Appendix

This appendix presents the formal model. To aid in the presentation, we will assume the principles and results of the domestic model demonstrated in the Appendix to Chapter 3 of Bueno de Mesquita et al. (2003) and focus on the international aspects of the model.

The Model

We refer to the two states as **A** and **B**, their initial leaders as *A* and *B*, challengers as *a* and *b*, and selectors as α and β. W_A, S_A, W_B, S_B give the sizes of the winning coalition of **A**, the selectorate of **A**, the winning coalition of **B**, and the selectorate of **B** respectively. R_A, R_B are the resources of **A** and the resources of **B** respectively. Private benefits are denoted by g (for goodies), public goods by x, and resources devoted to foreign policy by p, with subscripts denoting state **A** or **B**. The foreign policy outcome is given by $f(p_A, p_B) = \pi_A p_A + \pi_B p_B$ with $\pi_A > 0$ and $\pi_B < 0$. There are N_A selectors in **A** and N_B in **B**. Let Φ_A be the set of all N_A-dimensional vectors corresponding to all permutations of the integers from 1 to N_A; for example, if $N_A = 3$, then $\Phi_A = \{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}$. Φ_B is the set of all N_B-dimensional vectors composed of all permutations of the integers from 1 to N_B. Let

$\varphi_A \in \Phi_A$ and $\varphi_B \in \Phi_B$, and $\varphi_A(i)$ denotes the i-th element of φ_A with $\varphi_B(i)$ giving the i-th element of φ_B.

The time line of the model is as follows:

1. A chooses whether to replace B with a puppet at cost C_D, take territory r from **B** at cost C_r, do both at cost $C_D + C_r$, or do neither. If A installs a puppet, A sets W_B and S_B.
2. If this is the first round of the game or if challenger a was selected in the previous round, select φ_A randomly from Φ_A with uniform probability across all members of Φ_A. If a was selected in the previous round, relabel a as A and new challenger as a.
3. A selects w_A and allocates g_A, x_A, and p_A with $|w_A|g_A + x_A + p_A \leq R_A$ ($R_A + r$ if A took territory from **B**).
4. a selects w_a and allocates g_a, x_a, and p_a with $|w_a|g_a + x_a + p_a \leq R_A$ ($R_A + r$ if A took territory from **B**).
5. All selectors α choose between A and a, with a winning if $|\{\alpha \in w_A | \alpha \text{ chose } A\}| < W_A|$ and $|\{\alpha \in w_a | \alpha \text{ chose } a\}| \geq W_A|$ and A winning otherwise (the constructive rule of deposition).
6. If this is the first round of the game or if challenger b was selected in the previous round, select φ_B randomly from Φ_B with uniform probability across all members of Φ_B. If b was selected in the previous round, relabel b as B and new challenger as b.
7. B selects w_B and allocates g_B, x_B, and p_B with $|w_B|g_B + x_B + p_B \leq R_B$ ($R_B - r$ if A took territory from **B**).
8. b selects w_b and allocates g_b, x_b, and p_b with $|w_b|g_b + x_b + p_b \leq R_B$ ($R_B - r$ if A took territory from **B**).
9. All selectors β choose between B and b with the winner decided by the constructive rule of deposition. The round ends.

Moves 2 through 9 define one round of the game, with the rounds infinitely repeated and payoffs (as defined in the chapter) from a round received at the end of that round. All players discount future payoffs by common discount factor δ.

Payoffs for a round are as follows:

For A, $\Psi_A + R_A - |w_A|g_A - x_A - p_A + \sum_{i \in w_A} \varphi_A(i)\varepsilon$ if A reselected and did not take territory from **B**; $\Psi_A + R_A + r - |w_A|g_A - x_A - p_A + \sum_{i \in w_A} \varphi_A(i)\varepsilon$ if A reselected and did take territory from **B**; and 0 if not reselected. $\varepsilon > 0$ and $\varepsilon << \min\left(\frac{R_A}{N_A(N_A+1)}, \frac{R_B}{N_B(N_B+1)}\right)$; the summation gives A's affinities for selectors in its winning coalition. These values are assumed to be close to 0 so they only influence which selectors A includes in W_A not how many selectors are included. a has parallel payoffs without the affinity summation as her affinities are not selected until she is the incumbent.

