In situ determination of Zeeman content of collective atomic memories

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Abstract

Knowledge and control of atomic Zeeman populations is necessary for the realization of useful, long-lived quantum memories. We propose and implement a method to determine atomic state population distributions for atomic spin waves. Zeeman composition of single atomic spin waves of a cold atomic gas, confined in a one-dimensional optical lattice, is inferred with high precision by measurements of signal–idler polarization correlations as a function of spin-wave storage time.

(Some figures may appear in colour only in the online journal)

1. Introduction

The control of an atom’s external and internal degrees of freedom is a feature of modern research in ultracold atomic physics. For applications to quantum information processing, such as quantum networks based on the idea of the quantum repeater [1–12], magneto-optically trapped and cooled atoms can be used to store information in spin-wave excitations [3, 13–16]. The latter are characterized by a grating structure induced by laser Raman scattering, superimposed on the frozen atomic ensemble. These spin waves are distinguished by their internal spin structure, being defined by pairs of states in the hyperfine doublet of the atom’s electronic ground level. A spin wave based on the \( m = 0 \leftrightarrow m = 0 \) clock transition is only second-order sensitive to ambient magnetic fields and is thus attractive for the realization of long-lived quantum memory [17–22]. Alkali atoms prepared in magneto-optical traps are nominally created in an internal mixed quantum state in which population is equally distributed across the Zeeman states of one of the hyperfine ground levels. Larmor precession of magnetically sensitive Zeeman states leads to characteristic collapse and damped revivals of the optically retrieved signal [23, 24]; the damping is a result of inhomogeneous broadening of the Zeeman components due to variations of the ambient magnetic field [17]. An atomic ensemble prepared in an unpolarized magnetic distribution of one of the hyperfine ground levels exhibits this characteristic signal over tens of \( \mu s \) storage time [17].

Long storage is also challenged by thermal motion of cold atoms on ms timescales, as determined by the time for displacement across a spin-wave grating period. Optical lattice confinement augmented by compensation schemes for the lattice-induced differential ac Stark shift of the clock doublet has resulted in storage times in the 100 ms range [20–22]. In such experiments, optical pumping into the \( m = 0 \) hyperfine Zeeman state may be employed in an attempt to empty other Zeeman states and thereby reduce the excitation of magnetically sensitive spin waves that dephase in less than a ms. Another form of optical pumping, of particular interest here, may occur in protocols such as the DLCZ protocol [2], which is the standard method for creating single atomic spin-wave excitations in a quantum memory. The protocol involves Raman scattering of a sequence of weak off-resonant write laser pulses between hyperfine ground levels of an initially prepared atomic ensemble until a Raman signal photon is detected. Specifically, in the experiments to be reported here, a trial begins with a 50 ns write pulse of power \( \approx 1 \mu W \) incident on \(^{87}\text{Rb} \) atoms prepared in the \( \mid b \rangle = \mid 5S_{1/2}, F_B = 1 \rangle \) ground hyperfine level and detuned \(-20 \text{ MHz} \) from the \( \mid c \rangle = \mid 5P_{1/2}, F_c = 1 \rangle \) excited level. If a signal photon emitted on the \( \mid c \rangle \leftrightarrow \mid a \rangle \) (\( \mid a \rangle = \mid 5S_{1/2}, F_a = 2 \rangle \)) transition is not detected, the write pulse is followed 700 ns
after by a 200 ns clean pulse (power 270 μW) resonant with the |a⟩ ↔ |c⟩ transition. Both the write and clean pulses have a transverse Gaussian intensity profile with a 1/e² waist of 230 μm. After a quiescent interval, the next trial begins; the trial period is 1.5 μs. Figure 1 shows results for the populations of the m = −1, 0, 1 Zeeman states, |b, m⟩, of the initially unpolarized hyperfine level (b) over 2000 trials. These results were found by numerically integrating the optical Bloch equations for a single ⁸⁷Rb atom including all the relevant hyperfine structure. A few hundred trials are sufficient to noticeably alter the population distribution, whereas typically 10⁷–10⁸ trials are needed to generate high-quality, single (i.e. with double excitations strongly suppressed) spin-wave excitations in experiments. A conventional way to discuss the distribution of population in the Zeeman states (b, 1) (dashed line) and (b, −1) (dot-dashed line) as a function of the number of write/read trials, in the presence of a 1.0 G bias magnetic field.

The distribution of atomic populations across Zeeman states of hyperfine levels has been studied by means of polarization spectroscopy (see for example [25, 26]). In this paper, we show how the degree of orientation and alignment, or Zeeman content, of the atomic spin waves in a ⁸⁷Rb quantum memory can be ascertained in situ from suitable measurements. Specifically signal–idler polarization correlations are studied as a function of spin-wave storage time Tₚ. We demonstrate experimentally that the Zeeman content can be measured with high precision. As pointed out above, a spin wave based on the clock doublet dephases slowly in an inhomogeneous magnetic field. Magnetic Zeeman states have a much faster dephasing rate 1/τₚ fast. In ⁸⁷Rb, there is another class of hyperfine coherences associated with an anomalously small g-factor, which have a much slower decay rate 1/τₚ slow. Our technique exploits the regimes Tₚ ≪ τₚ fast ≪ τₚ slow. In the former microsecond storage regime, the atomic state depends sensitively on the instant the spin wave is written. In the longer storage time limit, we take advantage of the long coherence time of the magnetically insensitive spin waves and the relatively benign effects of atomic motion over a sub-millisecond timescale in a cold atomic rubidium gas. While here we apply this technique to an ensemble of ⁸⁷Rb atoms, we point out in the appendix that this method is applicable to any ensemble of three-level atoms whose ground hyperfine levels |b⟩ and |a⟩ have total respective angular momenta, Fₐ and Fₐ, that satisfy Fₐ = Fₐ + 1, and whose excited level |c⟩ has a total angular momentum Fₐ = Fₐ.

The remainder of this paper is organized as follows. Storage and retrieval of spin waves are described in section 2. Theoretical analysis for the situation of pure atomic alignment is presented in section 3. Experimental methods are described in section 4; the observed data are presented in section 5. The details of the full theoretical model are given in the appendix.

2. Storage and retrieval of spin waves

In this section, we summarize the basic physical principles involved in the storage and retrieval process in ⁸⁷Rb. As we are interested in extracting information on the atomic population distribution produced by the protocol that generates single atomic spin waves, the role of hyperfine structure, laser polarization and external magnetic field are important factors. A detailed theoretical treatment of the system, including the latter features, is presented in the appendix: this is written in a rather general way so as to be applicable to other alkali atoms. In this section, we summarize some of the salient features useful for understanding of polarization correlations of the detected light in the ⁸⁷Rb system that are discussed in the following section.

2.1. Writing a spin wave

The Raman scattering of the write laser results in a joint density operator for the electromagnetic field-atomic ensemble system of the form

\[ \hat{\rho} \approx \left(1 - i\chi \sum_{\lambda} \hat{a}_{\lambda}^{\dagger} \hat{A}_{\lambda}^{\dagger}\right) \hat{\rho}_0 \left(1 + i\chi \sum_{\lambda} \hat{a}_{\lambda} \hat{A}_{\lambda}\right), \tag{1} \]

where \( \chi \ll 1 \) is an effective Raman coupling constant, and \( \hat{a}_{\lambda} \) and \( \hat{a}_{\lambda}^{\dagger} \) are the annihilation and creation operators for the detected signal field mode of polarization \( \lambda \). The spin-wave operators associated with the write process \( \hat{A}_{\lambda}, \hat{A}_{\lambda}^{\dagger} \) are quasi-bosonic [\( \hat{A}_{\lambda}, \hat{A}_{\lambda}^{\dagger} \] = 1 + O(1/√N)]. The operators are linear combinations of the elementary spin-wave operators \( \delta_{m_b,m_a} \) (c, t) \( \{A,29a\} \) and depend on the polarization of the write field and the initial populations of the Zeeman states |b, m_b⟩, m_b = −1, 0, 1, where \( m_a \) is a magnetic quantum number of the |a, m_a⟩ state.

2.2. Spin-wave storage

The spin wave evolves in time under the influence of a magnetic field. During storage, the atomic hyperfine coherences undergo Larmor precession in the magnetic field
with the $z$-component $B_0 + B'z$. The homogeneous field $B_0$ induces Larmor precession with frequencies given by $\omega_{0,m} = (\mu B_0/h) [g_0 (m_a + m_b) - \delta g_m]$ where $g_0$ and $g_0$ are the Landé $g$-factors for levels $|a\rangle$ and $|b\rangle$, respectively, and $\delta g \equiv g_e - g_0$; numerically for $^{87}$Rb, $g_0 = -0.5018$ and $\delta g = -0.002$, $|\omega_{0,\pm 2}| \approx |\omega_{\pm 1,\pm 1}| \approx 500 |\omega_{\pm 1,\pm 1}|$. The Larmor frequencies $|\omega_{0,\pm 2}|$ and $|\omega_{\pm 1,\pm 1}|$ correspond to fast elementary spin waves $\hat{s}_{m_a, m_b}(z, t)$ (with $(m_a, m_b) = (\pm 1, \pm 1)$ or $(0, \pm 2)$) and $|\omega_{\pm 1,\pm 1}|$ correspond to slow elementary spin waves $\hat{s}_{\pm 1,\pm 1}(z, t)$.

Magnetic field inhomogeneities cause Zeeman shifts that vary in space across the atomic cloud causing the Heisenberg operator $\hat{s}_{m_a, m_b}(z, t)$ to pick up a spatially varying phase factor $e^{-i\omega_{m_a, m_b} t B'z}$ resulting in dephasing of the stored excitations. After storage for time $T_s$ in an ensemble with a Gaussian density profile $f(z)$, the elementary distributed spin wave $\hat{s}_{m_a, m_b}(z, t)$ suffers a Gaussian dephasing factor $e^{-\frac{1}{2} \left( \frac{z - z_s}{\delta z_s} \right)^2}$, where $\delta n_{m_a} = \langle m_a | -B' / B_0 \rangle$ and $l$ is the sample length. The fast and slow elementary spin waves dephase in proportion to their precession rates. For short enough storage time, both fast and slow spin waves may be observed, but for longer storage periods, only the slow spin waves remain.

