bar{\epsilon}$	11	21
3	$14-\bar{\epsilon}$	15	26

Figure 9: Match surpluses.

7.2 Nonmonotonic values

The main takeaway of Lemma 2, namely the rise of favorable selection in certain complete-information blocks, cannot be avoided by assuming that ϕ_{wf} is instead decreasing in w. Consider the following variant of Assumption 1, and the result that follows:

Assumption 4. *Workers' premuneration value ν_{wf} is increasing in f and strictly increasing in w, and firms' premuneration value ϕ_{wf} is strictly increasing in f and decreasing in w.*

Lemma 3. *Suppose that Assumptions 2 and 4 hold, and fix any $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \sum^1$. If (i, j) is a complete-information block at $\mathbf{w}$, then $\mathbf{f}_{\mu_i} \leq \mathbf{f}_j$.*

The proof can be found in the Appendix, but the intuition goes along the same lines than that for Lemma 2. Notice, however, that Lemma 3 does not obtain $\mu_i \neq \emptyset$, like Lemma 2, because that is not longer necessary. To see this, suppose that $\mu_i = \emptyset$. By (2), and the fact that ν_{wf} increases in w, it follows that $\mathbf{w}_i^0 \geq \mathbf{w}_i$. Yet this no longer contradicts (1), because ϕ_{wf} now decreases with w. It follows that Assumptions 2 and 4 also deliver assortativeness and efficiency, but prescribe "less" favorable selection than Assumptions 1 and 2.

Like Lemma 2, Lemma 3 is not necessarily true under Assumptions 2 and 3. This can be seen by going back to Example 1, because at $\mathbf{w}'$ the only complete-information block is the one formed by the pair (i_2, j_1), but $(\mu, \mathbf{p}, \mathbf{w}', \mathbf{f}) \in \sum^1$, so that Lemma 3 does not bite.

What does it follow from Lemmas 2 and 3? At the heart of the matter rests the idea that incomplete-information stability is not consistent with the existence of favorable selection. In the presence of Assumption 2, however, Assumptions 1 and 4 imply that certain complete-information blocks (including, but not only, those that arise from failures of efficiency and positive assortativeness) lead to favorable selection, and are therefore inconsistent with incomplete-information stability. To avoid reaching a contradiction with incomplete-information stability *because* the given complete-information block leads to favorable selection, firms' values must be neither increasing nor decreasing. This is the key feature of Examples 1 and 2, which feature, respectively, firms with single-peaked and single-dipped preferences.

7.3 Tightness

Liu *et al.* (2014) show by example (p. 557) that Proposition 1 does not necessarily hold when Assumption 1 is satisfied and S_{wf} is strictly supermodular, but ν_{wf} exhibits constant differences. Since Assumption 3 is weaker than Assumption 1, the strict supermodularity of ν_{wf} is also necessary to guarantee Proposition 2. Moreover, a similar argument can be made if ν_{wf} does not depend on w, given that in that case workers' values would exhibit constant differences.

The argument in Sections 6.1.1 and 6.1.2 shows that the underlying reason for the necessity of Assumptions 2 and 3 is the selection of workers. Plainly, if ν_{wf} does not depend on w or, more generally, exhibits constant differences, the role played by firms' higher-order inferences in Proposition 2 can be disrupted. To see this, notice that (2) and (3) are satisfied regardless of whether $\mathbf{w}_i^0$ is higher or lower than $\mathbf{w}_i$ whenever ν_{wf} does not depend on w or exhibits constant differences, respectively. Thus, any firm involved in a complete-information block that arises from a failure of worker assortativeness might be able to say "no" by means of an array at which the given matching is positive assortative. Indeed, the underlying matching in Sections 6.1.1 and 6.1.2 could be positive assortative at $\mathbf{w}^0$, if $\mathbf{w}_{i'}^0 < \mathbf{w}_{i'}$. Hence, the incomplete-information stability hypothesis would not fail.

7.4 Why do we obtain sorting and efficiency?

At first glance, the efficiency of every incomplete-information stable matching delivered by Proposition 1 should be somewhat surprising. After all, it is well known that the presence of incomplete information tends to create inefficiencies (see, e.g., Akerlof (1970)). The surprise seemed to vanish, given that Assumptions 1 and 2 imply that the selection of workers involved in most complete-information blocks that arise from failures of positive assortativeness and efficiency is favorable to the given firms (see Lemma 2), but that was not really so because Proposition 2 implies—and Example 1 illustrates—that assortativeness and efficiency can be obtained even in the presence of adverse selection. How come, then, incomplete-information stable outcomes are efficient?

A natural move is to try and scrutinize the underlying assumptions. For example, in the standard model of lemons the outside option of the seller (the counterpart of workers) depends on the quality of the car she owns (the counterpart of workers' types). That's not the case in this paper, since the payoff of every worker that is unmatched (the counterpart of a seller who does not trade) is zero, and thus does not vary with her own type. It is possible to relax the assumption that unmatched agents receive no payoff, but doing so will not change the efficiency of incomplete-information stable outcomes. To see this, imagine that there is one firm and one worker, with types w and f and premuneration values $\phi_{wf} = \nu_{wf} = wf$, who are unmatched. Let the outside option of the firm be zero no matter its type, but assume that the outside option of the worker is given by $O(w) = 3w$. Thus, the worker's value of being unmatched increases with her own type. Since $S_{wf} > 0$, the worker and the firm form a complete-information block. Indeed, efficiency demands "trade." Let $f = 2$, and consider the "smallest" transfer the worker would be willing to accept; i.e.,

$$p_w^\epsilon == 3w - 2w + \epsilon = w + \epsilon.$$

Now the smallest transfer the worker is willing to accept *increases* with her type, so that only types that are lower than w would accept p^ϵ_w.[9] Thus, if the value of the firm increases with the worker's type, the selection of the worker is adverse to the firm. Moreover, if the surplus created by the firm and the lower-type worker is positive again, they would form a complete-information block once more, and the same argument could be repeated. It follows that if the surplus is positive for every pair of types—so that there are gains of trade at every state—the firm will eventually infer the true type of the worker, and the *original* complete-information block would take place. This is the same unraveling that occurs in a standard lemons market, in the sense that the presence of adverse selection drives the worker's type "down," but here one ends up with efficiency, as there is "trade at every state."