For B, $\Psi_B + R_B - |w_B|g_B - x_B - p_B + \sum_{i \in w_B} \varphi_B(i)\varepsilon$ if B reselected and A did not take territory from **B**; $\Psi_B + R_B - r - |w_B|g_B - x_B - p_B + \sum_{i \in w_B} \varphi_B(i)\varepsilon$ if B reselected and A did take territory from **B**; and 0 if not reselected or removed for a puppet. b has parallel payoffs.

For α, $V_A(x_A, g_A, f(p_A, p_B))$ using the allocations of the winning candidates with $D_x V_A > 0$, $D^2_{xx} V_A < 0$, $D_g V_A > 0$, $D^2_{gg} V_A < 0$, $D_f V_A > 0$, and $D^2_{ff} V_A < 0$. β has parallel utility function with $D_f V_B < 0$, and $D^2_{ff} V_B < 0$. Both V_A and V_B are assumed to be additively separable in x, g, and f. We also assume $D^3_{fff} V_A, D^3_{fff} V_B > 0$.

Proof of Equilibrium

The equilibrium is found by backwards induction; first, we find the optimal decisions in **B**, then those in **A**, and finally examine *A*'s decsion whether to take territory or depose *B*. The selection of the winning coalition parallels that in Bueno de Mesquita et al. (2003), so we refer the reader there for the details of that argument. We first show that p_B, *B*'s allocation to foreign policy, is increasing in p_A, *A*'s allocation to foreign policy. Because *B* allocates resources after *A* does, *B*'s allocation takes p_A as fixed. However, $D_{p_A} p_B < -\frac{\pi_A}{\pi_B}$, implying that *B*'s allocation to foreign policy does not match the effect of *A*'s allocation on *f*, and so *f* is increasing in p_A even after *B*'s response to p_A. With this observation, equilibrium allocations for *A* and *B* are straightforward optimization problems, and we can compare *A*'s equilibrium payoffs from taking territory and deposing *B* to determine which option *A* adopts in the first move of the game.

From the model in Chapter 3 of Bueno de Mesquita et al. (2003), it follows that *B* will choose the $|w_B|$ selectors for which he has highest affinity for his winning coalition (i.e. the selectors with the greatest values in φ_B) and can always allocate sufficient resources to guarantee his reselection. *B* maximizes $R_B - x_B - W_B g_B - p_B$ subject to the constraint of the incumbency condition I_B,

$$(1-\delta)[V_B(x_B, g_B, f(p_A, p_B)) \; - v_B(R_B, W_B, p_A)] +$$

$$\delta\left(1-\tfrac{W_B}{S_B}\right)[V_B(x_B, g_B, f(p_A, p_B)) \; - V_B(x_B, 0, f(p_A, p_B))] = 0$$

where $v_B(R_B, W_B, p_A)$ is the maximum utility that can be produced by allocating R_B across the three goods. The incumbency condition compares the payoffs for staying loyal to B and defecting to b for a pivotal member of B's winning coalition. If pivotal member β remains loyal, B is retaimed, and β receives $V_B(x_B, g_B, f(p_A, p_B))$ in this and all future periods. If β defects, b becomes leader, and β receives $v_B(R_B, W_B, p_A)$ this period, the most that b can offer, and has a $\frac{W_B}{S_B}$ chance of being in b's winning coalition in the future, producing a payoff of $V_B(x_B, g_B, f(p_A, p_B))$ and a $1-\frac{W_B}{S_B}$ chance of not being in b's winning coalition in the future, producing a payoff of $V_B(x_B, 0, f(p_A, p_B))$. In equilibrium, B has sufficient resources to meet this condition for a coalition of size W_B, so B will choose the W_B selectors it has highest affinity for as w_B.