2.3. Spin-wave retrieval.

After a storage time $T_s$, a linearly polarized read field, counterpropagating with respect to the write laser field, is applied near resonance with the $|a\rangle \leftrightarrow |c\rangle$ transition. The read laser converts the stored atomic excitation into an idler field by Raman emission on the $|b\rangle \leftrightarrow |c\rangle$ transition. The idler field wavevector $k_c$ is determined by the four-wave-matching condition $k_c = -k_a + k_w - k_b = 0$; hence, the idlers are counterpropagating to the signal field in our geometry. Idler and signal photon pairs are strongly correlated in this geometry as they result from a cyclic two-photon emission process in which each atom starts and ends in the same Zeeman state. When the read laser is polarized in the $z$-direction, parallel to the magnetic field, the influence of dephasing on the retrieval process is straightforward to analyse. In the Appendix, we discuss the more general case in which the interaction Hamiltonian is expressed in terms of eigenstates of angular momentum along the laser polarization direction. In both cases, the retrieval process can be described in terms of dark-state polaritons.

3. Analysis of pure atomic alignment $p_1 = p_{-1}$

We discuss a limiting case that does not require the general theory of the Appendix, in which the atoms are prepared in an aligned state (zero orientation $\hat{O} = 0$, corresponding to $p_1 = p_{-1}$).

To compute the dynamics of the signal–idler correlations, we need to follow the fast and slow rotation of the various Zeeman spin waves over the storage period. Some bookkeeping is required to keep track of various sets of bosonic spin waves naturally associated with the write and read processes with linearly polarized laser fields. Each of these subsidiary spin waves may be written in terms of the elementary fast and/or slow Zeeman spin waves. The subsidiary spin-wave dynamics clarifies characteristic features of the signal–idler photon correlations. Features due to interference of fast and slow spin waves and their experimental observation will be discussed later.

Recall that the elementary clock $\hat{s}_{0,0}$ and Zeeman coherences $\hat{s}_{\pm 1,\pm 1}$ are slowly varying in time, while $\hat{s}_{\pm 1,\pm 1}$ and $\hat{s}_{0,\pm 2}$ are rapidly varying. We identify ‘linearly polarized’ slow and fast spin-wave annihilation operators

$$s_{1, H}^{\text{slow}} = \hat{s}_{1, -1, \pm 1}^{\text{slow}} = \frac{1}{\sqrt{2}} (\hat{s}_{1, -1} + \hat{s}_{-1, 1})$$

$$s_{1, H}^{\text{fast}} = \hat{s}_{1, -1, \pm 1}^{\text{fast}} = \frac{1}{\sqrt{2}} (\hat{s}_{1, -1} + \hat{s}_{-1, 1})$$

$$s_{0, H}^{\text{slow}} = \hat{s}_{0, 0, \pm 2}^{\text{slow}} = \frac{1}{\sqrt{2}} (\hat{s}_{0, 0, 2} + \hat{s}_{0, -2})$$

$$s_{0, H}^{\text{fast}} = \hat{s}_{0, 0, \pm 2}^{\text{fast}} = \frac{1}{\sqrt{2}} (\hat{s}_{0, 0, 2} - \hat{s}_{0, -2}).$$

These evolve in time as a fixed axis rotation. For example, the slow spin waves $s_{1, H}^{\text{slow}}$ and $s_{1, V}^{\text{slow}}$ evolve in a constant magnetic field on the slow frequency scale as

$$s_{1, V}^{\text{slow}}(z, t) = e^{-i\omega_{0,1,0} t} (s_{1, V}^{\text{slow}}(z, 0), s_{0,1,0}^{\text{slow}}(z, 0)).$$

A photoelectric detection event of a signal photon with polarization $\lambda$ ($\lambda = H$ or $V$) imprints a single-mode collective write-spin-wave excitation $\hat{A}_1$ onto the ensemble. The atomic density matrix conditioned on this detection is $\hat{A}_1^{\dagger} \hat{A}_1$, and the idler field is written inhomogeneously over the length of the sample and may be expressed in terms of local spin-wave operators $\hat{u}_{0, \pm 1}^{\text{write}}(z)$, arising from the initial atomic states $|b, 0\rangle$, with probability $p_1$ and $|b, \pm 1\rangle$, with probability $p_{-1}$, respectively ($p_1 + 2p_{-1} = 1$). By tracing over undetected field modes [27], we find

$$\hat{A}_1(z) = \int dz' \sqrt{f(z')} (\mu_1 \hat{u}_{0, \pm 1}(z) + \sqrt{1 - \mu_2^2} \hat{u}_{1, \pm 1}(z)).$$

where $f(z) = \tilde{n}(z)/\int dz' \tilde{n}(z')$, $\mu_H = \sqrt{2p_0 / (1 + p_0)}$ and $\mu_V = \sqrt{2p_0 / (3 + 2p_0)}$. The local spin-wave operators $\hat{u}_{m, \pm 1}(z)$ ($w$ for write) are quasi-bosonic with commutation relations $[\hat{u}_{m, \pm 1}(z), \hat{u}_{m', \pm 1}(z')] = \delta_{m, m'} \delta(z - z') + O(1/\sqrt{N})$, and may be written explicitly, in terms of the subsidiary fast and slow spin waves,

$$\hat{u}_{1, H} = \frac{1}{\sqrt{2}} (s_{1, H}^{\text{slow}} - s_{1, H}^{\text{fast}}).$$
\[
\dot{\hat{w}}_{1,V} = -\frac{1}{\sqrt{2}} \left( \hat{\sigma}_{1,H}^{\text{low}} + \hat{\sigma}_{1,H}^{\text{fast}} \right),
\]
(11)
\[
\dot{\hat{w}}_{0,H} = \hat{\sigma}_{0,\text{V}}^{\text{fast}} \text{ and}
\]
(12)
\[
\dot{\hat{w}}_{0,V} = \sqrt{\frac{2}{3}} \hat{\sigma}_{0,\text{H}}^{\text{fast}} - \frac{i}{3} \hat{\sigma}_{0,0}. \quad (13)
\]
The detection of an H/V signal photon produces spin waves that will be read out as V/H idlers. Note that \( \hat{w}_{0,V} \) has a slowly varying contribution from the clock transition \( \hat{\sigma}_{0,0} \), in addition to fast spin waves associated with \( \hat{\sigma}_{0,\pm 1} \), while \( \hat{w}_{0,H} \) has only the latter.

The retrieval dynamics are governed by the Heisenberg–Langevin equations for the orthogonally polarized idler fields, \( \hat{\phi}_h(z, t) \) and \( \hat{\phi}_v(z, t) \), which obey the equal time commutation relations \( \{ \hat{\phi}_h(z, t), \hat{\phi}_v^\dagger(z', t) \} = \delta_{z, z'} \delta(t - t') \), and those for the atomic excitations. For our particular atomic configuration, we take advantage of symmetry to arrive at a set of just three equations (see appendix A.1), which identify the hyperfine read \( \hat{r}_{m,\lambda} \) and optical \( \hat{\theta}_{m,\lambda} \) spin waves that couple to the retrieved field,
\[
(\hat{\partial}_t + c\hat{\partial}_z) \hat{\phi}_h = i\kappa \sqrt{\frac{\mu_0}{\hbar c}} \left( \sqrt{\mu_0 C_{0,\lambda} \hat{\theta}_{0,\lambda}} + \sqrt{\mu_0 C_{1,\lambda} \hat{\theta}_{1,\lambda}} \right),
\]
(14a)
\[
(\hat{\partial}_t + \Gamma / 2) \hat{\theta}_{m,\lambda} = i\kappa \sqrt{\frac{\mu_0}{\hbar c}} \left( \hat{\omega}_{1,\lambda} \hat{\theta}_{0,\lambda} + i\Omega C_{m,\lambda} \hat{r}_{m,\lambda} + \hat{\theta}_{m,\lambda} \right),
\]
(14b)
\[
\hat{r}_{m,\lambda} = i\Omega^* C_{m,\lambda} \hat{\theta}_{m,\lambda}. \quad (14c)
\]

Here \( \Gamma \) is the spontaneous decay rate of level \( |c\rangle \), and \( C_{0,\lambda} = 1 - 1/\sqrt{2} \) and \( C_{1,\lambda} = 1 - (1)^{\nu} \mu / \sqrt{2} \) are relative coupling strengths of the idler field to optical coherences \( \hat{\theta}_{m,\lambda} \), \( m = 0, 1 \). Similarly, \( C_{0,\lambda} = \sqrt{(3 + \delta_{\lambda, H})/10} \) and \( C_{1,\lambda} = 3/10 \) determine the relative coupling of the optical coherences and read spin waves; \( \Omega \) is the read field Rabi frequency, \( \kappa \) is the single-photon Rabi frequency on the idler transition and \( \hat{\sigma}_{0,\lambda} \) is a noise operator. The collective optical coherence operators are given by \( \hat{\epsilon}_{1,\lambda} = -i (\hat{\theta}_{1,0} + (1)^{\nu} \hat{\theta}_{1,0}) / \sqrt{2} \), and
\[
\hat{\theta}_{m,\lambda}(z) = \sum_{\mu = 1}^N \frac{n^2}{m} r_{(\mu)} e^{-i k_{\mu} t} r_{\mu} z \mu |b, m\rangle_{\mu} |c, m\rangle,
\]
and \( k_{\mu} \) is the idler wavevector. The read spin-wave operators are expressed in terms of the same fast and slow coherences imprinted in the write process, \( \hat{r}_{1,\lambda} = \left( \hat{\sigma}_{1,\lambda}^{\text{fast}} + \hat{\sigma}_{1,\lambda}^{\text{slow}} \right) / \sqrt{2} \) and \( \hat{r}_{0,\lambda} = \hat{\sigma}_{0,0} \).

