Degree bounds for semi-invariant rings of quivers [☆]

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ABSTRACT

We use recent results on matrix semi-invariants to give degree bounds on generators for the ring of semi-invariants for quivers with no oriented cycles.

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1. Introduction

In [3], we studied the left-right action of $\mathrm{SL}_n \times \mathrm{SL}_n$ on m-tuples of $n \times n$ matrices. Among other things, we proved bounds for the degree of generators for the invariant ring. This ring of invariants can be seen as the ring of semi-invariants for the m-Kronecker quiver for the dimension vector (n, n). In this paper, we obtain bounds for the degree of generators for the ring of semi-invariants for a quiver with no oriented cycles.

1.1. Quiver representations

A quiver is just a directed graph. Formally, a quiver Q is a pair (Q_0, Q_1), where Q_0 is a finite set of vertices, and Q_1 is a finite set of arrows. For each arrow $a \in Q_1$, we denote its head and tail by ha and ta respectively. We fix an infinite field K. A representation V of Q over K is a collection of finite dimensional K-vector spaces $V(x)$, $x \in Q_0$ together with a collection of K-linear maps $V(a) : V(ta) \to V(ha)$, $a \in Q_1$. The dimension vector of V is the function $\alpha : Q_0 \to \mathbb{Z}_{\geq 0}$ such that $\alpha(x) = \dim V(x)$ for all $x \in Q_0$. We say a dimension vector is sincere if $\alpha(x) \neq 0 \ \forall \ x \in Q_0$.

Let $\mathrm{Mat}_{p,q}$ denote the set of $p \times q$ matrices over K. For a dimension vector $\alpha \in \mathbb{Z}_{\geq 0}^{Q_0}$, we define its representation space by:

$$\mathrm{Rep}(Q, \alpha) = \prod_{a \in Q_1} \mathrm{Mat}_{\alpha(ha), \alpha(ta)}.$$

If V is a representation with dimension vector α and we identify $V(x) \cong K^{\alpha(x)}$ for all x, then V can be viewed as an element of $\mathrm{Rep}(Q, \alpha)$. Consider the group $\mathrm{GL}(\alpha) = \prod_{x \in Q_0} \mathrm{GL}_{\alpha(x)}$ and its subgroup $\mathrm{SL}(\alpha) = \prod_{x \in Q_0} \mathrm{SL}_{\alpha(x)}$. The group $\mathrm{GL}(\alpha)$ acts on $\mathrm{Rep}(Q, \alpha)$ by:

$$(A(x) \mid x \in Q_0) \cdot (V(a) \mid a \in Q_1) = (A(ha)V(a)A(ta)^{-1} \mid a \in Q_1).$$

For $V \in \mathrm{Rep}(Q, \alpha)$, choosing a different basis means acting by the group $\mathrm{GL}(\alpha)$. The $\mathrm{GL}(\alpha)$-orbits in $\mathrm{Rep}(Q, \alpha)$ correspond to isomorphism classes of representations of dimension α.

1.2. Invariants for quiver representations

The group GL_n acts by simultaneous conjugation on $\mathrm{Mat}_{n,n}^m$, the space of m-tuples of $n \times n$ matrices. Artin conjectured that in characteristic 0, the invariant ring is generated by traces of words in the matrices. Procesi proved this conjecture, and also showed that invariants of degree $\leq 2^n - 1$ generate the ring of invariants (see [23]). It was shown by Razmyslov that invariants of degree $\leq n^2$ suffice (see [24, final remark]). A concise account of the above results can also be found in [9].

Le Bruyn and Procesi generalized the results to arbitrary quivers. They proved that the ring of invariants $K[\mathrm{Rep}(Q, \alpha)]^{\mathrm{GL}(\alpha)}$ is generated by traces along oriented cycles. Using the aforementioned bound, they showed that the invariants of degree $\leq N^2$ generate the ring, where $N = \sum_i \alpha_i$ (see [19]).

1.3. Semi-invariants for quiver representations

From Le Bruyn and Procesi's results described above, we see that a quiver with no oriented cycles has no non-trivial invariants. However, the ring of semi-invariants $\mathrm{SI}(Q, \alpha) = K[\mathrm{Rep}(Q, \alpha)]^{\mathrm{SL}(\alpha)}$ could still be interesting.