The surprise really disappears when one notices the implicit dynamic nature of stability. An insightful discussion is given in Liu (2020) (p. 2644), but a context-specific framing is provided below, for the interested reader. Plainly, Assumptions 2 and 3 allow firms to screen for any given worker's type, upon failures of assortativeness or efficiency, because stability allows them to make more than one offer. Thus, firms can keep making offers for as long as the worker says "no."

Consider, first, the failure of worker assortativeness described in Section 6.1.1, and imagine that firm μ_i makes the following offers to worker i', based on $p^\epsilon_{\mathbf{w}_{i'}} = -\nu_{\mathbf{w}_{i'}\mathbf{f}_{\mu_i}} + \epsilon$: It offers first $p^\epsilon_{w^K}$, the minimum wage the highest possible type would be willing to accept. If—and only if—the offer is rejected, the firm offers $p^\epsilon_{w^{K-1}}$, the minimum wage the second-highest possible type would be willing to accept, and so on. The key observation here is that no type would accept the offer intended for a type that is higher whenever Assumption 3 is in place. That is, offer $p^\epsilon_{w^K}$ is rejected by every $w < w^K$, offer $p^\epsilon_{w^{K-1}}$ is rejected by every $w < w^{K-1}$, and so on. To see why, take any k and suppose that, being of type $w < w^k$, worker i' were to say "yes." Thus, we would then have

[9] Notice that if $O(w)$ is weakly decreasing in w, then p^ϵ_w will be decreasing in w, whenever ν_{wf} is increasing in w. It follows that if ϕ_{wf} increases in w, the worker's selection would be favorable to the firm. See Jovanovic (1982).

$$\nu_{w\mathbf{f}_{\mu_i}} + p^{\epsilon}_{w^k} > 0 \Leftrightarrow \nu_{w\mathbf{f}_{\mu_i}} - \nu_{w^k\mathbf{f}_{\mu_i}} + \epsilon > 0 \Leftrightarrow \nu_{w\mathbf{f}_{\mu_i}} - \nu_{w^k\mathbf{f}_{\mu_i}} \geq 0.$$

Since the last inequality contradicts Assumption 3, because $w < w^K$, it follows that $p^{\epsilon}_{w^K}$ is accepted if and only if i' is in fact of type w^K. If the offer is accepted, firm μ_i would infer that the type is w^K and the complete-information block would take place. If, instead, the offer is rejected, stability grants firm μ_i the ability to counteroffer $p^{\epsilon}_{w^{K-1}}$, which would only be accepted by w^K and w^{K-1}. But since $p^{\epsilon}_{w^K}$ has been rejected, the firm can infer, if $p^{\epsilon}_{w^{K-1}}$ is accepted, that the type of i' must be w^{K-1}. As a consequence, the complete-information block would again take place. The underlying reason is simple: Assumption 3 implies that $p^{\epsilon}_{w^k}$ is decreasing in k, and so ensures that no type would accept an offer meant for a type that is higher. By offering this sequence of contingent transfers, the firm is thus—eventually—able to infer what she wanted to learn.

It is not hard to see that the exact same argument—replacing i' with i and μ_i with j—can be used to deal with the failure of efficiency described in Section 6.3.

If worker i' is instead matched, as in Section 6.1.2, a similar argument can be made because Assumption 2 implies, again, that the offer the firm is making is decreasing in k. To see this, suppose that (i', μ_i) forms a complete-information block, and notice that the relevant offers is $p^{\epsilon}_{\mathbf{w}_{i'}} = \nu_{\mathbf{w}_{i'}\mathbf{f}_{\mu_{i'}}} + \mathbf{p}_{i',\mu_{i'}} - \nu_{\mathbf{w}_{i'}\mathbf{f}_{\mu_i}} + \epsilon$. Take any k and suppose that i' says "yes" to $p^{\epsilon}_{w^k}$, being of type $w < w^k$. Then,

$$\nu_{w\mathbf{f}_{\mu_i}} + p^{\epsilon}_{w^k} > \nu_{w\mathbf{f}_{\mu_{i'}}} + \mathbf{p}_{i',\mu_{i'}} \Leftrightarrow \nu_{w\mathbf{f}_{\mu_i}} + \nu_{w^k\mathbf{f}_{\mu_{i'}}} + \mathbf{p}_{i',\mu_{i'}} - \nu_{w^k\mathbf{f}_{\mu_i}} + \epsilon > \nu_{w\mathbf{f}_{\mu_{i'}}} + \mathbf{p}_{i',\mu_{i'}},$$

which is equivalent to

$$\nu_{w\mathbf{f}_{\mu_i}} + \nu_{w^k\mathbf{f}_{\mu_{i'}}} - \nu_{w^k\mathbf{f}_{\mu_i}} \geq \nu_{w\mathbf{f}_{\mu_{i'}}}.$$

This last inequality contradicts the strict supermodularity of ν_{wf}, because $w^k > w$ and $\mathbf{f}_{\mu_i} > \mathbf{f}_{\mu_{i'}}$. Notice that a similar argument would work, in turn, with failures of firm assortativeness, as in Section 6.2.

If $(i, \mu_{i'})$ is instead the pair that forms the complete-information block, the same argument goes through, although the order of the offers needs to be reversed. Indeed, the strict supermodularity of ν_{wf} now entails that $p^{\epsilon}_{w^k}$ is increasing in k. So, now the blocking firm—$\mu_{i'}$—would start with $p^{\epsilon}_{w^1}$, the minimum wage the second-lowest possible type would be willing to accept. As before, however, the selection of workers' types enables the firm to perfectly screen them. Now $p^{\epsilon}_{w^1}$ is only accepted by w^1, $p^{\epsilon}_{w^2}$ is only accepted by w^1 and w^2, etc. Thus, each subsequent rejection informs the firm of the type of the worker, at the moment of acceptance.