We solve equations (14) adiabatically and find dark-state polariton solutions with the annihilation operator
\[
\hat{\Psi}_k \propto \Omega^* \hat{\phi}_h - \kappa^* \sqrt{n} \sum_{m = 1}^N \frac{\sqrt{m}}{\mu} C_{m,\lambda} \hat{r}_{m,\lambda} / C_{m,\lambda}.
\]
(15)
The atomic density matrix, conditioned on the detection of a signal with polarization \( \lambda_s \), is \( \hat{A}_{\lambda,\lambda}^\dagger \hat{A}_{\lambda,\lambda} \), leading to order. The spatial distribution of the spin waves imprinted in the write process determines the bosonic idler mode annihilation operator \( \hat{T}_{\lambda}(T_s) = \int dz \sqrt{\hat{\nu}(\hat{\chi}(z, T_s) / \hat{\Omega} = 0} \). The retrieval efficiency of an idler of polarization \( \lambda \) given the detection of a signal of polarization \( \lambda_s \) is \( \langle \hat{T}_{\lambda}^\dagger / \hat{A}_{\lambda,\lambda} \rangle \). Using the conditioned atomic density operator, this reduces to [17]
\[
\eta_{\lambda,\lambda}(T_s) = \langle \hat{T}_{\lambda}^\dagger(1, \hat{A}_{\lambda,\lambda}) \rangle. \quad (16)
\]

Figure 2. Essential elements of the experimental setup. $^{87}$Rb atoms are loaded in a 1D optical lattice of 6 μm period, formed by interfering two 1064 nm beams with an angular separation 10°; (1/e²) beam waists are 90 and 120 μm, respectively, giving a high trap depth of 150 μK. The write and read fields share a single spatial mode of waist 230 μm, while signal and idler mode waist is 110 μm. The write/read and signal/idler modes intersect at the position of the atomic sample at an angle \( \theta_w = 3^\circ \). PBS is a polarizing beam splitter, D1–D4 are detectors. The inset shows the atomic level scheme and write/read protocol, see the text.

### 4. Experimental methods

A sample of \( N \sim 10^7 \) $^{87}$Rb atoms is prepared in the lowest energy hyperfine ground level \( |b\rangle \) with angular momentum \( F_b = 1 \) (inset of figure 2). We consider the Raman configuration with ground levels \( |b\rangle \) and \( |a\rangle \) (\( F_a = 2 \)) and excited level \( |c\rangle \) with energies \( \hbar\omega_{0b}, \hbar\omega_{0a}, \text{and} \hbar\omega_{0c} \), respectively. The level \( |c\rangle \) is the $5S_{1/2}$ hyperfine level with angular momentum \( F_c = 1 \). A linearly polarized write laser pulse with wavevector \( k_w \) and nearly resonant on the \( D_1 \) (\( |b\rangle \leftrightarrow |c\rangle \) transition is Raman scattered to produce a signal field on the \( |a\rangle \leftrightarrow |c\rangle \) transition. The write pulse polarized in \( x-z \) plane travels at a small angle with respect to the negative \( z \)-direction. A magnetic field produced by a pair of coils in a Helmholtz configuration is applied in the \( z \)-direction. A single transverse signal mode \( u \) (\( r \), centred about the wavevector \( k_r = -k_z \)) is collected with horizontal (H) and vertical (V) components in the directions \( -\hat{x} \) and \( \hat{y} \), respectively. Detection of a signal photodetector event heralds the creation of an atomic spin-wave excitation, which is stored for a duration \( T_s \). A vertically (\( y \)) polarized read laser pulse counter-propagating to the write pulse retrieves the excitation by generating an idler field which is detected in the spatial mode \( u \) (\( r \)) in the positive \( z \)-direction.

Atoms are collected and cooled in a magneto-optical trap for a period of 0.4 s. The trap laser is then detuned by up to 95 MHz below atomic resonance, the quadrupole magnetic field is turned off and the repump laser intensity is lowered for 40 ms, in order to optimize sub-Doppler cooling and loading of the optical lattice. As a result, the lattice contains about \( 10^3 \) atoms in the $5S_{1/2}, F = 1$ hyperfine level (level \(|b\rangle \)), with magnetic field \( B_0 = 1.1 \) G applied along the
conditioned retrieval efficiencies, \( η \) and populations of the Zeeman state \( \omega \).

In this section, we show how the dependence of retrieval efficiencies, we optically pump with an additional laser field tuned to the \( |b\rangle \leftrightarrow |c\rangle \) transition, propagating in the \( x\)-direction. The temperature of the cloud was measured to be 20 \( \mu \text{K} \). The lattice parameters, see caption of figure 2, result in the oscillation frequencies \( 1.4 \times 10^4 \), 400 and 30 Hz. The lattice pancakes coincide with the constant phase planes of atomic spin waves, thereby reducing motional decoherence. The write/read process is phase-matched according to the condition \( k_x + k_y = k_x + k_y \) \( (k_x = -k_x, k_y) \).

The spin waves are written by a 72 ms long sequence utilizing a measurement-based feedback protocol [28]. This begins with a 50 ns long write pulse of power \( \simeq 1 \mu\text{W} \), detuned by \(-20 \text{MHz}\) from the free-space \( |b\rangle \leftrightarrow |c\rangle \) transition frequency. If a signal photoelectric event is not detected, the write pulse is followed by a strong clean pulse to transfer population back from \( |a\rangle \) to \( |b\rangle \); the sequence is repeated at a rate of 0.66 MHz. The clean pulse has duration 200 ns, power 270 \( \mu\text{W} \) and is linearly polarized orthogonal to the write field. Detection of a signal event by \( D_1 \) or \( D_2 \) heralds excitation of the desired atomic spin wave, and halts the write/clean sequence. Under our experimental conditions, the fast coherence decay time \( \tau_{fast} \sim 100 \mu\text{s} \), consistent with milli-Gauss variations in the magnetic field [17]. Numerical calculations of the write/clean sequence suggest that \( p_0 \) falls from its initial value of 1/3 to a steady state value of 0.15 within the first few milliseconds of the 72 ms protocol duration (figure 1). The alignment symmetry \( p_m = p_{-m} \) is broken by the presence of the bias magnetic field, at the level of about 10% under our experimental conditions.

5. Results

5.1. Retrieval efficiencies for aligned atoms

In this section, we show how the dependence of retrieval efficiencies on storage times can be exploited to infer the populations of the Zeeman state \( |b, 0\rangle \) given that the atomic populations are balanced, i.e. \( p_1 = p_{-1} \). Figure 3 illustrates the sensitivity of the read-out efficiency \( η_{WH} \) on the population \( p_0 \) for storage times \( T_s \) much less than the fast coherence decay time \( \tau_{fast} \). When an \( H \)-polarized signal photon is detected, the spin waves \( \hat{w}_{0,H} \) and \( \hat{w}_{1,H} \) are imprinted on the ensemble. At \( T_s = 0 \) and \( ω_{0,H}T_s = 2π \), only atoms originating in the \( m = 0 \) state contribute to \( η_{WH} \). This can be traced to destructive interference between the \( V \)-polarized fast and slow spin waves originating from the \( m = \pm 1 \) states, \( [\hat{r}_{1,V}(\cdot), \hat{w}_{1,H}^{\dagger}]_{T_s=0} = 0 \).

At intermediate times, the fast spin-wave rotation modulates the interference, and the read-out contributions of \( ω_{1,H} \) and \( ω_{0,H} \) are determined by the populations of the \( m = \pm 1 \) and \( m = 0 \) states. The sensitivity of this and the other three conditioned retrieval efficiencies, \( η_{HL,H}, η_{LV,H} \) and \( η_{LV,V} \), allows us to accurately infer the population \( p_0 \) from experimental observations.

For \( T_s \ll \tau_{fast} \), we evaluate equation (16) with \( ω_{0,2} = ω_{1,1} \) and \( ω_{1,-1} \to 0 \) to give

\[
η = \begin{bmatrix}
\frac{(1+2p_0)^2 \sin^2(ω_{0,2}T_s)}{2(p_0+1)(p_0+1)} & \frac{3(1+2p_0) \cos(ω_{0,2}T_s)}{25p_0^2+31(p_0+1)} \\
\frac{(1-p_0-(1-5p_0) \cos(ω_{0,2}T_s)}{4p_0+1)^2} & \frac{3(1-5p_0)^2 \sin^2(ω_{0,2}T_s)}{45p_0^2+31(p_0+1)}
\end{bmatrix},
\]

where \( η \) is the matrix with elements \( η_{\lambda \lambda}, \lambda, \lambda' \in \{H, V\} \).

Figure 3. The sensitivity of retrieval efficiencies to the population \( p_0 \) in an ensemble without orientation \( (p_1 = p_{-1}) \). The efficiency \( η_{WH}(T_s) \) with which an \( H \)-polarized idler photon is retrieved from the ensemble given the detection of a \( V \)-polarized signal for storage times \( T_s \) much less than the fast spin-wave decoherence time. This conditioned efficiency is calculated over one oscillation period of the fast spin waves \( (0 \leq ω_{0,2}T_s/(2π) \leq 1) \) for the range of populations \( 0 \leq p_0 \leq 1 \).

Figure 4. The measured retrieval efficiencies \( η_{H,V}(λ, λ' = H, V) \) for \( T_s \ll \tau_{fast} \). Left column: without optical pumping; right column: with optical pumping. The circles, diamonds, squares and triangles are for \( H, H', HH \) and \( VH \) idler–signal combinations, respectively. The solid curves are theoretical, fitted according to the text.