A multiplicative character of the group GL_α is of the form

$$\chi_\sigma : (A(x) \mid x \in Q_0) \in \mathrm{GL}_\alpha \mapsto \prod_{x \in Q_0} \det(A(x))^{\sigma(x)} \in K^\star,$$

where $\sigma : Q_0 \to \mathbb{Z}$ is called the weight of the character χ_σ. Define

$$\mathrm{SI}(Q, \alpha)_\sigma = \{f \in K[\mathrm{Rep}(Q, \alpha)] \mid \forall A \in \mathrm{GL}(\alpha) \ A \cdot f = \chi_\sigma(A)f\}.$$

Then the ring of semi-invariants has a weight space decomposition

$$\mathrm{SI}(Q, \alpha) = \bigoplus_\sigma \mathrm{SI}(Q, \alpha)_\sigma.$$

If $\sigma \cdot \alpha = \sum_{x \in Q_0} \sigma(x)\alpha(x) \neq 0$, then $\mathrm{SI}(Q, \alpha)_\sigma = 0$. Assume that $\sigma \cdot \alpha = 0$. We can write $\sigma = \sigma_+ - \sigma_-$ where $\sigma_+(x) = \max\{\sigma(x), 0\}$ and $\sigma_-(x) = \max\{-\sigma(x), 0\}$. Define $|\sigma|_\alpha = \sigma_+ \cdot \alpha = \sigma_- \cdot \alpha$.

Now we define an $n \times n$ linear matrix

$$A : \bigoplus_{x \in Q_0} V(x)^{\sigma_+(x)} \to \bigoplus_{x \in Q_0} V(x)^{\sigma_-(x)}$$

where each block $\mathrm{Hom}(V(x), V(y))$ is of the form $t_1 V(p_1) + \cdots + t_r V(p_r)$, $t_1, t_2, \ldots, t_r$ are indeterminates and $p_1, p_2, \ldots, p_r$ are all paths from x to y. We use different indeterminates for the different blocks, so the linear matrix has $m = \sum_{x \in Q_0} \sum_{y \in Q_0} \sigma_+(x) b_{x,y} \sigma_-(y)$ indeterminates where $b_{x,y}$ is the number of paths from x to y. We can write $A = t_1 X_1 + \cdots + t_m X_m$ with $X_1, \ldots, X_m \in \mathrm{Mat}_{n,n}$. We have the following result (see [4, Corollary 3], [8] and [25]).

Theorem 1.1. *The space* $\mathrm{SI}(Q, \alpha)_\sigma$ *is spanned by* $\det(t_1 X_1 + \cdots + t_m X_m)$ *with* $t_1, \ldots, t_m \in K$.

The ring $\mathrm{SI}(Q, \alpha)$ is the ring of invariants for the action of the reductive group SL_α on the vector space $\mathrm{Rep}(Q, \alpha)$. The ring of regular functions on $\mathrm{Rep}(Q, \alpha)$ is a polynomial ring, and has a natural grading by total degree. The group SL_α acts on each graded piece, and thus the ring $\mathrm{SI}(Q, \alpha)$ inherits the grading:

$$\mathrm{SI}(Q, \alpha) = \bigoplus_{d \in \mathbb{Z}_{\geq 0}} \mathrm{SI}(Q, \alpha)_{[d]}.$$

Note that we use $\mathrm{SI}(Q, \alpha)_{[d]}$ to denote the dth graded piece with respect to the total degree grading, whereas we use $\mathrm{SI}(Q, \alpha)_\sigma$ to denote the weight space corresponding to the weight σ. Theorem 1.1 gives a determinantal description for the semi-invariants of a given weight. However, such a description of the invariants in terms of the total degree grading is not readily available. On the other hand, several results in Computational Invariant Theory apply to the total degree grading, and not directly to the weight space decomposition. The null cone $\mathcal{N} \subseteq \mathrm{Rep}(Q, \alpha)$ is defined as the zero set of the non-constant homogeneous invariants. From the description in Theorem 1.1, it follows that the null cone is the zero set of the semi-invariants for all nonzero weights. The null cone is an important tool in Computational Invariant Theory.

Recall that $||\alpha||_1 = \sum_{i \in Q_0} |\alpha_i|$ and $||\alpha||_2 = (\sum_{i \in Q_0} \alpha_i^2)^{1/2}$. We prove:

Theorem 1.2. *Let* $Q = (Q_0, Q_1)$ *be a quiver with no oriented cycles, and let* $|Q_0| = n$. *Then the null cone for the action of* SL_α *on* $\mathrm{Rep}(Q, \alpha)$ *is defined by semi-invariants for nonzero weights* σ *such that*
$$|\sigma|_\alpha \leq \frac{||\alpha||_1^{2n}}{4(n-1)^{2n-2}}.$$

In characteristic 0, bounds on the degree of the invariants defining the null cone can be translated into bounds for the degree of generating invariants.

Theorem 1.3. *Let* $Q = (Q_0, Q_1)$ *be a quiver with no oriented cycles, and let* $|Q_0| = n$. *Assume* $\mathrm{char}\, K = 0$ *and let* r *be the Krull dimension of* $\mathrm{SI}(Q, \alpha)$. *The ring* $\mathrm{SI}(Q, \alpha)$ *is generated by semi-invariants of weights* σ *with*
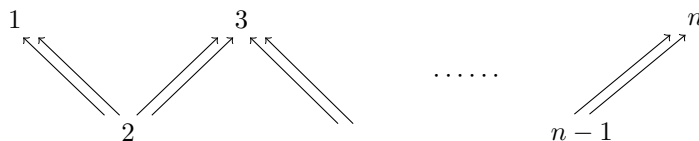
$$|\sigma|_\alpha \leq \frac{3rn^2 ||\alpha||_1^{4n}}{128(n-1)^{4n-4}}.$$

Note that $\dim(\mathrm{SI}(Q, \alpha)) \leq \dim \mathrm{Rep}(Q, \alpha)$, which depends on Q_0 and Q_1. We show that using a theorem of Weyl, we can give a bound that depends only on $n = |Q_0|$ and α.