Notice, however, that these arguments do not work if the underlying matching is efficient and positive assortative. Thus, the screening ability granted by stability "bites" only in the presence of Assumptions 2 and 3 and some failure of efficiency or positive assortativeness. To see why, suppose that $\mathbf{w}_{i'} > \mathbf{w}_i$, as before, but now $\mathbf{f}_{\mu_{i'}} > \mathbf{f}_{\mu_i}$. Imagine that transfers are such that (i', μ_i), say, forms a complete-information block. Firm μ_i would like to perfectly screen the type of i' as before, by using the same "mechanism" described above. This not possible, however, because Assumption 2 does not imply that $p^{\epsilon}_{w^k}$ decreases with k anymore. Indeed, $p^{\epsilon}_{w^k}$ now increases. Thus, the very first offer μ_i would make—$p^{\epsilon}_{w^K}$—would not only be accepted by w^K, but also by every other type.

7.5 Higher-order inferences

The assumption that firms can draw higher-order inferences plays an important, but somewhat limited role in Proposition 1, because of Lemma 2. Indeed, Sections 5.1 and 5.2 reveal that the only failure of assortativeness and efficiency that invokes firms' higher-order inferences are those in which both workers are matched and (the transfers are such that) the only complete-information block is formed by the low-low pair. Thus, the resulting unraveling would only involve the higher-order inferences made by the low-type firm, about the type of the low-type worker.

Instead, Proposition 2 makes use of firms' higher-order inferences in *every* failure of assortativeness and efficiency. In particular, under Assumptions 2 and 3 both the low-low and high-high type pairs can form a complete-information block, under failures of assortativeness in which both workers are matched, and so the resulting unraveling might involve a sequence of arrays that alternate the blocking pair that takes place (see Example 1 and Figure 6). This observation points not only to the need for a somewhat different argument in proving that these failures of assortativeness contradict incomplete-information stability, but also, and perhaps more importantly, to a discussion of how sophisticated firms must be. This discussion is outside the scope of the present paper, but is the focus of a companion paper (Peralta (2024)).

7.6 Related literature

The literature on matching under incomplete information is far from new. To my knowledge, the earliest attempt to embed incomplete information in the theory of matching markets goes back to Roth (1989), who studies stable mechanisms in the presence of preference uncertainty (see also Ehlers & Massó (2007)). Unlike the private values model analyzed by Roth (1989), Chakraborty *et al.* (2010) examine a one-sided incomplete-information model with interdependent values and show that the existence of a stable—and strategy-proof—mechanism depends on whether the mechanism makes the allocation public or not. Both of these papers analyze stability in centralized markets. In contrast, the present paper belongs to the set of recent papers that seek to understand what constitutes a, and what are the properties of, stable matching in decentralized markets.

Also motivated by the seminal contribution of Liu *et al.* (2014), a recent set of papers have analyzed matching and stability in decentralized markets with incomplete information, transfers, and interdepedent values.[10] These papers analyze variations in either the environment or the stability notion, or offer "foundations" for incomplete information. For example, Chen & Hu (2023) extend incomplete-information stability to markets with two-sided uncertainty—first analyzed in Bikhchandani (2014) for markets without transfers—and Liu (2020) proposes a Bayesian notion of stability (see also Alston (2020)). Closer to the spirit of this paper, Chen & Ho Cher Sien (2020) investigate what conditions deliver assortativeness and efficiency in markets with two-sided uncertainty. In terms of foundations, Pomatto (2022) offers a non-cooperative counterpart to incomplete-information stability and Chen & Hu (2019) show that market allocations must "eventually" be incomplete-information stable.

This paper goes back to the environment and stability notion introduced by Liu *et al.* (2014) to revisit the question of when stable matchings satisfy the standard properties of positive assortativeness and efficiency. Since incomplete-information stability captures necessary conditions for stability, the analysis and results in this paper shed light on when these properties are satisfied, in markets with one-sided uncertainty, by—pretty much—every stability notion. On the other hand, however, Peralta (2024) reveals that Assumption 1 and 2 deliver efficiency and assortativeness only if firms can draw an arbitrarily large number of inferences, but shows that only second-order inferences suffice within well known preference domains.

[10] There is also a literature that analyzes markets without transfers. See, e.g., Bikhchandani (2017), Jeong (2019), and Peralta (2022).

8 Appendix

8.1 Proof of Lemma 2

Take any $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \sum^1$, and any (i, j) that forms a complete-information block at $\mathbf{w}$. If $\mu_i = \emptyset$, then consider

$$p^{\epsilon}_{\mathbf{w}_i} := -\nu_{\mathbf{w}_i \mathbf{f}_j} + \epsilon,$$

where $\epsilon > 0$. Since unmatched agents get a payoff of 0, the reader should interpret $p^{\epsilon}_{\mathbf{w}_i}$, when ϵ is "small," as the "smallest" transfers for which (i, j) forms a complete-information block at $\mathbf{w}$. Since $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \sum^1$, there must be, for every $\epsilon > 0$, some $\mathbf{w}' \in \Omega$ with $(\mu, \mathbf{p}, \mathbf{w}', \mathbf{f}) \in \sum^0$ and $\mathbf{w}'_{\mu_j} = \mathbf{w}_{\mu_j}$ such that

$$\nu_{\mathbf{w}'_i \mathbf{f}_j} + p^{\epsilon}_{\mathbf{w}_i} > 0 \quad \text{and} \quad \phi_{\mathbf{w}'_i \mathbf{f}_j} - p^{\epsilon}_{\mathbf{w}_i} \leq \phi_{\mathbf{w}_{\mu_j} \mathbf{f}_j} - \mathbf{p}_{\mu_j, j},$$

where the first inequality makes use of the fact that unmatched agents get a payoff of 0, and the second inequality uses the fact that $\mathbf{w}'_{\mu_j} = \mathbf{w}_{\mu_j}$. Since W is finite, and $\Omega \subseteq W^{|I|}$, we can assume, without loss of generality, that these two inequalities are true, at $\mathbf{w}'$, for every $\epsilon > 0$. Thus, the inequality on the left holds if and only if $\nu_{\mathbf{w}'_i \mathbf{f}_j} - \nu_{\mathbf{w}_i \mathbf{f}_j} \geq 0$, which is true if and only if $\mathbf{w}'_i \geq \mathbf{w}_i$, by Assumption 1.