Allocation (g_B, x_B, p_B) solves the following system:

$$\begin{cases} \lambda(1-\delta)D_x V_B - 1 = 0 \\ \lambda\left(1-\delta\frac{W_B}{S_B}\right)D_g V_B - W_B = 0 \\ \lambda(1-\delta)(D_f V_B)\pi_B - 1 = 0 \\ I_B = 0 \end{cases}$$

As p_A is fixed when B allocates resources, there are no strategic considerations in B's allocation. Because V_B is decreasing and convex in f, there is a unique solution to B's optimization problem, and so p_B is a function of p_A, which we write as $PB(p_A)$. $PB(p_A)$ is increasing in p_A; $D_{p_A}PB > 0$. From the Lagrangian system above (all partial derivatives are found by applying Cramer's Theorem to the matrices of partial derivatives of the equations defining the implicit functions),

$$\mathrm{D}_{p_A}\mathrm{p}_B = -\left(\frac{(1-\delta)^2\lambda^2\left(1-\delta\frac{W_B}{S_B}\right)\pi_A\pi_B}{(1-\delta)^2\lambda^2\left(1-\delta\frac{W_B}{S_B}\right)\pi_B^2} \right)$$

$$\left(\frac{(D_{xx}^2V_B)\left(-\left(1-\delta\frac{W_B}{S_B}\right)(D_gV_B)^2\left(D_{ff}^2V_B\right)-(1-\delta)(D_fV_B)(D_fV_B(p_B)-D_fV_B(R))(D_{gg}^2V_B)\right)-(1-\delta)(D_xV_B)^2\left(D_{ff}^2V_B\right)(D_{gg}^2V_B)}{(D_{xx}^2V_B)\left(-\left(1-\delta\frac{W_B}{S_B}\right)(D_gV_B)^2\left(D_{ff}^2V_B\right)-(1-\delta)(D_fV_B)^2(D_{gg}^2V_B)\right)-(1-\delta)(D_xV_B)^2\left(D_{ff}^2V_B\right)(D_{gg}^2V_B)} \right) > 0$$

or simplifying, we have

$$D_{p_A}p_B = -\frac{\pi_A}{\pi_B}\left(1-\frac{(1-\delta)(D_{xx}^2V_B)(D_fV_B)(D_fV_B(R))(D_{gg}^2V_B)}{Denom_B}\right)$$

where $Denom_B = \left(1-\delta\frac{W_B}{S_B}\right)(D_{xx}^2V_B)(D_{ff}^2V_B)(D_gV_B)^2$

$$+(1-\delta)(D_{xx}^2V_B)(D_fV_B)^2(D_{gg}^2V_B)+(1-\delta)(D_xV_B)^2(D_{ff}^2V_B)(D_{gg}^2V_B) > 0$$

Note that all the mixed partials in the determinants equal 0 because V_B is additively separable. All the first derivatives except D_fV_B are positive, all the second derivates are less than 0 by concavity of V_B, $\pi_A > 0$, and $\pi_B < 0$. Further, the numerator of the second term is less than the denominator because $D_fV_B(p_B) \geq D_fV_B(R)$ by concavity of

V_B. Similarly, we have the following:

$$D_{p_A} x_B = \frac{(1-\delta)(D_g V_B)\pi_A \left(D^2_{ff} V_B\right)(D^2_{xx} V_B)(D_f V_B(R))}{Denom_B} < 0$$

$$D_{p_A} g_B = \frac{(1-\delta)(D_x V_B)\pi_A \left(D^2_{ff} V_B\right)(D^2_{gg} V_B)(D_f V_B(R))}{Denom_B} < 0$$

In both cases, the numerator is negative and the demoninator is positive, making the overall partial negative. As *A* commits more resources to foreign policy, *B* commits added resources in response. Rises in *A*'s resources committed to foreign policy forces *B* to spend more in foreign policy but less on public goods and private benefits.

Further, *B* allocates more resources overall; if $m_B = x_B + W_B g_B + p_B$,

$$D_{p_A} m_B = -\frac{\pi_A}{\pi_B}\left(\frac{1}{(D^2_{xx} V_B)\left(-\left(1-\delta\frac{W_B}{S_B}\right)(D_g V_B)^2\left(D^2_{ff} V_B\right)-(1-\delta)(D_f V_B)^2(D^2_{gg} V_B)\right)-(1-\delta)(D_x V_B)^2\left(D^2_{ff} V_B\right)(D^2_{gg} V_B)}\right)$$