The short-time retrieval efficiencies are shown in figure 4, left column. By fitting the matrix \( η \) to the four measured efficiencies with a single set of three adjustable parameters (an overall amplitude, \( ω_{0,2} \) and \( p_0 \)), we obtain \( p_0 = 0.183 \pm 0.003 \). The quoted uncertainty is determined by the analysis of distributions of \( p_0 \) generated from data sets that are Poisson-distributed around the measured means. To account for non-equal detection efficiencies, we rotate signal and idler polarizations by 90° every 20 min. The total acquisition time is 160 min. A full scan over \( T_s \) (2.1–3.7 ms) is performed within 48 s to avoid slow systematic drifts.

To confirm the predicted influence of \( p_0 \) on the retrieval efficiencies, we optically pump with an additional laser field tuned to the \( |b\rangle \leftrightarrow |c\rangle \) transition, propagating in the \( x \)-direction.
and linearly polarized in the z-direction. During the optical pumping, atoms accumulate in the $|b, m = 0\rangle$ state, whereas the write sequence reverses this tendency. The optical pumping period, during which a cleaning field empties level $a$, is 30 $\mu$s, followed by 80 write/clean cycles before repeating the sequence. The measured efficiencies are shown in the bottom row of figure 4. Using the procedure described above, we extract the value $p_0 = 0.47 \pm 0.01$. By varying the optical pumping, we have obtained $p_0$ as high as 0.75.

We have not employed a completely independent method to verify the correctness of the inferred Zeeman distributions. However, as an additional check we look at the dynamics of the slow hyperfine coherences.

For $T_{\text{fast}} \ll T_r \ll T_{\text{slow}}$, the retrieval efficiencies can be written in the general form $\eta_{\lambda';\lambda}(T_r) = |p_0^2 \delta_{\lambda;\lambda'} + p_1 \chi_{\lambda';\lambda}(T_r)|^2$, where $\chi_r$ is a constant contribution from the clock spin wave and $\chi_{\lambda';\lambda}(T_r)$ is the amplitude of the slow spin waves $\chi_{\lambda';\lambda}(T_r)$. We note that the clock spin wave contributes only to $\eta_{HV}$. The probability amplitudes for Raman excitation of the clock transition interfere constructively and destructively for $V$- and $H$-polarized signal emission, respectively. Explicitly, the retrieval efficiency matrix is

$$\eta = \frac{(1 - p_0^2 \sin^2(\omega_0 T_r))^2}{2(p_0 + 1)(p_0 + 1)} \frac{3(p_0 + (1 - p_0) \cos(\omega_0 T_r))^2}{2(3p_0 + 1)(3p_0 + 1)} .$$

We note that in a limit of $p_0 \approx 0$ for storage times $T_{\text{fast}} \ll T_r \ll T_{\text{slow}}$, only $\eta_{VV} \chi_{VV}$ and $\eta_{HV} \chi_{HV}$ contribute to the idler retrieval. For intermediate storage times, this pair of excitations can serve as a qubit where the Larmor spin-wave evolution approximates qubit rotation about a fixed axis.

In figure 5, we show efficiencies measured in the millisecond storage time regime, in the absence of optical pumping. By fitting the data with the above theoretical expressions, we extract $p_0 = 0.16 \pm 0.02$, in agreement with the value inferred from the dynamics of the fast coherences. This value is sufficiently small that $\eta_{HV}$ and $\eta_{HH}$ show sinusoidal rotation, whereas $\eta_{VV}$ and $\eta_{HV}$ are additionally modulated, as expected. The relative efficiencies at short and long times, shown in figures 4 and 5, are consistent with our theory, when adjusted at long times by an additional factor of $3 / 5$. This factor is in agreement with our measurements of the effects of atomic motion on the clock spin-wave dynamics, cf. [17].

Although the calculations leading to figure 1 indicate that repeated write/read trials produce a slight orientation (on the order of 10%) of the atomic sample, here we have presented in situ measurements of $p_0$ assuming that the populations are perfectly balanced ($p_1 = p_{-1}$) in order to illustrate the theory presented in section 3. From the general theory presented in the appendix, we can compute the retrieval efficiencies presented in this section for arbitrary orientation. However, for the combination of field polarizations considered in this subsection, a small orientation does not result in a statistically significant effect. A non-zero orientation would best be observed by measuring conditioned retrieval efficiencies in a basis where the signal and idler are not linearly polarized. We discuss such a scenario in the following subsection.

5.2. Photon correlations for oriented and aligned atoms

The presence of atomic orientation slightly complicates the picture. Whereas the spin waves $\hat{S}_{m_0 m_0}$ and $\hat{S}_{-m_0 -m_0}$ contribute equally to the written excitations and dark-state polaritons when the populations $p_1$ and $p_{-1}$ are balanced, an imbalance no longer permits one to exploit the treatment in section 3. In that section, the symmetry allowed us to identify specific optical and hyperfine coherences that coupled to either horizontally or vertically polarized light. For each polarization, there existed a corresponding dark-state polariton which the read process retrieves. In general, the two polaritons propagate at different group velocities and experience walk-off. Although we neglected walk-off in our calculation of retrieval efficiencies, identification of the dark-state polariton polarizations was essential in determining the specific linear combinations of hyperfine coherences of which they are comprised.

When the populations are unbalanced, the system still supports two distinct dark-state polaritons that correspond to idlers of orthogonal polarizations that experience walk-off. However, these polarizations are no longer linear. One can readily see that this is the case in the extreme limit where all of the atoms are initially in the state $|b, 1\rangle$ ($p_1 = 1, p_{-1} = 0$). Here, because $F_r = F_0$, right circularly polarized light would propagate through the sample at speed $c$ and not interact with the atoms, while the left circularly polarized idlers experience a diminished group velocity and can be stored in the ensemble. In that special case, detection of a signal photon can only prepare an idler polariton with left circular polarization. These polaritons, however, still contain fast and slow coherences and the retrieval efficiency would therefore oscillate with storage time, as they did when the populations were balanced. For arbitrary orientation and alignment, the idler polarizations corresponding to the dark-state polaritons supported by the ensemble must be calculated using the general formalism presented in the appendix. Detection of an idler along a certain axis then represents a detection of a superposition of the two dark-state polaritons.

When the imbalance in the population is large, the effects of the atomic orientation better manifest themselves when the signal and idler polarizations are observed in the circular basis $|\xi_{+}, \xi_{-}\rangle \in \{\xi_1, \xi_{-1}\}$ as shown in figure 6. To create the atomic orientation, we optically pump the ensemble with circularly polarized fields prior to the first
write trial. Circular polarizations are observed by inserting quarter-wave plates into the signal and idler paths (see figure 2). Figure 6 shows measurement probabilities of circularly polarized signal conditioned on the detection of a signal after a storage time $T_s$. The data are fit to equation (A.42) using the population $p_0$ and orientation $(p_1 - p_{-1})/2$ as adjustable parameters. Each of the two dark-state polaritons supported at different group velocities, which, in principle, results in differing temporal envelopes $\phi_j(t)$ ($j = 1, 2$) for photons retrieved from either of these modes. For simplicity, in our calculations, we assumed that these envelopes are identical, and we have neglected group velocity walk-off. This approximation contributes to imperfections of the fits of our model to experimental data. The fits of these to the data are plotted in figure 6, resulting in inferred values of the Zeeman level populations. For the data in the left column, we performed optical pumping to more heavily populate the state $|b, 1\rangle$, while for the right column, the state $|b, -1\rangle$ was more heavily populated.

6. Conclusion

In conclusion, we have presented a method to determine Zeeman populations of an atomic sample used as a quantum memory. Repeated attempts to produce a collective excitation within the quantum memory alter the populations of the Zeeman states. By examining the retrieval efficiencies of an idler conditioned on the detection of a signal for various field polarizations and storage times, one can measure the resulting Zeeman populations in situ. The theory is in agreement with measurements on a cold atomic sample in a 1D optical lattice.

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Appendix. Theoretical model

We consider an optically thick ensemble of $N \gg 1$ alkali atoms with ground state hyperfine levels $|a\rangle$ and $|b\rangle$. The atoms are prepared in the hyperfine ground level $|b\rangle$ with the lowest total angular momentum $F_b$. The initial single-atom density matrix is a mixed state in which an atom finds itself in the Zeeman state $|b, m\rangle$ with probability $p_m$. In alkali atoms, the second ground level $|a\rangle$ has the total angular momentum $F_a = F_b + 1$.

To generate and subsequently retrieve collective spin-wave excitations within the ensemble, we consider a $\Lambda$ configuration where all field frequencies (with the exception of those used for optical trapping) interact primarily with an excited level $|c\rangle$ with total angular momentum $F_c = F_b$. The atoms, labelled by an index $\mu = 1, \ldots, N$, are distributed in the ensemble (independent identically distributed) random positions $\mathbf{r}_\mu$ such that the mean number density is $n(\mathbf{r})$.

We apply a magnetic bias field with magnitude $B$ along the $z$-axis, which shifts the energies of the Zeeman states $|j, m\rangle$ within a hyperfine level $|j\rangle$ by $g_j/\mu_B B$ where $g_j$ is the anomalous Landé $g$ factor and $\mu_B$ is the Bohr magneton. However, imperfections in the field coils as well as residual ambient magnetic fields could result in inhomogeneities in the bias field. These inhomogeneities, as we discuss later in this appendix, introduce spatial variations in the Zeeman energy shifts, and consequently could result in dephasing of any spin waves during the storage process.