On the other hand, the fact that (i, j) forms a complete-information block at $\mathbf{w}$ is equivalent to

$$\nu_{\mathbf{w}_i \mathbf{f}_j} - \nu_{\mathbf{w}_i \mathbf{f}_{\mu_i}} + \phi_{\mathbf{w}_i \mathbf{f}_j} - \phi_{\mathbf{w}_{\mu_j} \mathbf{f}_j} > -\mathbf{p}_{j, \mu_j},$$

because $\mathbf{p}_{\mu_i, i} = 0$. Thus, given the weak inequality on the right above, it follows that

$$\phi_{\mathbf{w}_i \mathbf{f}_j} > \phi_{\mathbf{w}'_i \mathbf{f}_j},$$

which holds true, by Assumption 1 if and only if $\mathbf{w}'_i < \mathbf{w}_i$, a contradiction. Thus,

$\sum^1$ implies that we must have $\mu_i \neq \emptyset$ in every complete-information block (i, j).

Suppose now that $\mu_i \neq \emptyset$, but $\mathbf{f}_j > \mathbf{f}_{\mu_i}$, and consider

$$p^{\epsilon}_{\mathbf{w}_i} := \nu_{\mathbf{w}_i \mathbf{f}_{\mu_i}} + \mathbf{p}_{i,\mu_i} - \nu_{\mathbf{w}_i \mathbf{f}_j} + \epsilon.$$

Since $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \sum^1$, there must be, for every $\epsilon > 0$, some $\mathbf{w}' \in \Omega$ with $(\mu, \mathbf{p}, \mathbf{w}', \mathbf{f}) \in \sum^0$ and $\mathbf{w}'_{\mu_j} = \mathbf{w}_{\mu_j}$ such that

$$\nu_{\mathbf{w}'_i \mathbf{f}_j} + p^{\epsilon}_{\mathbf{w}_i} > \pi^{\mathbf{w}'}_i \quad \text{and} \quad \phi_{\mathbf{w}'_i \mathbf{f}_j} - p^{\epsilon}_{\mathbf{w}_i} \leq \phi_{\mathbf{w}_{\mu_j} \mathbf{f}_j} - \mathbf{p}_{\mu_j, j},$$

Again, because W is finite, and $\Omega \subseteq W^{|I|}$, we can assume without loss that these two inequalities are true, at $\mathbf{w}'$, for every $\epsilon > 0$. The inequality on the right again implies that $\mathbf{w}'_i < \mathbf{w}_i$, by Assumption 1, and the inequality on the left is equivalent to

$$\nu_{\mathbf{w}'_i \mathbf{f}_j} + \nu_{\mathbf{w}_i \mathbf{f}_{\mu_i}} + \mathbf{p}_{i,\mu_i} - \nu_{\mathbf{w}_i \mathbf{f}_j} + \epsilon > \pi^{\mathbf{w}'}_i.$$

It is not hard to see that this inequality is true for every $\epsilon > 0$ if and only if

$$\nu_{\mathbf{w}'_i \mathbf{f}_j} + \nu_{\mathbf{w}_i \mathbf{f}_{\mu_i}} - \nu_{\mathbf{w}_i \mathbf{f}_j} - \nu_{\mathbf{w}'_i \mathbf{f}_{\mu_i}} \geq 0.$$

Since $\mathbf{f}_j > \mathbf{f}_{\mu_i}$ and ν_{wf} is strictly supermodular, because of Assumption 2, it follows that $\mathbf{w}'_i \geq \mathbf{w}_i$, a contradiction.□

8.2 Proof of Lemma 3

Take any $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \sum^1$, and any (i, j) that forms a complete-information block at $\mathbf{w}$. Suppose, contrary to hypothesis, that $\mu_i \neq \emptyset$ and $\mathbf{f}_{\mu_i} > \mathbf{f}_j$, and consider

$$p^{\epsilon}_{\mathbf{w}_i} := \nu_{\mathbf{w}_i \mathbf{f}_{\mu_i}} + \mathbf{p}_{i,\mu_i} - \nu_{\mathbf{w}_i \mathbf{f}_j} + \epsilon.$$

Since $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \sum^1$, there must be, for every $\epsilon > 0$, some $\mathbf{w}' \in \Omega$ with $(\mu, \mathbf{p}, \mathbf{w}', \mathbf{f}) \in \sum^0$ and $\mathbf{w}'_{\mu_j} = \mathbf{w}_{\mu_j}$ such that

$$\nu_{\mathbf{w}_i'\mathbf{f}_j} + p^{\epsilon}_{\mathbf{w}_i} > \pi_i^{\mathbf{w}'} \quad \text{and} \quad \phi_{\mathbf{w}_i'\mathbf{f}_j} - p^{\epsilon}_{\mathbf{w}_i} \leq \phi_{\mathbf{w}_{\mu_j}\mathbf{f}_j} - \mathbf{p}_{\mu_j,j},$$