Corollary 1.4. *Let* $Q = (Q_0, Q_1)$ *be a quiver with no oriented cycles, and let* $|Q_0| = n$ *Assume* $\mathrm{char}\, K = 0$. *The ring* $\mathrm{SI}(Q, \alpha)$ *is generated by semi-invariants of weights* σ *with*

$$|\sigma|_\alpha \leq \frac{3}{256} \left(||\alpha||_1^2 - ||\alpha||_2^2 \right) \frac{n^2 ||\alpha||_1^{4n}}{(n-1)^{4n-4}}.$$

Even though our bounds are not polynomial in $n = |Q_0|$, we give an example to show that it is not possible to obtain general bounds that are polynomial. Indeed consider the quiver Q_n shown below.

Proposition 1.5. *For the quiver Q_n, and dimension vector $\alpha = (2, 3, \ldots, 3, 1)$, the semi-invariants of weights σ with $|\sigma|_\alpha < 2^n - 2$ do not define the null cone, and hence do not generate $\mathrm{SI}(Q_n, \alpha)$.*

1.4. Organization

We recall relevant results from Computational Invariant Theory in Section 2 and some recent results from [3] in Section 3. In Section 4, we recall King's criterion for semi-stability and stability. We then use King's criterion to get bounds for the null cone in Section 5. We use the bounds on the null cone to generate bounds for the degree of generators in Section 6, and in Section 7, we remove the dependence on $\dim(\mathrm{SI}(Q, \alpha))$. Finally in Section 8, we prove Proposition 1.5.

2. Computational Invariant Theory

Let V be a rational representation of a linearly reductive group G. Then $K[V]$ is a polynomial ring, and has a natural grading by total degree. The ring of invariants $K[V]^G$ inherits this grading. Further, we know that it is finitely generated since G is reductive (see [11,12,20,10]).

Definition 2.1. We define $\beta(K[V]^G)$ to be the smallest integer D such that invariants of degree $\leq D$ generate $K[V]^G$.

A set of homogeneous invariants $f_1, f_2, \ldots, f_r$ is called a homogeneous system of parameters (hsop) if they are algebraically independent, and $K[V]^G$ is a finite module over $K[f_1, f_2, \ldots, f_r]$. In particular, the number of invariants in any hsop must be equal to the Krull dimension of $K[V]^G$. The invariant ring $K[V]^G$ is Cohen–Macaulay by the Hochster–Roberts theorem, which implies that $K[V]^G$ is in fact a free module over any hsop.

Definition 2.2. Null Cone : The null cone $\mathcal{N}$ is the zero set of all homogeneous invariant polynomials of positive degree

$$\mathcal{N} = \{v \in V | f(v) = 0 \ for \ all \ f \in K[V]_+^G\}.$$

If K is algebraically closed, then a set of algebraically independent homogeneous invariants $f_1, f_2, \ldots, f_r$ form a hsop if and only if their zero set coincides with the null cone (see [2, Lemma 2.4.5]). Kempf showed that the Hilbert series of $K[V]^G$ is a rational function of degree ≤ 0 (see [14]). From this, one can deduce that if $\deg(f_i) = d_i$, then $\beta(K[V]^G) \leq d_1 + d_2 + \cdots + d_r$. In [17,18] Knop further proved that the degree of the Hilbert series is in fact $\leq -r$ if G is a semisimple connected group in characteristic 0 (see also [2, Theorem 2.6.2]).

Definition 2.3. We define $\gamma(K[V]^G)$ to be the smallest integer D such that the non-constant homogeneous invariants of degree $\leq D$ define the null cone.

Popov argued that one could use bounds for $\gamma(K[V]^G)$ to get bounds for $\beta(K[V]^G)$. Hence one can use Kempf and Knop's results on the Hilbert series to get bounds for $\beta(K[V]^G)$ given bounds on $\gamma(K[V]^G)$. The first author in [1] improved these bounds to one that is polynomial in $\gamma(K[V]^G)$.

Theorem 2.4 *([1]). Assume K is algebraically closed. For a rational representation V of a linearly reductive group G, we have*

$$\beta(K[V]^G) \le \max\{2, \tfrac{3}{8}r(\gamma(K[V]^G))^2\}.$$

Note that in characteristic 0, linearly reductive groups are precisely the reductive groups. A more detailed treatment of the above results, as well as several other techniques for finding degree bounds can be found in [2], see also [21,22].