To generate a correlated state between a scattered signal photon and a stored collective atomic excitation, one shines write field propagating in the direction $\hat{k}_w$ nearly resonant on the $|b, 1\rangle \leftrightarrow |c\rangle$ transition with detuning $\Delta$; the write field frequency is $\Omega_w = (\omega_c - \omega_b) + \Delta$. For simplicity, we assume that the write field is uniform over the ensemble such that we may take its electric field at position $\mathbf{r}$ to be $\hat{E}_w(\mathbf{r}, t) = E_w^{(+) - (\mathbf{r}, t)} + E_w^{(-) - (\mathbf{r}, t)}$, where the positive frequency component $E_w^{(+) - (\mathbf{r}, t)}(\mathbf{r}, t) \equiv \hat{E}_w^{+}(\mathbf{r}, t) \exp(i(\mathbf{k}_w \cdot \mathbf{r} - \omega_c t ))$ and $E_w^{(-) - (\mathbf{r}, t)} \equiv \hat{E}_w^{(-)}(\mathbf{r}, t)$. We collect a signal field with wavenumber $\mathbf{k}_s$ and frequency $\omega_s = \omega_c - \omega_b + \Delta$ nearly resonant on the $|a\rangle \leftrightarrow |c\rangle$ transition with transverse profile $u(\mathbf{r}_\perp)$, where we have neglected diffraction so that $\mathbf{u}$ only depends on $\mathbf{r}_\perp \equiv \mathbf{r} - \mathbf{k}_w \mathbf{k}_w \cdot \mathbf{r}$. In the paraxial and slowly varying envelope approximations, the quantized electric signal field is given by $\hat{E}_s^{(+) - (\mathbf{r}, t)} + \hat{E}_s^{(-) - (\mathbf{r}, t)}$ where the positive frequency component is given in the interaction picture by

$$\hat{E}_s^{(+) - (\mathbf{r}, t)} = \frac{\hbar k_s}{2\epsilon_0} e^{i(\mathbf{k}_s \cdot \mathbf{r} - \omega_s t)} u^*(\mathbf{r}_\perp) \times \sum_{j \in \{H, V\}} \hat{\epsilon}_{j}(\mathbf{k}_s) \hat{\psi}_j(t - \mathbf{k}_s \cdot \mathbf{r}/c), \quad (A.1)$$

the negative frequency component $\hat{E}_s^{(-) - (\mathbf{r}, t)} = \hat{E}_s^{(+)}(\mathbf{r}, t)$, and $\hat{\epsilon}_{H}(\mathbf{k})$ and $\hat{\epsilon}_{V}(\mathbf{k})$ are respectively the horizontal and
vertical polarization vectors associated with a propagation direction $\mathbf{k}$ such that $[\hat{e}_h(\mathbf{k}), \hat{e}_v(\mathbf{k}), \hat{e}_p(\mathbf{k})]$ form a right-handed coordinate system and $\psi_\alpha(t)$ are photon field operators that satisfy the approximate slowly varying bosonic commutation relations $[\hat{\psi}_\alpha(t), \hat{\psi}_\beta(t')] = \delta_{\alpha,\beta} \delta(t-t')$.

These fields interact with the atomic electric dipoles for the atoms labelled by an index $\mu = 1, \ldots, N$. When one adiabatically eliminates the excited state, one arrives at the Raman scattering Hamiltonian governing the write process

$$\hat{H}_w = \hat{V} + \hat{V}^\dagger,$$

where

$$\hat{V} = \sum_{\nu=1}^N \frac{\mathbf{E}_\nu(\mathbf{r}_\mu, t) \cdot \hat{\mathbf{d}}_{\nu,\mu}(t) \hat{\mathbf{d}}_{\nu,\mu}^\dagger(t)}{\hbar \Delta},$$

the dipole transition operator between arbitrary levels $|f\rangle$ and $|g\rangle$ is $\hat{d}_{f,g}(t) = \hat{p}^\mu_f \hat{d}_{\mu}^\dagger(t) \hat{p}^\mu_g$, and $\hat{p}^\mu_f \equiv \sum_{m=-F_f}^{F_f} |f, m\rangle \langle f, m| \mu$ is the projection onto the level $|f\rangle$ with the total angular momentum $F_f$ for atom $\mu$.

We find it useful to express these dipole transition operators between the levels $|f\rangle$ and $|g\rangle$ in terms of a vector of Clebsch–Gordan coefficient matrices

$$C_{F_f F_g} = \sum_{\mu=1}^N \xi^\mu_{f} \xi^\dagger_{g} C_{F_f F_g}^\mu,$$

where $\xi_{f,g} \equiv \tilde{T}(\mathbf{x} \pm i\mathbf{y})/\sqrt{2}$ and $\tilde{T}$ are the spherical basis vectors and $C_{F_f F_g}^\mu$ are $(2F_f + 1) \times (2F_g + 1)$ matrices with elements

$$[C_{F_f F_g}^\mu]_{m_f, m_g} = C_{F_f F_g}^\dagger_{m_g, m_f}$$

and $m_f = -F_f, \ldots, F_f$ and $m_g = -F_g, \ldots, F_g$ correspond to possible magnetic quantum numbers. Between any two levels $|f\rangle$ and $|g\rangle$, we also define the matrix of slowly varying transition operators $\tilde{\sigma}_f^\mu$ whose elements are the transition operators between individual Zeeman states. Explicitly, these matrix elements are

$$[\tilde{\sigma}_f^\mu(t)]_{m_f, m_g} = \tilde{\sigma}_f^{\mu}_{m_f, m_g}(t),$$

where $\tilde{\sigma}_f^{\mu}_{m_f, m_g}(t) \equiv \exp(i(\omega_g - \omega_f)t) \tilde{\sigma}_f^{\mu}_{m_f, m_g}(t)$ so that, in the interaction picture, the operators $\tilde{\sigma}_f^{\mu}_{m_f, m_g}(t)$ are $|f, m_f\rangle \langle g, m_g|$ time independent. By the Wigner–Eckart theorem, one can show that the dipole transition operators between the levels $|f\rangle$ and $|g\rangle$ are given by

$$\hat{d}_{f,g}^\mu(t) = \gamma_{f,g} e^{-i(\omega_g - \omega_f)t} \text{Tr}[C_{F_f F_g}^\mu \tilde{\sigma}_f^\mu(t)],$$

where $\text{Tr}$ denotes a trace over the atomic angular momentum degree of freedom, and $\gamma_{f,g} \equiv \langle f | \hat{d}_f | g \rangle$ are the reduced matrix elements of the dipole operator. To illustrate the mechanics of the matrices of Clebsch–Gordan coefficients and operators, we can express the dipole operators in terms of coherences between individual Zeeman levels as

$$\hat{d}_{f,g}^\mu(t) = \gamma_{f,g} e^{-i(\omega_g - \omega_f)t} \sum_{\alpha=1}^1 \xi^\dagger_{\alpha} \text{Tr}[C_{F_f F_g}^\mu \tilde{\sigma}_f^\mu(t)].$$

where we employed the cyclic property of the trace in the first line.

Scattering of the signal fields produces the formation of collective coherences between various Zeeman states within levels $|b\rangle$ and $|a\rangle$. These take the form of collective spin-wave operators where each atom $\mu$ at position $\mathbf{r}_\mu$ has imprinted on it an associated phase $\exp[i(\mathbf{k}_0 \cdot \mathbf{r}_\mu) - \omega_\mu]$ and $\omega_\mu$ as we will see later, it is the evolution and ultimately preservation of this position-dependent phase that is key to being able to retrieve any imprinted information after some storage time. The collective spin-wave operator connecting the states $|b, m_b\rangle$ and $|a, m_a\rangle$ is given by

$$\hat{S}_{m_a, m_b} = \sum_{m=1}^N |u^*(\mathbf{r}_\mu) e^{i(\mathbf{k}_0 \cdot \mathbf{r}_\mu) - \omega_\mu} \tilde{\sigma}_f^{\mu}_{m_a, m_b} \rangle \langle \tilde{\sigma}_f^{\mu}_{m_a, m_b} |\sqrt{p_{m_a} \frac{\delta E n(r) |u(r)|^2}{2\Delta}}.$$}

When the total number of photons scattered into either the signal field mode or other modes is much less than the number of atoms in the ensemble, by the central limit theorem, these spin-wave operators satisfy the quasi-bosonic commutation relations

$$[\hat{S}_{m_a, m_b}, \hat{S}_{m_c, m_d}^\dagger] = \delta_{m_a, m_c} \delta_{m_b, m_d} (1 + O(1/\sqrt{N})).$$

where the $O(1/\sqrt{N})$ correction term is a Gaussian random variable resulting from the statistical distribution of the atomic positions. Substituting for the form of the dipole operators in equation (A.7) into the write process interaction of equation (A.3), one finds that this interaction can be written exactly in terms of the spin waves as

$$\hat{V}(t) = h \gamma \phi(t) \sum_{\lambda} \xi^\lambda \hat{\psi}_{\lambda}(t) \hat{\lambda}_{\lambda},$$

where $\hat{\lambda}_{\lambda}$ is the collective atomic excitation imprinted by a signal photon with polarization $\mathbf{e}_\lambda$, the interaction parameter

$$\chi = \frac{\gamma_{g,f} \sqrt{\int dr |E_w(t)|^2}}{\hbar \Delta} \sqrt{\frac{\hbar \kappa}{2 \epsilon_0} \int d^3 r n(r) |u(r)|^2}$$

and the temporal envelope of the emitted signal photon is

$$\phi(t) = \frac{\gamma_{g,f} |E_w(t)|}{\sqrt{\int dr |E_w(t)|^2}}.$$

The collective atomic operator is expressed in terms of Zeeman spin waves as

$$\hat{\lambda}_\lambda \equiv \frac{\text{Tr}[\mathcal{P} \hat{S}(\mathbf{e}_\lambda \cdot \mathbf{C}_{F_f F_g})(\mathbf{e}_\lambda \cdot \mathbf{C}_{F_f F_g})]}{X_{\lambda}}.$$

where $\mathcal{P}$ is the matrix of atomic populations with $\mathcal{P}_{m,m'} = p_{m_a} \delta_{m,m'}$, $\hat{S}$ has the matrix elements $\hat{S}_{m_a, m_b} = \hat{S}_{m_a, m_b}$ and the normalization factor

$$X_{\lambda} \equiv \text{Tr}[\left((\mathbf{e}_\lambda \cdot \mathbf{C}_{F_f F_g})(\mathbf{e}_\lambda \cdot \mathbf{C}_{F_f F_g})\right)(\mathbf{F}_{F_f F_g} \cdot \mathbf{e}_\lambda)(\mathbf{C}_{F_f F_g} \cdot \mathbf{e}_\lambda)^{1/2}].$$