As before, because W is finite, and $\Omega \subseteq W^{|I|}$, we can assume without loss that these two inequalities are true, at $\mathbf{w}'$, for every $\epsilon > 0$. The same argument made above shows that the inequality on the right implies $\mathbf{w}_i' > \mathbf{w}_i$, by Assumption 4. Since the inequality on the left is equivalent to

$$\nu_{\mathbf{w}_i'\mathbf{f}_j} + \nu_{\mathbf{w}_i\mathbf{f}_{\mu_i}} - \nu_{\mathbf{w}_i\mathbf{f}_j} - \nu_{\mathbf{w}_i'\mathbf{f}_{\mu_i}} \geq 0,$$

and we have $\mathbf{f}_j < \mathbf{f}_{\mu_i}$, we reach a contradiction with Assumption 2, which demands ν_{wf} to be strictly supermodular.□

8.3 Proof of Proposition 2

Given Lemma 1, the proof starts starts with positive assortativeness, and proceeds by contradiction. Suppose that $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \sum$, but μ fails to be positive assortative at $(\mathbf{w}, \mathbf{f})$.

Worker assortativeness

If the failure corresponds to worker assortativeness, there two workers, i and i', with $\mu_i \neq \emptyset$, such that

$$\mathbf{w}_{i'} > \mathbf{w}_i$$
$$\mathbf{f}_{\mu_{i'}} < \mathbf{f}_{\mu_i},$$

where either $\mu_{i'} = \emptyset$ or $\mu_{i'} \neq \emptyset$.

i' is unmatched: If $\mu_{i'} = \emptyset$, then (i', μ_i) forms a complete-information block at $\mathbf{w}$, because otherwise

$$\begin{aligned} S_{\mathbf{w}_{i'}\mathbf{f}_{\mu_i}} &\leq \pi_{i'}^{\mathbf{w}} + \pi_{\mu_i}^{\mathbf{f}} \\ &= \pi_{\mu_i}^{\mathbf{f}} \\ &= S_{\mathbf{w}_i\mathbf{f}_{\mu_i}} - \pi_i^{\mathbf{w}} \end{aligned}$$

where the second line uses the fact that the payoff of unmatched agents is zero, and the third the fact that the sum of the payoffs of any two agents that are matched exhausts the surplus they create. Since the surplus is strictly increasing in w, by Assumption 3, it follows that

$$
\begin{aligned}
\pi_i^{\mathbf{w}} &\leq S_{\mathbf{w}_i \mathbf{f}_{\mu_i}} - S_{\mathbf{w}_{i'} \mathbf{f}_{\mu_i}} \\
&< 0,
\end{aligned}
$$

contradicting that the given allocation is individually rational. Hence, (i', μ_i) forms a complete-information block at $\mathbf{w}$. For "small" $\epsilon > 0$, the transfer $p^{\epsilon}_{\mathbf{w}_{i'}} := -\nu_{\mathbf{w}_{i'} \mathbf{f}_{\mu_i}} + \epsilon$ should be interpreted as the "smallest" transfer that i' is willing to accept to match with μ_i, at $\mathbf{w}$. Since $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \sum$, there must be, for each $\epsilon > 0$, some $\mathbf{w}' \in \Omega$, with $(\mu, \mathbf{p}, \mathbf{w}', \mathbf{f}) \in \sum^0$ and $\mathbf{w}'_i = \mathbf{w}_i$, such that the block involving $p^{\epsilon}_{\mathbf{w}_{i'}}$ is still profitable to i', but not to μ_i; i.e., such that

$$
\phi_{\mathbf{w}'_{i'} \mathbf{f}_{\mu_i}} - p^{\epsilon}_{\mathbf{w}_{i'}} \leq \phi_{\mathbf{w}_i \mathbf{f}_{\mu_i}} - \mathbf{p}_{\mu_i, i} \quad \text{and} \quad \nu_{\mathbf{w}'_{i'} \mathbf{f}_{\mu_i}} + p^{\epsilon}_{\mathbf{w}_{i'}} > 0, \tag{5}
$$

where the right-hand side of the first inequality makes use of $\mathbf{w}'_i = \mathbf{w}_i$, and the right-hand size uses the fact that the payoff of unmatched agents is zero. Since W is finite, and $\Omega \subseteq W^{|I|}$, there must in fact be one such $\mathbf{w}'$ satisfying both inequalities in (5) for every $\epsilon > 0$. The fact that (i', μ_i) forms a complete-information block at $\mathbf{w}$ is equivalent to:

$$
\nu_{\mathbf{w}_{i'} \mathbf{f}_{\mu_i}} + \phi_{\mathbf{w}_{i'} \mathbf{f}_{\mu_i}} - \phi_{\mathbf{w}_i \mathbf{f}_{\mu_i}} > -\mathbf{p}_{\mu_i, i},
$$

because $\mathbf{p}_{i', \mu_{i'}} = \nu_{\mathbf{w}_{i'} \emptyset} = 0$. Thus, given the weak inequality on the left above, it follows that

$$
\phi_{\mathbf{w}_{i'} \mathbf{f}_{\mu_i}} > \phi_{\mathbf{w}'_{i'} \mathbf{f}_{\mu_i}}. \tag{6}
$$

Hence, (6) delivers $\mathbf{w}_{i'} \neq \mathbf{w}'_{i'}$. The fact that the right inequality in (5) must be true for every $\epsilon > 0$ delivers $\nu_{\mathbf{w}'_{i'}\mathbf{f}_{\mu_i}} \geq \nu_{\mathbf{w}_{i'}\mathbf{f}_{\mu_i}}$, which implies that $\mathbf{w}'_{i'} \geq \mathbf{w}_{i'}$, by Assumption 3. Thus, $\mathbf{w}'_{i'} > \mathbf{w}_{i'}$. Here we use induction. If, as base case, we assume that $\mathbf{w}_{i'} = w^K$, then the complete-information block would take place and the incomplete-information stability hypothesis would fail. Suppose that the same conclusion is obtained if $\mathbf{w}_{i'} \in \{w^r, ..., w^K\}$, for some $1 < r < K$, and consider the case in which $\mathbf{w}_{i'} = w^{r-1}$. Given that (i', μ_i) forms a complete-information block, the argument above implies that $\mathbf{w}'_{i'} \geq \mathbf{w}_{i'}$ we would have $\mathbf{w}'_{i'} \in \{w^r, ..., w^K\}$. Thus, the inductive hypothesis delivers the desired contradiction.