$$\left[\begin{array}{l} (D^2_{xx} V_B)(D^2_{ff} V_B)(D_g V_B)\left((1-\delta)\pi_B D_f V_B(R) - \left(1-\delta\frac{W_B}{S_B}\right)D_g V_B\right) - \\ (D^2_{xx} V_B)(D^2_{gg} V_B)(1-\delta)(D_f V_B)(D_f V_B(p_B) - D_f V_B(R)) - \\ (D^2_{xx} V_B)(D^2_{gg} V_B)(1-\delta)(D_f V_B)(D_f V_B(p_B) - D_f V_B(R)) \end{array}\right]$$

$$> 0$$

The numerator is positive because in equilibrium the expression in the last pair of parentheses in the first term above reduces to $(1-\delta)\pi_B(D_f V_B(R) - W(D_f V_B)) < 0$ and the last expression in the third term reduces to $(1-\delta)\pi_B(D_f V_B(R) - D_f V_B) < 0$. Then each of these three terms is negative as is the remainder of the numerator in the first line above. Because *B*'s payoff is $\Psi_B + R_B - m_B$, it decreases as *A* commits more resources to foreign policy.

Because B's allocation of R_B across x_B, g_B, p_B has a unique solution given W_B, S_B, p_A, we can define function $PB(p_A)$ as the equilibrium allocation to p_B for a given value of p_A. As shown above, $D_{p_A}PB > 0$. Further, $D^2_{p_A p_A}PB < 0$ by taking the partial of $D_{p_A}p_B$ above to arrive at the following:

$$D^2_{p_A p_A}p_B = \left(\frac{\pi_A^2}{\pi_B}\right)\frac{(1-\delta)(D^2_{xx}V_B)(D^2_{gg}V_B)}{(Denom_B)^2}$$

$$\left[\begin{array}{c} \left(1-\delta\frac{W_B}{S_B}\right)(D^2_{xx}V_B)(D_gV_B)^2 \\ \left[(D^2_{ff}V_B)^2(D_fV_B(R)) + (D_fV_B)(D^2_{ff}V_B)(D^2_{ff}V_B(R)) - (D_fV_B)(D_fV_B(R))(D^3_{fff}V_B)\right] \\ + (1-\delta)(D^2_{xx}V_B)(D^2_{gg}V_B)\left[(D_fV_B)^2(D^2_{ff}V_B)(D_fV_B(p_B) - D_fV_B(R))\right] \\ + (1-\delta)(D_xV_B)^2(D^2_{gg}V_B) \\ \left[(D^2_{ff}V_B)^2(D_fV_B(R)) + (D_fV_B)(D^2_{ff}V_B)(D^2_{ff}V_B(R)) - (D_fV_B)(D_fV_B(R))(D^3_{fff}V_B)\right] \end{array}\right] < 0$$

With $PB(p_A)$ well-defined, we would like to know the properties of $f(p_A, PB(p_A))$, the foreign policy outcome that results from A's foreign policy, including B's reaction to that allocation. Compared to the absence of a reaction by B, A commits more resources to foreign policy in equilibrium. The effective price of f for A is $\frac{1}{D_{p_A}f(p_A, PB(p_A))}$, where

$$D_{p_A}f(p_A, PB(p_A)) = D_{p_A}f + (D_{p_B}f)(D_{p_A}PB) = \pi_A + \pi_B(D_{p_A}PB) < \pi_A = D_{p_A}f,$$

implying that the price of foreign policy rises and A allocates more to foreign policy than it would in the absence of B's anticipated reaction. Similarly, this effective increase in price leads A to increase its total allocation, m_A, over what it would be in absence of a reaction by B. Further, $f(p_A, PB(p_A))$ is increasing in p_A.

$D_{p_A} f(p_A, PB(p_A)) = \pi_A + \pi_B (D_{p_A} PB) =$

$$\pi_A + \pi_B \left(-\frac{\pi_A}{\pi_B} \left(1 - \frac{(1-\delta)(D^2_{xx} V_B)(D_f V_B)(D_f V_B(R))(D^2_{gg} V_B)}{Denom_B} \right) \right)$$

$= \pi_A \left(\frac{(1-\delta)(D^2_{xx} V_B)(D_f V_B)(D_f V_B(R))(D^2_{gg} V_B)}{Denom_B} \right) > 0.$ Also as shown above,

$(1-\delta)(D^2_{xx} V_B)(D_f V_B)(D_f V_B(R))(D^2_{gg} V_B) < Denom_B$, so $D_{p_A} f(p_A, PB(p_A)) < \pi_A$.