3. Degree bounds on matrix semi-invariants

In [3], we studied the left-right action of $\mathrm{SL}_n \times \mathrm{SL}_n$ on m-tuples on $n \times n$ matrices. Let $R(n, m) = K[\mathrm{Mat}_{n,n}^m]^{\mathrm{SL}_n \times \mathrm{SL}_n}$ be the ring of invariants. Studying linear subspaces of matrices, and the behavior of ranks in tensor blow-ups, we proved:

Theorem 3.1 *([3]). We have the following bounds for invariants defining the null cone:*

$$n\lfloor \sqrt{n^2 - 1} \rfloor \le \gamma(R(n, m)) \le n(n - 1).$$

In characteristic 0, we show the existence of a hsop in degree $n(n-1)$, and using the bounds on Hilbert series given by Knop, we get bounds for $\beta(R(n, m))$.

Theorem 3.2 *([3]). Let* char $K = 0$. *We have:*

(1) $\beta(R(n, m)) \le mn^4$;
(2) *For all m, $\beta(R(n, m)) \le n^6$.*

The second result follows from the first using Weyl's theorem on polarization of invariants (see [16, Section 7.1, Theorem A]).

Given a character $\sigma : Q_0 \to \mathbb{Z}$, we can consider the subring of semi-invariants $\mathrm{SI}(Q, \alpha, \sigma) = \bigoplus_{d \in \mathbb{Z}_{\ge 0}} \mathrm{SI}(Q, \alpha)_{d\sigma}$. The projective variety $\mathrm{Proj}(\mathrm{SI}(Q, \alpha, \sigma))$, if non-empty, is a moduli space for α-dimensional representations of Q (see [15]). A representation $V \in \mathrm{Rep}(Q, \alpha)$ is called σ-semistable if there exists an semi-invariant $f \in \mathrm{SI}(Q, \alpha)_{d\sigma}$ with $f(V) \ne 0$ (see [15]). The results on matrix semi-invariants mentioned above readily generalize to the subrings $\mathrm{SI}(Q, \alpha, \sigma)$.

Proposition 3.3 *([3]). Let $Q = (Q_0, Q_1)$ be a quiver with no oriented cycles. Let $\sigma \in \mathbb{Z}^{Q_0}$ be a weight. Then*

(1) *If V is σ-semistable, and $d \ge |\sigma|_\alpha - 1$, then there exists a semi-invariant $f \in \mathrm{SI}(Q, \alpha)_{d\sigma}$ with $f(V) \ne 0$;*
(2) *If* char $K = 0$, *then the ring $\mathrm{SI}(Q, \alpha, \sigma)$ is generated in degree $\le |\sigma|_\alpha^5$.*

Even though we know the bounds for these subrings, they do not immediately give bounds for the entire ring of semi-invariants, since there is no universal bound for $|\sigma|_\alpha$. Such a bound does not exist even if we restrict to indivisible weights. In the next two sections, we rectify this by showing that it suffices to consider only a finite subset of the weights.

4. Stability conditions and the null cone

Fix a quiver $Q = (Q_0, Q_1)$ with no oriented cycles and let $|Q_0| = n$. There is a criterion for deciding σ-semistability of a representation in terms of the dimension vectors of subrepresentations due to King (see [15]). We use the conventions in [6]. Given a weight σ and a dimension vector β, we define $\sigma(\beta) = \sum_{i \in Q_0} \sigma_i \beta_i$.

Theorem 4.1 ([15]). *Let $Q = (Q_0, Q_1)$ be a quiver with no oriented cycles, α be a sincere dimension vector, and σ be a weight. Then we have:*

(1) *A representation $V \in \text{Rep}(Q, \alpha)$ is σ-semistable if and only if $\sigma(\underline{\dim}V) = 0$ and $\sigma(\underline{\dim}W) \leq 0$ for all subrepresentations $W \subset V$;*
(2) *A representation $V \in \text{Rep}(Q, \alpha)$ is σ-stable if and only if $\sigma(\underline{\dim}V) = 0$ and $\sigma(\underline{\dim}W) < 0$ for all proper subrepresentations $0 \neq W \subsetneq V$.*

The set of σ-semistable representations form an abelian subcategory of the category of finite dimensional representations of a quiver Q. The simple objects in the category are precisely the σ-stable representations.

Corollary 4.2. *If V is σ-semistable and $\sigma(\underline{\dim}W) = 0$ for some non-zero proper subrepresentation W of V, then W and V/W are also σ-semistable.*

In fact, we have a Jordan–Hölder filtration $0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_m = V$. The composition factors V_i/V_{i-1} are unique up to rearrangement and isomorphism. Further these composition factors are σ-stable representations. We can define $\text{gr}_\sigma(V) = \bigoplus_i V_i/V_{i-1}$.

Remark 4.3. Let $d \in \mathbb{Z}_{>0}$. From Theorem 4.1, it follows that a representation V is σ-semistable (resp. stable) if and only if V is $d\sigma$-semistable (resp. stable). Hence, in particular, we have $\text{gr}_\sigma(V) = \text{gr}_{d\sigma}(V)$.