i' **is matched:** If $\mu_{i'} \neq \emptyset$, then the strict supermodularity of S_{wf} implies that, at $\mathbf{w}$, either (i', μ_i) or $(i, \mu_{i'})$ must form a complete-information block. Suppose not. That is, suppose that

$$S_{\mathbf{w}_{i'}\mathbf{f}_{\mu_i}} \leq \pi^{\mathbf{w}}_{i'} + \pi^{\mathbf{f}}_{\mu_i} \text{ and } S_{\mathbf{w}_i\mathbf{f}_{\mu_{i'}}} \leq \pi^{\mathbf{w}}_{i} + \pi^{\mathbf{f}}_{\mu_{i'}}.$$

Adding up, and using $\pi^{\mathbf{w}}_{i} + \pi^{\mathbf{f}}_{\mu_i} = S_{\mathbf{w}_i\mathbf{f}_{\mu_i}}$ and $\pi^{\mathbf{w}}_{i'} + \pi^{\mathbf{f}}_{\mu_{i'}} = S_{\mathbf{w}_{i'}\mathbf{f}_{\mu_{i'}}}$, we get

$$S_{\mathbf{w}_{i'}\mathbf{f}_{\mu_i}} + S_{\mathbf{w}_i\mathbf{f}_{\mu_{i'}}} \leq S_{\mathbf{w}_i\mathbf{f}_{\mu_i}} + S_{\mathbf{w}_{i'}\mathbf{f}_{\mu_{i'}}}.$$

Since $\mathbf{w}_{i'} > \mathbf{w}_i$ and $\mathbf{f}_{\mu_i} > \mathbf{f}_{\mu_{i'}}$, we contradict the strict supermodularity of S_{wf}.

If the complete-information block is formed by (i', μ_i), consider

$$p^{\epsilon}_{\mathbf{w}_{i'}} := \nu_{\mathbf{w}_{i'}\mathbf{f}_{\mu_{i'}}} + \mathbf{p}_{i',\mu_{i'}} - \nu_{\mathbf{w}_{i'}\mathbf{f}_{\mu_i}} + \epsilon.$$

Since $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \sum$, there must be, for each $\epsilon > 0$, some $\mathbf{w}' \in \Omega$, with $(\mu, \mathbf{p}, \mathbf{w}', \mathbf{f}) \in \sum^0$ and $\mathbf{w}'_i = \mathbf{w}_i$, such that

$$\nu_{\mathbf{w}'_{i'}\mathbf{f}_{\mu_i}} + p^{\epsilon}_{\mathbf{w}_{i'}} > \pi^{\mathbf{w}'}_{i'} \text{ and } \phi_{\mathbf{w}'_{i'}\mathbf{f}_{\mu_i}} - p^{\epsilon}_{\mathbf{w}_{i'}} \leq \phi_{\mathbf{w}_i\mathbf{f}_{\mu_i}} - \mathbf{p}_{\mu_i,i}, \tag{7}$$

where the right-hand side of the inequality on the right uses the fact that $\mathbf{w}'_i = \mathbf{w}_i$. Since W is finite, and $\Omega \subseteq W^{|I|}$, there must in fact be some such $\mathbf{w}'$ at which both inequalities in (7) are satisfied for every $\epsilon > 0$. Hence, the inequality on the left is equivalent to

$$\nu_{\mathbf{w}'_{i'}\mathbf{f}_{\mu_i}} + \nu_{\mathbf{w}_{i'}\mathbf{f}_{\mu_{i'}}} - \nu_{\mathbf{w}_{i'}\mathbf{f}_{\mu_i}} - \nu_{\mathbf{w}'_{i'}\mathbf{f}_{\mu_{i'}}} \geq 0.$$

Given that $\mathbf{f}_{\mu_{i'}} < \mathbf{f}_{\mu_i}$, the strict supermodularity of ν_{wf} granted by Assumption 2 implies that this inequality is satisfied if and only if $\mathbf{w}'_{i'} \geq \mathbf{w}_{i'}$. The fact that (i', μ_i) forms a complete-information block at $\mathbf{w}$ is equivalent to:

$$\nu_{\mathbf{w}_{i'}\mathbf{f}_{\mu_i}} - \nu_{\mathbf{w}_{i'}\mathbf{f}_{\mu_{i'}}} + \phi_{\mathbf{w}_{i'}\mathbf{f}_{\mu_i}} - \phi_{\mathbf{w}_i\mathbf{f}_{\mu_i}} > \mathbf{p}_{i',\mu_{i'}} - \mathbf{p}_{\mu_i,i}.$$

Hence, the inequality on the right in (7) delivers, once again, the strict inequality (6). Hence, $\mathbf{w}_{i'} \neq \mathbf{w}'_{i'}$. Thus, once more, $\mathbf{w}'_{i'} > \mathbf{w}_{i'}$.