From the model in Chapter 3, it follows that A will choose the $|w_A|$ selectors for which he has highest affinity as his winning coalition and can always allocate sufficient resources to guarantee his reselection. A maximizes $R_A - x_A - W_A g_A - p_A$ subject to the constraint of the incumbency condition I_A,

$$(1-\delta)[V_A(x_A, g_A, f(p_A, PB(p_A))) - v_A(R_A, W_A)] +$$

$$\delta \left(1 - \frac{W_A}{S_A} \right) [V_A(x_A, g_A, f(p_A, PB(p_A))) - V_A(x_A, 0, f(p_A, PB(p_A)))] = 0$$

where $v_A(R_A, W_A)$ is the maximum utility that can be produced by allocating R_A

across the three goods. Because V_A is concave in all three goods and $f(p_A, PB(p_A))$ is increasing in p_A, there is a unique solution to this problem.

Allocation (g_A, x_A, p_A) solves the following system:

$$\begin{cases} \lambda(1-\delta)D_x V_A - 1 = 0 \\ \lambda\left(1-\delta\frac{W_A}{S_A}\right)D_g V_A - W_A = 0 \\ \lambda(1-\delta)(D_f V_A)(\pi_A)\left(\frac{(1-\delta)(D^2_{xx}V_B)(D_f V_B)(D_f V_B(R))(D^2_{gg}V_B)}{Denom_B}\right) - 1 = 0 \\ I_A = 0 \end{cases}$$

The unique solution to this system gives A's optimal allocation of R_A to g_A, x_A, p_A; call them g^*_A, x^*_A, p^*_A. A's equilibrium payoff if it does not take territory from **B** nor depose B is $\Psi_A + R_A - |w_A|g^*_A - x^*_A - p^*_A$. In parallel fashion, define g^+_A, x^+_A, p^+_A be A's optimal allocations if A takes territory from **B**, giving **A** $R_A + r$ resources to allocate and **B** $R_B - r$ resources, $g^\wedge_A(W,S), x^\wedge_A(W,S), p^\wedge_A(W,S)$ be A's optimal allocations if A removes B and changes W_B to W and S_B to S, and $g^\sim_A(W,S), x^\sim_A(W,S), p^\sim_A(W,S)$ be A's optimal allocations if A takes territory from **B** and removes B and changes W_B to W and S_B to S. Corresponding equilibrium payoffs are then

$\Psi_A + R_A + r - |w_A|g^+_A - x^+_A - p^+_A - C_r$ for taking territory,

$\Psi_A + R_A - |w_A|g^\wedge_A(W,S) - x^\wedge_A(W,S) - p^\wedge_A(W,S) - C_D$ for deposing B, and

$\Psi_A + R_A + r - |w_A|g^\sim_A - x^\sim_A - p^\sim_A - C_r - C_D$ for doing both. A will choose the course of action with the highest payoff. Pulling together all the best replies, we have the equilibrium.

Comparative Statics

Taking Territory

The value to A of taking territory from **B** depends on how much of the added resources will be retained by A,

$r - |w_A|(g_A^+ - g_A^*) - (x_A^+ - x_A^*) - (p_A^+ - p_A^*) = r - m_A^+ + m_A^*$. Taking territory has two effects, a direct effect of increasing R_A and an indirect effect of decreasing R_B. We analyze these separately because changing W_B, S_B mirros the indirect effect by changing p_B in equilibrium.

Begin with the direct effect. Using the same rule for finding the partial derivative of the implicit function $m_A(r) = |w_A|g_A + x_A + p_A$, we have the following:

$$D_{R_A} m_A = |w_A| D_{R_A} g_A + D_{R_A} x_A + D_{R_A} p_A =$$

$$\left(\frac{(1-\delta) D_R V_A}{\pi_A \chi_B} \right) \left(\frac{\pi_A \chi_B D_{gg}^2 V_A D_x V_A D_{ff}^2 V_A + W_A \pi_A \chi_B D_g V_A D_{xx}^2 V_A D_{ff}^2 V_A + D_{gg}^2 V_A D_{xx}^2 V_A D_f V_A}{(1-\delta) D_{gg}^2 V_A (D_x V_A)^2 D_{ff}^2 V_A + \left(1-\delta \frac{W_A}{S_A}\right) (D_g V_A)^2 D_{xx}^2 V_A D_{ff}^2 V_A + (1-\delta) D_{gg}^2 V_A D_{xx}^2 V_A (D_f V_A)^2} \right) > 0$$