Lemma 4.4. *A representation $V \in \text{Rep}(Q, \alpha)$ is not in the null cone if and only if there exists a nonzero weight σ such that V is σ-semistable.*

Proof. We have already remarked that the null cone is the zero set of the semi-invariants of nonzero weights. Thus if a representation V is not in the null cone, then there is a semi-invariant $f \in \text{SI}(Q, \alpha)_\sigma$ for some nonzero weight σ such that $f(V) \neq 0$. Consequently for this σ, f is σ-semistable. Conversely, if $V \in \text{Rep}(Q, \alpha)$ is σ-semistable for some nonzero weight σ, then there is an invariant $f \in \text{SI}(Q, \alpha)_{d\sigma}$ such that $f(V) \neq 0$ for some $d \in \mathbb{Z}_{>0}$. Hence V is not in the null cone. $\quad\square$

5. Bounds for the null cone

5.1. A result from linear algebra

We first discuss some linear algebra that we require. For any vector $w = (w_1, w_2, \ldots, w_n) \in \mathbb{Q}^n$, recall that

$$||w||_1 = |w_1| + |w_2| + \cdots + |w_n|,$$

and

$$||w||_2 = (w_1^2 + w_2^2 + \cdots + w_n^2)^{1/2}.$$

We have the inequalities

$$||w||_2 \leq ||w||_1 \leq \sqrt{n}||w||_2.$$

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{n-1} \in \mathbb{Z}_{\geq 0}^n$ be linearly independent over $\mathbb{Q}$, with $\vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_{n-1} = \vec{v} \in \mathbb{Z}_{\geq 0}^n$. Considering each $\vec{v}_i$ as a row vector, we can write a $(n-1) \times n$ matrix M whose i^{th} row is $\vec{v}_i$. Since the $\vec{v}_i$ are linearly

independent over $\mathbb{Q}$, the rank of this matrix is $n-1$. Hence it has a 1-dimensional kernel. The following proposition bounds the smallest nonzero integral vector on this 1-dimensional kernel:

Proposition 5.1. *Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{n-1}, \vec{v}$ and M be as above. Then there is a nonzero integral vector $\vec{u} = (u_1, u_2, \ldots, u_n) \in \mathrm{Ker}(M)$ such that $|u_i| \leq \left(\dfrac{||\vec{v}||_1}{n-1} \right)^{n-1}$*

Proof. Let $\widehat{M}(i)$ denote the $(n-1) \times (n-1)$ minor of M obtained by removing the ith column. Then define $u_i = (-1)^i \widehat{M}(i)$. It is clear that $\vec{u} = (u_1, u_2, \ldots, u_n)$ is an integral vector and that it is in the kernel of M. Further, we have

$$
\begin{aligned}
|u_i| &\leq ||\vec{v}_1||_2 \cdot ||\vec{v}_2||_2 \cdots ||\vec{v}_{n-1}||_2 \\
&\leq ||\vec{v}_1||_1 \cdot ||\vec{v}_2||_1 \cdots ||\vec{v}_{n-1}||_1 \\
&\leq \left(\frac{||\vec{v}_1||_1 + ||\vec{v}_2||_1 + \cdots + ||\vec{v}_{n-1}||_1}{n-1} \right)^{n-1} \\
&= \left(\frac{||\vec{v}||_1}{n-1} \right)^{n-1} \qquad \square
\end{aligned}
$$

5.2. Bounds for the null cone

Without loss of generality, we can assume α is sincere, i.e., $\alpha(x) \neq 0 \; \forall x \in Q_0$. If not, one can work with the subquiver $\widetilde{Q}$ defined by $\mathrm{supp}(\alpha)$, to get better bounds. Given a representation $V \in \mathrm{Rep}(Q, \alpha)$ which is not in the null cone, we denote by $\mathrm{C}(V)$, the set of weights σ for which V is σ-semistable, i.e,

$$
\mathrm{C}(V) = \{ \sigma | V \text{ is } \sigma\text{-semistable} \}.
$$

Notice that $\mathrm{C}(V) \subset \mathbb{Z}^{Q_0}$ is cut out by a linear equation $\sigma(\alpha) = 0$ and by linear inequalities $\sigma(\underline{\dim} W) \leq 0$ for proper subrepresentations W of V. Let L be an extremal ray of $\mathrm{C}(V)$. It is clear that this extremal ray is defined by degenerating a subset of the linear inequalities to equalities. Hence, there exist subrepresentations $W_i, i \in I$ such that the equalities $\sigma(\underline{\dim} W_i) = 0$, and $\sigma(\alpha) = 0$ define $\mathbb{Q}L$.