(A.18)
Because light scattered into modes other than the collected signal mode leaves behind spin waves that for $N \gg 1$ are orthogonal to that generated by the signal [27], when one traces over the undetected signal modes, the density matrix describing the system after the write interaction is given by

$$\hat{\rho} = \hat{U}(\chi)\hat{\rho}_0\hat{U}^\dagger(\chi),$$

(A.19)

where the unitary operator

$$\hat{U}(\chi) = T \exp\left(-\frac{i}{\hbar} \int \mathrm{d}t \left(\hat{V}(t) + \hat{V}^\dagger(t)\right)\right),$$

and $T$ indicates time ordering. When the interaction parameter $\chi \ll 1$, as it is in the experiments under consideration, the density matrix may be written as

$$\hat{\rho} \approx \left(1 - i\chi \sum_{\lambda} \hat{a}_{\lambda}^\dagger \hat{a}_{\lambda}\right)\hat{\rho}_0 \left(1 + i\chi \sum_{\lambda} \hat{a}_{\lambda}^\dagger \hat{a}_{\lambda}\right),$$

(A.21)

where the single signal mode operator is given by $\hat{a}_{\lambda} \equiv \int \mathrm{d}t \hat{\phi}\dagger(t)\hat{\phi}(t)$. The emission of a signal photon of polarization $\hat{\phi}$, leaves behind an atomic excitation in the mode $\hat{A}_{\lambda}$. The collective atomic operators satisfy the commutation relations $[\hat{A}_{\lambda}, \hat{A}_{\lambda}^\dagger] = 1 + O(1/\sqrt{N})$. However, depending on the initial population distribution, the atomic excitations left behind by photons of orthogonal polarizations $\lambda_1, \lambda_2$ are not necessarily orthogonal. This will not affect the calculation of retrieval efficiencies based on the detection of a single excitation in the limit $\chi \ll 1$, however.

After a storage time $T_s$, a classical read pulse with linear polarization $\hat{\phi}$, travelling in the direction $\hat{k}_r = -\hat{k}_s$, is shined on the ensemble. This read pulse is resonant on the $|\alpha\rangle \leftrightarrow |\gamma\rangle$ transition. The interaction of the read field with the ensemble, in turn, leads to the emission of an idler photon with a central frequency resonant on the $|b\rangle \leftrightarrow |c\rangle$ transition. We collect a single transverse mode of the idler field propagating in the direction $\hat{k}_i = -\hat{k}_s$. This mode has a transverse profile conjugate to that of the signal field. The positive frequency component of the idler field is then given by

$$\hat{E}_i^{\dagger+}(r, t) = \sqrt{\frac{\hbar c}{2\varepsilon_0}} e^{i(k_r r - c k_r t)} \sum_{\lambda \in \{H,V\}} \hat{e}_\lambda(\hat{k}_i) \hat{\phi}_\lambda(r, t),$$

(A.22)

where $r_\parallel \equiv \hat{k}_r \cdot r$ and the slowly varying operators $\hat{\phi}_\lambda(r, t)$ satisfy the equal time commutation relations $[\hat{\phi}_\lambda(r_1, t), \hat{\phi}_\lambda^\dagger(r_2, t')] = \delta_{\lambda,\lambda'}\delta(r_1 - r_2)\delta(t - t')$.

When the length of the atomic ensemble $L \ll |k_s - k_w - k_i + k_i|^{-1}$, and in the absence of any dephasing mechanisms during storage, the idler field is phase matched with the stored atomic excitations [27, 29].

The read and idler fields interact with the atoms via the electric dipole interactions. In the rotating wave approximation, the interaction potential is given by

$$\hat{H}_I = \sum_{\mu=1}^{N} \left(\hat{V}_i^\mu + \hat{V}_r^\mu + \hat{V}_R^\mu\right) + \text{h.c.},$$

(A.23)

where the interaction of atom $\mu$ with the idler field is

$$\hat{V}_i^\mu = -\hat{E}_i^{\dagger+}(r_\mu, t) \cdot \hat{d}_{\mu,c}^\dagger(b)$$

and $\hat{V}_R^\mu = -\hat{d}_{\mu,c}^\dagger(r_\mu, t) \cdot \hat{E}_r^\mu$, $\hat{d}_{\mu,c}^\dagger(\alpha, \beta)$ is the interaction with the read field. The interaction with the undetected field modes $\hat{V}_R^\mu$ accounts for spontaneous emission.

Before examining the propagation dynamics of the idler field within the ensemble, let us look in detail at the form of the read field interaction. From equation (A.7), one can immediately write this interaction in terms of transitions between Zeeman states as

$$\hat{V}_r^\mu(t) = \hbar \Omega(t) e^{i(k_r r_\mu - c k_r t)} \mathrm{Tr}\left[\hat{\sigma}_\mu^{(c)} K^\dagger\right]$$

(A.24)

$$= \hbar \Omega(t) e^{i(k_r r_\mu - c k_r t)} \sum_{a=1}^N (\xi_a \cdot \hat{e}_r)$$

$$\times \sum_{m_a = -F_r}^{F_r} C_{ma} \Gamma F_r \cdot |c, m + \alpha\rangle \langle a, m_a|,$$

(A.25)

where $\Omega(t) \equiv \hat{q}_{\perp} E_r(t)/\hbar$ is the read field Rabi frequency, and we have defined the matrix $\mathcal{K} \equiv \hat{\xi}^\dagger \cdot \mathcal{C}_{F_r F_r}$. When the read field polarization is oriented along the $z$-axis, this interaction takes on a relatively simple form in which the read field couples a state in the excited level $|c, m\rangle$ to a single ground state $|a, m\rangle$. One can recover a similar simplification for an arbitrary linearly polarized read field by expressing this potential in terms of eigenstates of the angular momentum along the polarization direction. Such a state within a level $|f\rangle$ can be expressed as $|f, m\rangle \equiv \sum_{m'} D_{mf}(m') |f, m'\rangle$, where $D_{mf}(m')$ is a matrix element of the appropriate rotation operator acting on a level with angular momentum $F_r$. Explicitly,

$$\hat{V}_r^\mu(t) = \hbar \Omega(t) \sum_{m_a = -F_r}^{F_r} C_{ma} \Gamma F_r \cdot |c, m + \alpha\rangle \langle a, m_a|.$$  

(A.26)

Expressed in this way, one readily sees that the states $|\bar{a}, \pm F_a\rangle$ are dark with respect to the read field, and hence, that spin waves involving these states cannot be accessed during the read process. On the other hand, for the atomic level configuration under consideration, every excited state $|\bar{c}, m\rangle$ ($-F_s \leq m \leq F_s$) is resonantly coupled to a ground state $|\bar{a}, m\rangle$ via the read field. The presence of this resonant coupling for every excited Zeeman level, as we will see later, permits the formation of electromagnetically induced transparency for a resonant idler field. Thus, comparing the expressions for $\hat{V}_r$ in equations (A.24) and (A.26), one can deduce that

$$\mathcal{K} = D_{F_r F_r}^* |c, F_r\rangle \langle a, F_r|.$$  

(A.27)

In the weak idler limit, where the probability of any one atom to be displaced from its ground state is negligible (i.e. the populations $|\bar{a}, c, m\rangle, |\bar{a}, \bar{m}_c, m\rangle \ll 1$), the interaction Hamiltonian (equation (A.23)) leads to the following Heisenberg–Langevin equations describing the propagation of the idler field:

$$\left(\frac{\partial}{\partial t} + c \hat{k}_r \cdot \nabla\right) \hat{\phi} = i\sqrt{\hbar} n_{\phi}^\dagger \mathrm{Tr}\left[\mathcal{P}_{1/2} \hat{e}^\dagger(1 - \hat{k}_r \hat{k}_r^\dagger) \cdot \mathcal{C}_{F_r F_r}\right],$$

(A.28a)

$$\left(\frac{\partial}{\partial t} + \frac{\Gamma}{2}\right) \hat{\epsilon} = i\sqrt{\hbar} \left(\mathcal{P}_{1/2} \mathcal{C}_{F_r F_r}^\dagger \cdot \hat{\phi} + i\Omega^\dagger \mathcal{K}^\dagger + \xi\right),$$

(A.28b)

$$\frac{\partial \hat{\delta}}{\partial t} = i\Omega^\dagger \mathcal{K}^\dagger,$$

(A.28c)
where $\hat{\delta}(\zeta, t)$ and $\hat{\epsilon}(\zeta, t)$ are the matrices of hyperfine and optical local spin waves respectively with the matrix elements

$$\hat{\delta}_{\mu, \nu}(\rho_{\parallel}, t) \equiv \sum_{\mu=1}^{N} u^\ast \rho_{\parallel} e^{-(ik_{\parallel} - k_{J}) \cdot \rho_{\parallel}} \delta(\rho_{\parallel} - \rho_{\parallel, \mu}) \hat{\sigma}_{\mu, \nu}^{-1} \hat{\sigma}_{\mu, \nu},$$

(A.29a)

and

$$\hat{\epsilon}_{\mu, \nu}(\rho_{\parallel}, t) \equiv \sum_{\mu=1}^{N} u^\ast \rho_{\parallel} e^{-(ik_{\parallel} - k_{J}) \cdot \rho_{\parallel}} \delta(\rho_{\parallel} - \rho_{\parallel, \mu}) \hat{\sigma}_{\mu, \nu}^{-1} \hat{\sigma}_{\mu, \nu},$$

(A.29b)

$\hat{\delta}(\rho_{\parallel}, t)$ is a $(2F_{c} + 1) \times (2F_{e} + 1)$ matrix of Langevin noise terms and $\hat{n}(\rho_{\parallel}) \equiv \int d^{2}r_{\parallel} |u(r)|^2 n(r)$ is the average atomic density over the transverse idler profile. We assume that the atomic density varies over transverse length scales much larger than idler beam waist as it is for atoms in a magneto-optical trap [13, 24], or that the atoms are in an optical lattice [17] with each site roughly equally populated such that many lattice periods lie within the transverse profile. In this way, the collective atomic excitations would only coherently couple to the collected idler mode.