If the complete-information block is instead formed by $(i, \mu_{i'})$, we can consider

$$p^{\epsilon}_{\mathbf{w}_i} := \nu_{\mathbf{w}_i\mathbf{f}_{\mu_i}} + \mathbf{p}_{i,\mu_i} - \nu_{\mathbf{w}_i\mathbf{f}_{\mu_{i'}}} + \epsilon.$$

Since $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \sum$, there must be, for each $\epsilon > 0$, some $\mathbf{w}'' \in \Omega$, with $(\mu, \mathbf{p}, \mathbf{w}'', \mathbf{f}) \in \sum^0$ and $\mathbf{w}''_{i'} = \mathbf{w}_{i'}$, such that

$$\nu_{\mathbf{w}''{}_i\mathbf{f}_{\mu_{i'}}} + p^{\epsilon}_{\mathbf{w}_i} > \pi_i^{\mathbf{w}''} \text{ and } \phi_{\mathbf{w}''{}_i\mathbf{f}_{\mu_{i'}}} - p^{\epsilon}_{\mathbf{w}_i} \leq \phi_{\mathbf{w}_{i'}\mathbf{f}_{\mu_{i'}}} - \mathbf{p}_{\mu_{i'},i'}, \tag{8}$$

where the inequality on the right uses $\mathbf{w}''_{i'} = \mathbf{w}_{i'}$. Since W is finite, and $\Omega \subseteq W^{|I|}$, there must in fact be some such $\mathbf{w}'$ at which both inequalities in (8) are satisfied for every $\epsilon > 0$. Thus, the inequality on the left is equivalent to

$$\nu_{\mathbf{w}''_i\mathbf{f}_{\mu_{i'}}} + \nu_{\mathbf{w}_i\mathbf{f}_{\mu_i}} - \nu_{\mathbf{w}_i\mathbf{f}_{\mu_{i'}}} - \nu_{\mathbf{w}''_i\mathbf{f}_{\mu_i}} \geq 0.$$

Given that $\mathbf{f}_{\mu_{i'}} < \mathbf{f}_{\mu_i}$, the strict supermodularity of ν_{wf} granted by Assumption 2 implies that this inequality is satisfied if and only if $\mathbf{w}_i'' \leq \mathbf{w}_i$. On the other hand, the reader can easily check that the fact that $(i, \mu_{i'})$ forms a complete-information block at $\mathbf{w}$, given the inequality on the right of (8), implies that $\phi_{\mathbf{w}_i \mathbf{f}_{\mu_{i'}}} > \phi_{\mathbf{w}_i'' \mathbf{f}_{\mu_{i'}}}$. Hence, it follows that $\mathbf{w}_i'' < \mathbf{w}_i$.

At this point, one can use double induction on the type of both i and i'. Let $(\mathbf{w}_{i'}, \mathbf{w}_i) = (w^n, w^m)$, where $n > m$. The two base cases are, respectively, (w^K, w^m) and (w^n, w^1), and both can be shown by induction.

Induction on (w^K, w^m)**:** If $m = 1$, then we reach a contradiction because regardless of whether (i', μ_i) or $(i, \mu_{i'})$ forms a complete-information block at (w^K, w^1) the incomplete-information stability hypothesis would fail. Suppose, as induction hypothesis, that incomplete-information stability fails whenever $m \in \{1, ..., r\}$, for some $1 < r < K$, and let $m = r + 1$. Clearly, (i', μ_i) cannot form a complete-information block at (w^K, w^{r+1}), for otherwise the incomplete-information stability hypothesis would fail. Hence, $(i, \mu_{i'})$ must form a complete-information block at (w^K, w^{r+1}). But then, the argument above implies that $\mathbf{w}_i'' \in \{w^1, ..., w^r\}$. Thus, the induction hypothesis delivers the desired contradiction. □

Induction on (w^n, w^1)**:** If $n = K$, then again we reach a contradiction because regardless of whether (i', μ_i) or $(i, \mu_{i'})$ forms a complete-information block at (w^K, w^1) the incomplete-information stability hypothesis would fail. Suppose, as induction hypothesis, that incomplete-information stability fails whenever $n \in \{x, ..., K\}$, for some $1 < x < K$, and let $n = x - 1$. Clearly, $(i, \mu_{i'})$ cannot form a complete-information block at (w^{x-1}, w^1), for otherwise the incomplete-information stability hypothesis would fail. Hence, (i', μ_i) must form a complete-information block at (w^{x-1}, w^1). But then, the argument above implies that $\mathbf{w}_{i'}' \in \{w^x, ..., w^K\}$. Thus, the induction hypothesis delivers the desired contradiction.

□

Suppose, as induction hypothesis on the double induction argument, that incomplete-information stability fails at (w^n, w^m), whenever $m \in \{1, ..., r\}$, for some $1 < r < K$, and at (w^n, w^m), whenever $n \in \{x, ..., K\}$, for some $1 < x < K$, and consider (w^{x-1}, w^{r+1}). If (i', μ_i) forms a complete-information block at (w^{x-1}, w^{r+1}), then the argument above implies that $\mathbf{w}'_{i'} \in \{w^x, ..., w^K\}$. Thus, the induction hypothesis delivers the desired contradiction. If, instead, $(i, \mu_{i'})$ forms a complete-information block at (w^{x-1}, w^{r+1}), then the argument above implies that $\mathbf{w}''_i \in \{w^1, ..., w^r\}$. Thus, the induction hypothesis delivers the desired contradiction. □

Firm assortativeness

If the failure of positive assortativeness is due to a failure of firm assortativeness, there would be two firms, j and j', with $\mu_j \neq \emptyset$, such that

$$\mathbf{f}_{j'} > \mathbf{f}_j$$

$$\mathbf{w}_{\mu_{j'}} < \mathbf{w}_{\mu_j}.$$

As before, we can have either $\mu_{j'} = \emptyset$ or $\mu_{j'} \neq \emptyset$, but the latter would imply a failure of worker assortativeness and can therefore be dealt with using the same arguments described above.