where $\chi_B = \frac{(1-\delta)(D_{xx}^2 V_B)(D_f V_B)(D_f V_B(R))(D_{gg}^2 V_B)}{Denom_B}$, with $0 < \chi_B < 1$. To reduce clutter, we will also refer to

$$Denom_A = \left(1 - \delta \frac{W_A}{S_A}\right)(D^2_{xx}V_A)(D^2_{ff}V_A)(D_gV_A)^2 + (1-\delta)(D^2_{xx}V_A)(D_fV_A)^2(D^2_{gg}V_A)$$

$+(1-\delta)(D_xV_A)^2(D^2_{ff}V_A)(D^2_{gg}V_A)$. The attraction of increasing R_A depends on how fast m_A rises with it. We examine the mixed partials of $D_{R_A}m_A$:

$$D^2_{R_AW_A}m_A = \left(\frac{(1-\delta)D_RV_A}{\pi_A\chi_B}\right)\left[\frac{\pi_A\chi_BD_gV_AD^2_{xx}V_AD^2_{ff}V_A}{Denom_A} + \right.$$

$$\left.\frac{\delta}{S_A}(D_gV_A)^2D^2_{xx}V_AD^2_{ff}V_A\left(\frac{\pi_A\chi_BD^2_{gg}V_AD_xV_AD^2_{ff}V_A + W_A\pi_A\chi_BD_gV_AD^2_{xx}V_AD^2_{ff}V_A + D^2_{gg}V_AD^2_{xx}V_AD_fV_A}{(Denom_A)^2}\right)\right] > 0$$

as both terms in the parentheses are positive.

$$D^2_{R_AS_A}m_A = \left(\frac{(1-\delta)D_RV_A}{\pi_A\chi_B}\right)$$

$$\left[\frac{-\delta W_A}{(S_A)^2}(D_gV_A)^2D^2_{xx}V_AD^2_{ff}V_A\left(\frac{\pi_A\chi_BD^2_{gg}V_AD_xV_AD^2_{ff}V_A + W_A\pi_A\chi_BD_gV_AD^2_{xx}V_AD^2_{ff}V_A + D^2_{gg}V_AD^2_{xx}V_AD_fV_A}{(Denom_A)^2}\right)\right] < 0$$

Leaders gain less from territorial expansion as W_A increases and as S_A decreases. $\left|\frac{D^2_{R_AS_A}m_A}{D^2_{R_AW_A}m_A}\right| < \frac{W_A}{S_A}$, so the effect of W_A is larger than S_A as both increase in a fixed proportion.

$$D^2_{R_AR_A}m_A = \left(\frac{(1-\delta)D^2_{RR}V_A}{\pi_A\chi_B}\right)\left(\frac{\pi_A\chi_B D^2_{gg}V_A D_xV_A D^2_{ff}V_A + W_A\pi_A\chi_B D_gV_A D^2_{xx}V_A D^2_{ff}V_A + D^2_{gg}V_A D^2_{xx}V_A D_fV_A}{Denom_A}\right) < 0.$$

Turning to the indirect effect, changes in **B** alter m_A through χ_B, the effective price of *f* that *A* pays including *B*'s response in that price.

$$D_{\chi_B}m_A = |w_A|D_{\chi_B}g_A + D_{\chi_B}x_A + D_{\chi_B}p_A =$$

$$\left(\frac{D_fV_A}{\pi_A(\chi_B)^2}\right)\left(\frac{(1-\delta)D^2_{gg}V_A D_xV_A(\pi_A D_fV_A - D_xV_A) + D_gV_A D^2_{xx}V_A\left(W_A(1-\delta)\pi_A D_fV_A - \left(1-\delta\frac{W_A}{S_A}\right)D_gV_A\right)}{Denom_A}\right)$$