Lemma 5.2. *There exist dimension vectors $\beta(i)$, $1 \leq i \leq n-1$ with $|| \sum_{i=1}^{n-1} \beta(i) ||_1 \leq ||\alpha||_1$ such that the line $\mathbb{Q}L$ is defined by the linear equalities $\sigma(\beta(i)) = 0$, $1 \leq i \leq n-1$.*

Before we prove the lemma we remark that V is σ-stable for a weight σ precisely when σ is in the interior of $C(V)$.

Proof. Let $\widetilde{\sigma} \in L$. Consider any Jordan–Hölder series (in the abelian subcategory of $\widetilde{\sigma}$-semistable representations) $0 = V_0 \subseteq V_1 \subset V_2 \subset \cdots \subset V_k = V$, and let $Z_i = V_i / V_{i-1}$, $1 \leq i \leq k$ be the composition factors. Let $\alpha(i) = \dim Z_i$. We have $\widetilde{\sigma}(\alpha(i)) = 0$ since Z_i are $\widetilde{\sigma}$-semistable.

Let us look at the set of subrepresentations W_i such that the linear equalities $\sigma(\underline{\dim} W_i) = 0$ and $\sigma(\alpha) = 0$ define $\mathbb{Q}L$. Each W_j is a $\widetilde{\sigma}$-semistable subrepresentation of V, by Corollary 4.2. Therefore, the composition factors for W_j must be a subset of $\{ Z_i \mid 1 \leq i \leq k \}$, and hence we have $\dim W_j = \alpha(i_1) + \alpha(i_2) + \cdots + \alpha(i_l)$, for some subset $\{ i_1, i_2, \ldots, i_l \} \subseteq \{ 1, 2, \ldots, k \}$. Hence the condition $\sigma(\underline{\dim} W_i) = 0$ is a consequence of the conditions $\sigma(\alpha(i)) = 0$. In particular, we get that $\mathbb{Q}L$ is defined by the linear equalities $\sigma(\alpha(i)) = 0$ for $1 \leq i \leq k$.

However, some of these may be redundant. Since these equalities define a 1-dimensional subspace, there is a subset of the $\alpha(i)$'s of size $n-1$, say $\{\beta(1), \beta(2), \ldots, \beta(n-1)\}$, such that $\sigma(\beta(i)) = 0$ for $i = 1, 2, \ldots, n-1$ define $\mathbb{Q}L$. $\square$

Proposition 5.3. *Given a representation $V \in \mathrm{Rep}(Q, \alpha)$ such that $\mathrm{C}(V)$ is non-empty, V is σ-semistable for some weight σ such that each coordinate of σ has size $\leq \left(\dfrac{||\alpha||_1}{n-1} \right)^{n-1}$*

Proof. Pick an extremal ray L in $C(V)$. By Lemma 5.2, we have that $\mathbb{Q}L$ is the kernel of the $(n-1) \times n$ matrix whose rows are the dimension vectors $\beta(i)$. Let σ be the smallest integral vector in L. Then apply Proposition 5.1. $\square$

We can now translate this into a bound for $|\sigma|_\alpha$.

Corollary 5.4. *Given a representation $V \in \mathrm{Rep}(Q, \alpha)$ such that $\mathrm{C}(V)$ is non-empty, V is σ-semistable for some weight σ such that*

$$|\sigma|_\alpha \leq \left(\frac{||\alpha||_1}{n-1} \right)^{n-1} \left(\frac{||\alpha||_1}{2} \right) = \frac{||\alpha||_1^n}{2(n-1)^{n-1}}.$$

Proof. If every coordinate $|\sigma_i| \leq M$ for some M, then we have $|\sigma_i|\alpha_i \leq M\alpha_i$. Note further that since $\sigma(\alpha) = 0$, we have $\sum_{i=1}^{n} |\sigma_i|\alpha_i = \sigma_+ \cdot \alpha + \sigma_- \cdot \alpha = 2|\sigma|_\alpha$. Thus $|\sigma|_\alpha \leq \frac{1}{2}M||\alpha||_1$. $\square$

Proof of Theorem 1.2. Given $V \in \mathrm{Rep}(Q, \alpha)$ which is not in the null cone, we have that $C(V)$ is nonzero by Lemma 4.4. Hence there exists some σ with $|\sigma|_\alpha \leq \frac{||\alpha||_1^n}{2(n-1)^{n-1}}$, such that V is σ-semistable. Then by the first part of Proposition 3.3, there is a semi-invariant $f \in \mathrm{SI}(Q, \alpha)_{d\sigma}$ that does not vanish on V for each $d \geq |\sigma|_\alpha - 1$. Observe that $|d\sigma|_\alpha = d|\sigma|_\alpha$. Taking $d = |\sigma|_\alpha$ gives the required conclusion. $\square$

Remark 5.5. It might seem very wasteful to find bounds using an extremal ray L, as it is very likely that smaller weights lie in the interior of $\mathrm{C}(V)$. However, observe that if σ is an integral weight on an extremal ray L of $\mathrm{C}(V)$, then for $\mathrm{gr}_\sigma(V)$ we have $\mathrm{C}(\mathrm{gr}_\sigma(V)) = L$. Hence these extremal rays cannot be avoided.