If $\mu_{j'} = \emptyset$, then the strict monotonicity of S_{wf} with respect to f imposed by Assumption 3 implies that (μ_j, j') must form a complete-information block at $\mathbf{w}$. To see this, suppose not; i.e., assume that $S_{\mathbf{w}_{\mu_j}\mathbf{f}_{j'}} \leq \pi^{\mathbf{w}}_{\mu_j} + \pi^{\mathbf{f}}_{j'} = \pi^{\mathbf{w}}_{\mu_j}$, where the equality follows from the assumption that unmatched agents receive zero payoff. Since $\pi^{\mathbf{w}}_{\mu_j} = S_{\mathbf{w}_{\mu_j}\mathbf{f}_j} - \pi^{\mathbf{f}}_j$ and S_{wf} strictly increases with f, it follows that $\pi^{\mathbf{f}}_j < 0$, contradicting individual rationality. We can now consider

$$p^{\epsilon}_{\mathbf{w}_{\mu_j}} := \nu_{\mathbf{w}_{\mu_j}\mathbf{f}_j} + \mathbf{p}_{\mu_j,j} - \nu_{\mathbf{w}_{\mu_j}\mathbf{f}_{j'}} + \epsilon.$$

Since $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \sum$, there must be, for each $\epsilon > 0$, some $\mathbf{w}' \in \Omega$, with $(\mu, \mathbf{p}, \mathbf{w}', \mathbf{f}) \in \sum^0$ such that

$$\nu_{\mathbf{w}'_{\mu_j}\mathbf{f}_{j'}} + p^{\epsilon}_{\mathbf{w}_{\mu_j}} > \pi^{\mathbf{w}'}_{\mu_j} \text{ and } \phi_{\mathbf{w}'_{\mu_j}\mathbf{f}_{j'}} - p^{\epsilon}_{\mathbf{w}_{\mu_j}} \leq 0, \tag{9}$$

where the right-hand side of the inequality on the right uses the fact that the payoff of unmatched agents is zero. Since W is finite, and $\Omega \subseteq W^{|I|}$, there must in fact be some such $\mathbf{w}'$ at which both inequalities in (9) are satisfied for every $\epsilon > 0$. Thus, the former is equivalent to

$$\nu_{\mathbf{w}'_{\mu_j}\mathbf{f}_{j'}} + \nu_{\mathbf{w}_{\mu_j}\mathbf{f}_j} - \nu_{\mathbf{w}_{\mu_j}\mathbf{f}_{j'}} - \nu_{\mathbf{w}'_{\mu_j}\mathbf{f}_j} \geq 0,$$

which implies that $\mathbf{w}'_{\mu_j} \geq \mathbf{w}_{\mu_j}$, because $\mathbf{f}_{j'} > \mathbf{f}_j$ and ν_{wf} is strict supermodular, by Assumption 2. Since (μ_j, j') forming a complete-information block at $\mathbf{w}$ is equivalent to $\nu_{\mathbf{w}_{\mu_j}\mathbf{f}_{j'}} + \phi_{\mathbf{w}_{\mu_j}\mathbf{f}_{j'}} - \nu_{\mathbf{w}_{\mu_j}\mathbf{f}_j} > -\mathbf{p}_{\mu_j,j}$, the inequality on the right in (9) implies that $\mathbf{w}'_{\mu_j} \neq \mathbf{w}_{\mu_j}$. Hence, we must have $\mathbf{w}'_{\mu_j} > \mathbf{w}_{\mu_j}$. At this point one can use induction. As base case, notice that if $\mathbf{w}_{\mu_j} = w^K$ then incomplete-information stability fails. Suppose, as induction hypothesis, that incomplete-information stability fails whenever $\mathbf{w}_{\mu_j} \in \{x, ..., K\}$, for some $1 < x < K$, and suppose that $\mathbf{w}_{\mu_j} = w^{x-1}$. Hence, (μ_j, j') must form a complete-information block. But then, the argument above implies that $\mathbf{w}'_{\mu_j} \in \{x, ..., K\}$. Hence, the induction hypothesis leads to the desired contradiction. □

Efficiency

Fix any outcome $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \sum$ such that μ is positive assortative. Then, Lemma 1 implies that μ fails to be efficient if and only if there is some (i, j) with $\mu_i = \mu_j = \emptyset$ such that $S_{\mathbf{w}_i\mathbf{f}_j} > 0$. Hence, (i, j) forms a complete-information block at $\mathbf{w}$. Notice that $p^{\epsilon}_{\mathbf{w}_i} = -\nu_{\mathbf{w}_i\mathbf{f}_j} + \epsilon$. Since $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \sum$, there must be, for each $\epsilon > 0$, some $\mathbf{w}' \in \Omega$, with $(\mu, \mathbf{p}, \mathbf{w}', \mathbf{f}) \in \sum^0$ such that

$$\nu_{\mathbf{w}'_i \mathbf{f}_j} + p^{\epsilon}_{\mathbf{w}_{\mu_j}} > 0 \text{ and } \phi_{\mathbf{w}'_i \mathbf{f}_j} - p^{\epsilon}_{\mathbf{w}_{\mu_j}} \leq 0, \tag{10}$$

where the right-hand side in both inequalities uses the fact that the payoff of unmatched agents is zero. Since W is finite, and $\Omega \subseteq W^{|I|}$, there must in fact be some such $\mathbf{w}'$ at which both inequalities in (10) are satisfied for every $\epsilon > 0$. Hence, the inequality on the left of (10) implies, because ν_{wf} is increasing in w, by Assumption 3, that $\mathbf{w}'_i \geq \mathbf{w}_i$. The fact that (i, j) forms a complete-information block at $\mathbf{w}$, together with the inequality on the right in (10), implies that $\mathbf{w}'_i \neq \mathbf{w}_i$. Hence, it follows that $\mathbf{w}'_i > \mathbf{w}_i$. Once again, we can use induction. If $\mathbf{w}_i = w^K$, incomplete-information stability clearly fails. Suppose, as induction hypothesis, that incomplete-information stability fails whenever $\mathbf{w}_i \in \{x, ..., K\}$, for some $1 < x < K$, and suppose that $\mathbf{w}_i = w^{x-1}$. Since (i, j) forms a complete-information block, the argument above implies that $\mathbf{w}'_i \in \{x, ..., K\}$. Thus, the inductive hypothesis leads to the desired contradiction. □

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