In equilibrium, $D_xV_A = \frac{1}{\lambda(1-\delta)}, \pi_A D_fV_A = \frac{1}{\lambda(1-\delta)\chi_B}, \left(1-\delta\frac{W_A}{S_A}\right)D_gV_A = \frac{W_A}{\lambda}$, so

$D_{\chi_B}m_A = \left(\frac{D_fV_A}{\pi_A(\chi_B)^3}\right)\left(\frac{(1-\delta)(\chi_B-1)D^2_{gg}V_A D_xV_A}{Denom_A}\right) > 0.$ By the Chain Rule, we have the following:

$$D_{W_B}m_A = D_{\chi_B}m_A(D_{W_B}\chi_B) = D_{\chi_B}m_A\left(\frac{\delta}{S_B}\left[\frac{\chi_B}{Denom_B}\right](D^2_{xx}V_B)(D^2_{ff}V_B)(D_gV_B)^2\right) > 0,$$

$$D_{S_B}m_A = D_{\chi_B}m_A(D_{S_B}\chi_B) = D_{\chi_B}m_A\left(\frac{-\delta W_B}{(S_B)^2}\left[\frac{\chi_B}{Denom_B}\right](D^2_{xx}V_B)(D^2_{ff}V_B)(D_gV_B)^2\right) < 0,$$

and

$$D_{R_B}m_A = D_{\chi_B}m_A(D_{R_B}\chi_B) = D_{\chi_B}m_A\left(\chi_B\left(\frac{D^2_{ff}V_B(R)}{D_fV_B(R)}\right)\right) > 0.$$

As before, we examine $D^2_{R_B W_B} m_A$ and $D^2_{R_B S_B} m_A$ to determine how W_B, S_B influence A's willingness to take territory:

$$D^2_{R_B W_B} m_A = -\left(\frac{(1-\delta) D^2_{gg} V_A D_x V_A}{Denom_A} \right) \left(\frac{D^2_{ff} V_B(R)}{D_f V_B(R)} \right) \left(\frac{D_f V_A}{\pi_A (\chi_B)^3} \right) (\chi_B + 1) D_{W_B} \chi_B > 0 \text{ as}$$

$$D_{W_B} \chi_B = \left(\frac{\chi_B}{Denom_B} \right) \left(\delta \left(\frac{1}{S_B} \right) \right) (D_g V_A)^2 D^2_{xx} V_A D^2_{ff} V_A > 0, \text{ and}$$

$$D^2_{R_B S_B} m_A = -\left(\frac{(1-\delta) D^2_{gg} V_A D_x V_A}{Denom_A} \right) \left(\frac{D^2_{ff} V_B(R)}{D_f V_B(R)} \right) \left(\frac{D_f V_A}{\pi_A (\chi_B)^3} \right) (\chi_B + 1) D_{S_B} \chi_B < 0 \text{ as}$$

$$D_{S_B} \chi_B = \left(\frac{\chi_B}{Denom_B} \right) \left(-\delta \left(\frac{W_B}{(S_B)^2} \right) \right) (D_g V_A)^2 D^2_{xx} V_A D^2_{ff} V_A < 0.$$

Note that $\left| \frac{D^2_{R_B W_B} m_A}{D^2_{R_B S_B} m_A} \right| = \frac{W_B}{S_B}$, so the effects of simultaneous changes in W_B and S_B offset one another if the ratio $\frac{W_B}{S_B}$ does not change.

To summarize, A is more willing to take territory as $D_{R_A} m_A$ decreases. This means that A is more willing to take territory as W_A decreases and S_A increases, with the effect of W_A larger than that of S_A if they change in a fixed ratio. A is more willing to take territory as W_B increases and S_B decreases; however, the effects of both cancel out if they change in a fixed ratio. Obviously, A is more likely to take territory as C_r

decreases.

Overthrowing *B*

A's willingness to overthrow *B* and change W_B and S_B depends on $D_{W_B} m_A$ and $D_{S_B} m_A$; *A* compares a reduction in m_A to the cost of deposing $B,\ C_D$. *A* reduces m_A by decreasing W_B and increasing S_B as $D_{W_B} m_A > 0$ and $D_{S_B} m_A < 0$. As can be seen from the partial calculated above, $\left| \frac{D_{S_B} m_A}{D_{W_B} m_A} \right| = \frac{W_B}{S_B}$, so *A* reduces m_A by decreasing the ratio $\frac{W_B}{S_B}$. In equilibrium then, *A* changes W_B and S_B to reduce $\frac{W_B}{S_B}$. We do not analyze how low *A* reduces the ratio of $\frac{W_B}{S_B}$ because we do not model the range of freedom *A* has to set these values.