6. Bounds for generating semi-invariants

The ring $\mathrm{SI}(Q, \alpha)$ has two natural gradings. We have the weight space decomposition $\mathrm{SI}(Q, \beta) = \bigoplus_\sigma \mathrm{SI}(Q, \alpha)_\sigma$. We also have the natural grading inherited from viewing $K[\mathrm{Rep}(Q, \alpha)]$ as a polynomial ring. While the weight space decomposition is the more interesting one, all the results from Computational Invariant Theory hold for the latter grading. In the previous section, we found bounds for invariants defining the null cone in terms of the weight space decomposition. In order to use Theorem 2.4, we must switch to the latter grading.

Lemma 6.1. *Let $f \in \mathrm{SI}(Q, \alpha)_\sigma$, then its homogeneous components are non-trivial only for degrees between $|\sigma|_\alpha$ and $n|\sigma|_\alpha$.*

Proof. A set of semi-invariants spanning $f \in \mathrm{SI}(Q, \alpha)_\sigma$ was given in Theorem 1.1. A semi-invariant in this set is given by the determinant of a matrix, whose size is $|\sigma|_\alpha$. The matrix is described in block form, where each block defines a linear combinations of paths between two different vertices. Such paths have length at least 1 and at most n. Hence the entries of this matrix are polynomials whose homogeneous components are non-trivial only for degrees between 1 and n. $\square$

The above lemma can then be used to convert the bounds given in Theorem 1.2 with respect to weight spaces to one in the total degree grading.

Corollary 6.2. *The null cone for the action of* SL_α *on* $\mathrm{Rep}(Q, \alpha)$ *is defined by homogeneous invariants of degree* $\leq \dfrac{n||\alpha||_1^{2n}}{4(n-1)^{2n-2}}$, *i.e.,*

$$\gamma(\mathrm{SI}(Q, \alpha)) \leq \frac{n||\alpha||_1^{2n}}{4(n-1)^{2n-2}}.$$

Corollary 6.3. *Assume* $\mathrm{char}\, K = 0$ *and let* $r = \dim(\mathrm{SI}(Q, \alpha))$. *The ring of semi-invariants* $\mathrm{SI}(Q, \alpha)$ *is generated by invariants of degree* $\leq \dfrac{3}{8} r \left(\dfrac{n||\alpha||_1^{2n}}{4(n-1)^{2n-2}} \right)^2$.

Proof. When K is algebraically closed, we can apply Theorem 2.4 to get the required bound. If K is an arbitrary characteristic 0 field, then working over the algebraic closure $\overline{K}$, observe that the ring of semi-invariants is simply $\mathrm{SI}(Q, \alpha) \otimes_K \overline{K}$. Hence the same bound holds over K as well. $\square$

Proof of Theorem 1.3. This follows from Lemma 6.1 and Corollary 6.3. $\square$

Remark 6.4. The bounds given in Theorem 1.3 depend on $\dim(\mathrm{SI}(Q, \alpha))$. Kac gave a formula (see [13]) for $\dim(\mathrm{SI}(Q, \alpha))$ in terms of the canonical decomposition. There is in fact an efficient algorithm to compute the canonical decomposition due to the first author and Weyman, see [5]. More importantly, as remarked in the introduction, $\dim(\mathrm{SI}(Q, \alpha))$ is bounded by $\dim(\mathrm{Rep}(Q, \alpha)) = \sum_{a \in Q_1} \alpha(ha)\alpha(ta)$.

7. Removing dependence on $\dim \mathrm{SI}(Q, \alpha)$

The bounds in Theorem 1.3 depend on $|Q_0| = n$, α and $\dim(\mathrm{SI}(Q, \alpha))$. Note that $\dim(\mathrm{SI}(Q, \alpha))$ depends on Q_1. We now show how one can use Weyl's theorem on polarization of invariants to remove the dependence on $\dim(\mathrm{SI}(Q, \alpha))$, and get a bound which is purely in terms of $|Q_0| = n$ and α.

Given a quiver $Q = (Q_0, Q_1)$ with no oriented cycles, we can label the vertices $1, 2, \ldots, n$ so that for every arrow, $ta < ha$. Let $n(i, j)$ denote the number of arrows with tail i and head j. Fix a dimension vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Now, observe that

$$\mathrm{Rep}(Q, \alpha) = \bigoplus_{i < j} \mathrm{Mat}_{\alpha_j, \alpha_i}^{n(i,j)}.$$

Observe further that each $\mathrm{Mat}_{\alpha_j, \alpha_i}$ is a representation of GL_α as well as SL_α. Observe that $\dim \mathrm{Mat}_{\alpha_j, \alpha_i} = \alpha_i \alpha_j$. Hence, as a consequence of Weyl's theorem on polarization of invariants (see [26, II.5, Theorem 2.5A] and [16, Section 7.1, Theorem A]), we can obtain the semi-invariant ring $\mathrm{SI}(Q, \alpha)$ by the process of polarization from $K[\bigoplus_{i < j} \mathrm{Mat}_{\alpha_j, \alpha_i}^{\alpha_i \alpha_j}]^{\mathrm{SL}_\alpha}$. See also [7, Theorem 0.1] for a version that is better suited to our situation. In other words, for the purposes of finding a bound on the generating invariants, we can assume $n(i, j) = \alpha_i \alpha_j$.