Because, as mentioned above, every Zeeman level $|c, m_{c}\rangle$ is resonantly coupled to some state in level $|a\rangle$, one can invert equation (A.28c), expressing any collective optical coherence $\hat{\epsilon}_{\mu, \nu}$ in terms of some combination of hyperfine spin waves $\hat{\delta}_{\mu, \nu}$. Explicitly,

$$\hat{\epsilon} = \frac{1}{\Omega^2} \frac{\partial}{\partial t} \left[ K^{+} \right],$$

(A.30)

where the superscript $+$ denotes the Moore–Penrose pseudoinverse. From equation (A.27), one can exploit the unitary nature of the rotation matrices to write the pseudoinverse as

$$K^{+} = D^{\dagger}_{F_{e}F_{c}} e^{0_{F_{c}} \cdot F_{c}} D^{T}_{F_{e}},$$

(A.31)

where $[10^0_{F_{c}F_{e}m_{c}m_{e}} = 0$ for $m_{e} \neq m_{c}$ and $m_{c} = F_{e}$ and $[10^0_{F_{c}F_{e}m_{c}m_{e}} = 1\delta_{m_{c}m_{e}}$ for $|m_{c}| \leq F_{c}$]. This allows one to eliminate the appearance of the optical coherences from the idler propagation equation. Substituting equation (A.30) into equation (2.28a) yields

$$\left( \frac{\partial}{\partial t} + c \hat{\mathbf{k}} \cdot \nabla \right) \hat{\phi} = \frac{\Omega^{2} \hat{\mathbf{K}}}{\Omega^2} \frac{\partial}{\partial t} \text{Tr}[\mathbf{R}^{1/2}],$$

(A.32)

where

$$\mathbf{R} \equiv K^{+} \equiv (1 - \hat{\mathbf{k}} \hat{\mathbf{k}}^{\dagger}) \cdot C_{F_{c}F_{e}}.$$

(A.33)

Proceeding in a manner analogous to the adiabatic treatment of Fleischhauer and Lukin [33, 34], we assume that the bandwidth of the propagating field $\delta$ is much less than either the spontaneous emission rate $\Gamma$ or the Rabi frequency $\Omega$. In this adiabatic limit, we may take the time derivative of equation (A.28b), so that to first order in $\delta/\Omega$ and $\delta/\Gamma$, and neglecting any time variation in $\Omega$, we have

$$\frac{\partial}{\partial t} \hat{\delta} = -\frac{\sqrt{n_{e}}}{\Omega} \mathbf{R}^{1/2} \frac{\partial}{\partial t} \hat{\phi}.$$

(A.34)
signal of polarization $\lambda_j$ is therefore $(\hat{\Upsilon}_j(T)\hat{\Upsilon}_j(T))/\langle \hat{A}^\dagger_j\hat{A}_j \rangle$. Using the conditioned atomic density operator, this reduces to [17, 35]

$$[[\hat{\Upsilon}_j(T), \hat{A}_j]]]^2.$$

Equation (A.40)

More generally, the detection of an idler photon with polarization $\hat{e}_m$ conditioned on the detection of a signal with polarization $\hat{e}_n$ is given by

$$\eta_{n,m}(T) = \frac{\int dt \langle \hat{\psi}^{\dagger}_n(t)\hat{\psi}_{m}(t) \rangle/\langle \hat{A}^\dagger_j\hat{A}_j \rangle}{\langle \hat{A}^\dagger_j\hat{A}_j \rangle},$$

Equation (A.41)

where $\hat{\psi}_{m}(t) = \sum_j \hat{e}^\dagger_{m} \cdot \hat{v}_j \hat{\psi}_j(t)$. The polariton propagation dynamics imply that

$$\eta_{n,m}(T) = \sum_j \langle \hat{e}_{m}^\dagger \cdot \hat{v}_j \rangle^2[[\hat{\Upsilon}_j(T), \hat{A}_j]]^2 \times \langle [\hat{\Upsilon}_2(T), \hat{A}_j] \rangle \left( \int dt \phi^{\dagger}_n(t)\phi_m(t) \right).$$

Equation (A.42)

In the configuration we considered in this manuscript, the idler propagated in a direction parallel to the applied magnetic field, $\hat{\mathbf{k}} = \hat{z}$. During the storage process, the stored hyperfine coherences $\hat{S}_{m,m}(z, t)$ evolve under the influence of the magnetic field $B_0$ applied along the $z$-axis, precessing at the Larmor frequency $\omega_{m,m} = (\mu_B B_0/\hbar)[g_s m_s + m_p] - \delta g m_m$ where $g_s$ and $g_p$ are the Landé $g$-factors for levels $|a\rangle$ and $|b\rangle$ respectively, and $\delta g \equiv g_a - g_b$. The retrieval efficiencies (equation (A.42)) involve linear combinations of commutators $[\hat{S}_{m,m}(T), \hat{S}_{m',m'}(0)]$, each of which contains a phase factor $\exp(i\omega t)$. These superpositions of terms oscillating at various frequencies lead to collapses and revivals of the retrieval efficiency with storage time [23, 24]. Additionally, however, imperfections in the field coils as well as residual ambient magnetic fields result in inhomogeneities in the magnetic field across the sample, which we model as a gradient in $B'$ in the $z$-component of the field. In the presence of this gradient, the $\hat{\delta}_{m,m'}(z, T)$ constituents of $\hat{S}_{m,m}(T)$ pick up a spatially varying phase factor $e^{-i\omega t\cdot B'/\hbar}$, which can also be viewed as a temporal change in wave number $\Delta k_{m,m'} = \omega m, T B'/\hbar$:

$$\hat{S}_{m,m}(T) = e^{-i\omega t\cdot B'/\hbar} \int dz f(z) e^{-i\Delta k_{m,m'} z} \hat{S}_{m,m}(z).$$

Equation (A.43)

In evaluating the commutators, each spin wave contributes a term with a spatial Fourier transform factor $\int dz f(z) e^{-i\Delta k_{m,m'} z}$. Evaluating the commutators appearing in equation (A.42), one finds

$$\langle [\hat{\Upsilon}_j(T), \hat{A}_j] \rangle = \sum_{m,m'} R_{j,m,m'} \rho_{m,m'} \hat{S}_{m,m}(T)$$

$$\times \left( \hat{e}_m \cdot C_{J,F,\lambda} \hat{e}^\dagger_{m'} \cdot C_{J,F,\lambda} \right)_{m,m'},$$

Equation (A.44)

where $\hat{S}_{m,m}(T) \equiv \langle \hat{S}_{m,m}(T) \rangle$. The retrieval efficiency with storage time [23, 24]. Additionally, the idler field propagates in the $z$-direction and the detected signal that heralds the creation of the stored atomic excitations is collected from the negative $z$-direction. By exploiting symmetry relations of Clebsch–Gordan coefficients, and by virtue of symmetric populations, horizontal and vertical idler fields couple to balanced superpositions of optical coherences, $\hat{e}_{m,m}^\dagger$ and $\hat{e}_{m,-m}^\dagger$, and hyperfine coherences $\hat{S}_{m,m}$ and $\hat{S}_{-m,-m}$. For $^{87}$Rb, we then recover the Heisenberg–Langlevin equations (equation (14)). These equations suggest that when $p_l = p_{-l}$, there are two dark-state polariton operators corresponding to emission of an idler or either a horizontal or vertical polarization. This is consistent with the dark-state polariton operators obtained directly from the more general treatment: horizontal and vertical dark-state polaritons propagate at different velocities providing a group velocity walk-off.