The effect of W_A and S_A on *A*'s willingness to remove *B* is given by the following mixed partials:

$$D^2_{W_B W_A} m_A = D_{W_B} m_A \left(\frac{\delta}{S_A} \right) \left(\frac{(D^2_{xx} V_A)\left(D^2_{ff} V_A\right)(D_g V_A)^2}{Denom_A} \right) > 0$$

$$D^2_{W_B S_A} m_A = D_{W_B} m_A \left(\frac{-\delta W_A}{(S_A)^2} \right) \left(\frac{(D^2_{xx} V_A)\left(D^2_{ff} V_A\right)(D_g V_A)^2}{Denom_A} \right) < 0$$

$$D^2_{S_B W_A} m_A = D_{S_B} m_A \left(\frac{\delta}{S_A} \right) \left(\frac{(D^2_{xx} V_A)\left(D^2_{ff} V_A\right)(D_g V_A)^2}{Denom_A} \right) < 0$$

$$D^2_{S_B S_A} m_A = D_{S_B} m_A \left(\frac{-\delta W_A}{(S_A)^2} \right) \left(\frac{(D^2_{xx} V_A)\left(D^2_{ff} V_A\right)(D_g V_A)^2}{Denom_A} \right) > 0.$$

Again, we have $\left| \frac{D^2_{W_B S_A} m_A}{D^2_{W_B W_A} m_A} \right| = \frac{W_A}{S_A}$ and $\left| \frac{D^2_{S_B S_A} m_A}{D^2_{S_B W_A} m_A} \right| = \frac{W_A}{S_A}$, so the ratio $\frac{W_A}{S_A}$ determines *A*'s interest in deposing *B* and changing W_B and S_B.

Comparing Taking Territory and Deposing *B*

A is more willing to take territory and less willing to depose *B* compared to doing neither as W_A decreases and S_A increases from above. Clearly then, *A* is more willing to take territory over deposing *B* as W_A decreases and S_A increases. Increases in W_B and decreases in S_B make *A* more willing to take territory and to depose *B*, so their effect on *A*'s choice between these two is indeterminant.

Doing Both

If *A* takes territory and deposes *B*, it gains the benefits of both but pays both costs. One way to think about doing both is to separate it into the decisions of doing each separately; however, the benefits of one are lower if the other is also done. To see this, assume that *A* has taken territory, this means that $r + m^*_A - m^+_A > C_r$. *A* wants to depose *B* as well if $m^+_A - m^\sim_A > C_D$, and the claim that the benefits of deposing *B* are less after taking territory is that $m^+_A - m^\sim_A < m^*_A - m^\wedge_A$. Both sides of this inequality give the resources saved by A by deposing B, with the left hand side giving the benefit after A has taken resources r from **B** and the right hand side without territory being taken. The benefits depend on $D_{W_B}m_A$ and $D_{S_B}m_A$ and how these partial derivatives change with the shift in resources from **B** to **A**. $D_{W_B R_A}m_A = D_{S_B R_A}m_A = 0$, so the increases in **A**'s resources does not affect the benefits of deposing B. As shown above, $D^2_{R_B W_B}m_A > 0$ and $D^2_{R_B S_B}m_A < 0$. Taking r resources from **B** decreases R_B, so $0 < D_{W_B}m_A(m_A^+) < D_{W_B}m_A(m_A^*)$ and $0 > D_{S_B}m_A(m_A^+) > D_{S_B}m_A(m_A^*)$, and the benefits of deposing B are less when A also takes territory than when it does not.

The decision to take territory and depose B then requires A be willing to take territory and to depose B as separate decisions; $r + m_A^* - m_A^+ > C_r$ and $m_A^* - m_A^\wedge > C_D$ are both necessary but not sufficient for $r + m_A^* - m_A^\sim > C_r + C_D$. Because A is less likely to take territory but more likely to depose B as W_A increases and S_A decreases, doing both is unlikely for high and low values of W_A and S_A.