Define a quiver $\widetilde{Q}$ whose vertex set is $1, 2, \ldots, n$, and has $\alpha_i \alpha_j$ arrows from i to j. The above discussion can be summarized as follows:

Proposition 7.1. *Assume* $\mathrm{char}\, K = 0$, *then we have*

$$\beta(\mathrm{SI}(Q, \alpha)) \leq \beta(\mathrm{SI}(\widetilde{Q}, \alpha)).$$

Proof of Corollary 1.4. For $\widetilde{Q}$, we have:

$$\dim(\mathrm{SI}(\widetilde{Q}, \alpha)) \le \dim \mathrm{Rep}(\widetilde{Q}, \alpha)$$

$$= \sum_{i<j} \alpha_i \alpha_j$$

$$= \frac{\|\alpha\|_1^2 - \|\alpha\|_2^2}{2}.$$

Now, use this bound for r in Corollary 6.3, and apply Lemma 6.1. □

8. Exponential lower bound

We first recall some results on the 2-Kronecker quiver, the quiver with 2 vertices x and y and two arrows a, b from x to y

$$x \xrightarrow[\ b\ ]{\ a\ } y$$

We look at two particular indecomposable representations. The representation

$$V = \quad K \xrightarrow[\binom{0}{1}]{\binom{1}{0}} K^2$$

is an indecomposable representation for the dimension vector $(1, 2)$. It is easy to check that V is σ-semistable precisely when $\sigma \in \mathbb{Z}_{>0}(2, -1)$. Similarly, the representation

$$W = K^2 \xrightarrow[(\,0\ 1\,)]{(\,1\ 0\,)} K$$

is an indecomposable representation for the dimension vector $(2, 1)$. Once again, it is easy to check that W is σ-semistable precisely when $\sigma \in \mathbb{Z}_{>0}(1, -2)$.

Proof of Proposition 1.5. Consider the quiver Q_n, and observe that the odd vertices are sources and the even vertices are sinks. For any $i \in \{1, 2 \ldots, n-1\}$, one of i and $i+1$ is a source and the other is a sink. Let ψ_i be the embedding of the 2-Kronecker quiver, that maps the vertices to i and $i+1$, with source begin mapped to source and sink to sink. Under this embedding, we see that $\psi_i(V)$ and $\psi_i(W)$ are indecomposable representations of the quiver Q_n. We consider the representation

$$R = \psi_1(V) \oplus \psi_2(W) \oplus \psi_3(V) \cdots$$

$$= \bigoplus_{i \text{ odd}} \psi_i(V) \oplus \bigoplus_{i \text{ even}} \psi_i(W).$$

We have $\dim(R) = (2, 3, 3, \ldots, 3, 1)$. Moreover, R is σ-semistable for the indivisible integral weight $\sigma = (-1, 2, -4, 8, \ldots)$. Since R is a direct sum of indecomposables, it suffices to check σ-semistability of these indecomposables. That each of these indecomposables is σ-semistable follows from the above discussion above on 2-Kronecker quivers. Thus, in particular, $C(R)$ is non-empty, and R is not in the null cone.

Moreover, we have that R is a direct sum of $n-1$ indecomposables, and their dimension vectors are linearly independent vectors, and hence it follows from King's stability conditions that $C(R)$ is at most 1-dimensional. Since $C(R)$ is non-empty, and $(-1, 2, -4, 8, \dots)$ is indivisible, we have that $C(R) = \mathbb{Z}_{\geq 0}(-1, 2, -4, 8, \dots)$. More concretely, we have the condition that $\sigma \in C(R)$, then σ is in the kernel of

$$\begin{pmatrix} 2 & 1 & & & \\ & 2 & 1 & & \\ & & \ddots & \ddots & \\ & & & 2 & 1 \end{pmatrix}.$$

The kernel of the above matrix is precisely the 1-dimensional subspace spanned by the vector $(-1, 2, -4, 8, \dots)$, and the smallest integral vector in this 1-dimensional subspace is $(-1, 2, -4, 8, \dots)$ by virtue of being indivisible. For the weight $\sigma = (-1, 2, -4, 8, \dots)$, we get $|\sigma|_\alpha = 2^n - 2$ by computation. Thus in this case, the semi-invariants of weights σ with $|\sigma|_\alpha < 2^n - 2$ do not define the null cone. $\quad\square$

Remark 8.1. For any given quiver, one might be able to generate stronger bounds by improving the estimates we make at various stages of obtaining our bounds.

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