To understand which optical and hyperfine coherences the linearly polarized signals couple to, we explicitly evaluate the equation of motion for the field. We begin by noting that since $F_0 = F_1$, the Clebsch–Gordan coefficient matrices satisfy the relations $[C_{J,F,\lambda}]_{m,m} = [-C^\dagger_{J,F,\lambda}]_{m,-m}$. In the specific setup described in figure 2, and with $p_{l} = p_{-l}$, we can take the trace of equation (A.28a) to find the equations of motion for $\varphi = e^\dagger \cdot \varphi$ ($\lambda \in [H, V]$). $\varphi_0 = \hat{S}_{m,m}(0)$, which can also be viewed as a temporal change in wave number $\Delta k_{m,m'} = \omega m, T B'/\hbar$:

$$\hat{S}_{m,m}(T) = e^{-i\omega t\cdot B'/\hbar} \int dz f(z) e^{-i\Delta k_{m,m'} z} \hat{S}_{m,m}(z).$$

Equation (A.43)

$x \cdot \partial/\partial z + c \partial/\partial t = i\hbar \sqrt{\Omega} \hat{\delta}_{m,m}(\hat{e}_{m} \cdot \mathbf{C}_{J,F,\lambda})$ (A.46)

where $\hat{e}_{m} \cdot \mathbf{C}_{J,F,\lambda} = C_{0,\lambda} F_{0,\lambda} = -1/\sqrt{2}$ and $\hat{e}_{-m} \cdot \mathbf{C}_{J,F,\lambda} = (1/\sqrt{2}) C_{1,\lambda}$ are the relative coupling strengths of the idler field to optical coherences $\hat{e}_{\lambda}$. $\Omega$ is the real field Rabi frequency, $\kappa$ is the single-photon Rabi frequency on the idler transition and $\hat{\xi}_{\lambda}$ is a noise operator. The collective optical coherence operators are given by $\hat{e}_{0,\lambda} = (-i)^{\lambda-1} \hat{e}_{1,\lambda} + (-1)^{\lambda-1} \hat{e}_{\lambda}$, $\hat{e}_{\lambda} = (-i)^{\lambda-1} \hat{e}_{0,\lambda} + (-1)^{\lambda-1} \hat{e}_{1,\lambda}$. The horizontally polarized fields couple to optical coherences of the form $\hat{\delta}_{m,m}(\hat{e}_{m} \cdot \mathbf{C}_{J,F,\lambda}) = \hat{e}_{m} \cdot \mathbf{C}_{J,F,\lambda} = \hat{e}_{m} \cdot \mathbf{C}_{J,F,\lambda}$. To see how these optical coherences evolve, we use equation (A.28b). Owing to the symmetries of Clebsch–Gordan coefficients between the levels $|e\rangle$ and $|a\rangle$, $[C_{J,F,\lambda}]_{m,m} = [C^\dagger_{J,F,\lambda}]_{m,-m}$, and because $K = (\hat{\xi} \cdot \mathbf{C}_{J,F,\lambda}) = -i(C_{1,\lambda} + C_{0,\lambda}^\dagger)$, the matrix
governing the coupling between these levels satisfies the symmetry relation \( K_{-m,-m} = K_{m,m} \). So the symmetric and antisymmetric combinations of optical coherences that respectively couple to the horizontal and vertical idler fields obey the equations of motion

\[
\left( \frac{\partial}{\partial t} + \frac{\Gamma}{2} \right) \hat{\epsilon}_{m,m}^{(2)} = i\kappa \sqrt{n} \hat{p}_{m} \phi_{m} \sum_{a = \pm 1} (\alpha)^{m,a} \left[ \hat{C}_{a,\alpha}^{a,m} \right]_{m,m} \\
+ i\Omega \sum_{m,-m} \sum_{v} \rho_{m,v} \left( \hat{\delta}_{m,m} + \left( -1 \right)^{m,a} \hat{\delta}_{-m,m} \right) K_{m,m}^{\alpha,\alpha},
\]

(A.48)

Specifically, the coherences \( \hat{\delta}_{1,\lambda} \) for each polarization associated with atoms originating in the states \( |h, \pm 1\rangle \) couple to linear combinations of the fast and slow coherences \( \hat{s}_{1,\lambda}^{(\text{fast})} \) and \( \hat{s}_{1,\lambda}^{(\text{slow})} \). Similarly, for the atoms originating in the Zeeman state \( |h, 0\rangle \), the collective coherence \( \hat{s}_{0,0}^{(2)} \) couples to the fast spin waves \( \hat{s}_{0,0}^{(\text{fast})} \), but \( \hat{\rho}_{0,0} \) has contributions from both \( \hat{s}_{0,0}^{(\text{fast})} \) and the clock coherence \( \hat{\delta}_{0,0} \). Only the horizontal fields couple to the clock coherence because the paths leading from \( |h, 0\rangle \) to \( |\alpha, 0\rangle \) perfectly cancel when the idler is vertically polarized. For the coherences \( \hat{\delta}_{m,\lambda} \) with \( m \in \{0, 1\} \) and \( \lambda \in \{H, V\} \), we have

\[
\hat{\delta}_{m,\lambda} = i\kappa \sqrt{p_{m,i} \hat{C}_{m,\lambda}} \phi_{m,\lambda} + i\Omega \hat{C}_{m,\lambda} \hat{r}_{m,\lambda} + \hat{\epsilon}_{m,\lambda},
\]

(A.49)

where \( C_{0,\lambda} = \sqrt{(3 + \delta_{0,\lambda})/10} \) and \( C_{1,\lambda} = \sqrt{3}/10 \) determine the relative coupling of the optical coherences and read spin waves; \( \hat{\epsilon}_{m,\lambda} \) is a noise operator. The read spin-wave operators are expressed in terms of the same fast and slow coherences imprinted in the write process, \( \hat{\delta}_{1,\lambda} = \left( \hat{s}_{1,\lambda}^{(\text{fast})} + \hat{s}_{1,\lambda}^{(\text{slow})} \right) / \sqrt{2} \), \( \hat{\delta}_{0,0} = \hat{s}_{0,0}^{(\text{fast})} \) and \( \hat{\delta}_{0,0} = \sqrt{3}/2 \hat{s}_{0,0}^{(\text{fast})} - i\hat{\delta}_{0,0}/2 \).

From equation (A.28), we arrive at the equations of motion for the read spin waves

\[
\hat{\Delta}_{m,\lambda} = i\Omega \hat{C}_{m,\lambda} \hat{\delta}_{m,\lambda}.
\]

(A.50)

From these equations of motion for balanced populations, one can directly construct the dark-state polaritons associated with horizontally and vertically polarized idlers (equation (15)). To verify that \( \hat{x} \) and \( \hat{y} \) are indeed eigen-polarizations of the system associated with dark-state polaritons that propagate at distinct group velocities, we can examine the adiabatic diabatic matrices \( \mathcal{R}_{\lambda} \equiv \mathbf{e}_{\lambda}^{\ast} \cdot \mathbf{R} \).

Explicitly, for \(^{87}\text{Rb} \) with \( F_{x} = 0 \) and \( F_{x} = 1 \) and \( F_{y} = 2 \), these matrices are given by

\[
\mathcal{R}_{\lambda} = \begin{pmatrix}
0 & \frac{-1}{\sqrt{2}} C_{0,H} & C_{0,V} \\
\frac{i}{2} C_{1,H} & 0 & \frac{i}{2} C_{1,H} \\
0 & \frac{-i}{2} C_{0,V} & 0 \\
i C_{1,V} & \frac{-1}{\sqrt{2}} C_{0,H} & 0 \\
0 & \frac{-i}{2} C_{1,V} & 0 \\
i C_{1,V} & \frac{-1}{\sqrt{2}} C_{0,H} & 0
\end{pmatrix}
\]

(A.51)

where these matrices are arranged such that the index \( m_{b} \) increases from \(-1\) on the left to \( 1 \) on the right, and \( m_{c} \) increases from \(-2\) on the top row to \( 2 \) on the bottom row. In general, in a sample with both polarization and alignment, the spatial eigenvectors associated with each dark-state polaritons are the eigenvectors of the dyadic tensor

\[
\text{Tr}[\mathbf{R}^{\dagger} \mathbf{R}^{\dagger}] = \text{Tr}[\mathbf{R}^{\dagger} \mathbf{R}^{\dagger}] \hat{\mathbf{x}} \hat{\mathbf{x}} + \text{Tr}[\mathbf{R}^{\dagger} \mathbf{R}^{\dagger}] \hat{\mathbf{y}} \hat{\mathbf{y}} + \text{Tr}[\mathbf{R}^{\dagger} \mathbf{R}^{\dagger}] \hat{\mathbf{x}} \hat{\mathbf{y}} + \text{Tr}[\mathbf{R}^{\dagger} \mathbf{R}^{\dagger}] \hat{\mathbf{y}} \hat{\mathbf{x}},
\]

(A.52)

where in terms of the atomic populations, the various tensor components are

\[
\text{Tr}[\mathbf{R}^{\dagger} \mathbf{R}^{\dagger}] = p_{0} \frac{|C_{0,H}|^{2}}{|C_{0,H}|^{2}} + p_{1} + p_{-1} \frac{|C_{1,V}|^{2}}{|C_{1,V}|^{2}}
\]

(A.53)

\[
\text{Tr}[\mathbf{R}^{\dagger} \mathbf{R}^{\dagger}] = p_{0} \frac{C_{0,V}^{2}}{|C_{0,V}|^{2}} + p_{1} + p_{-1} \frac{|C_{1,V}|^{2}}{|C_{1,V}|^{2}}
\]

(A.54)

\[
\text{Tr}[\mathbf{R}^{\dagger} \mathbf{R}^{\dagger}] = p_{0} \frac{|C_{0,H}|^{2}}{|C_{0,H}|^{2}} + p_{1} + p_{-1} \frac{|C_{1,V}|^{2}}{|C_{1,V}|^{2}}
\]

(A.55)

\[
\text{Tr}[\mathbf{R}^{\dagger} \mathbf{R}^{\dagger}] = p_{0} \frac{C_{0,V}^{2}}{|C_{0,V}|^{2}} + p_{1} + p_{-1} \frac{|C_{1,V}|^{2}}{|C_{1,V}|^{2}}
\]

(A.56)

\[
\text{Tr}[\mathbf{R}^{\dagger} \mathbf{R}^{\dagger}] = p_{0} \frac{|C_{0,V}|^{2}}{|C_{0,V}|^{2}} + p_{1} + p_{-1} \frac{|C_{1,V}|^{2}}{|C_{1,V}|^{2}}
\]

(A.57)

When the populations are equal, both the tensor components \( \hat{\mathbf{x}} \hat{\mathbf{y}} \) and \( \hat{\mathbf{y}} \hat{\mathbf{x}} \) are identically zero, and the spatial eigenvectors of the dark-state polaritons correspond to horizontal and vertical polarizations.

Because the idler polariton polarizations are horizontal and vertical when the populations are balanced, we chose to investigate the efficiency with which a horizontally or vertically polarized idler is retrieved given the detection of the horizontally or vertically polarized signal. Applying the treatment of appendix A to \(^{87}\text{Rb} \) with balanced populations, we obtain the expressions for the written spin waves given in section 3.

References

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