Visualizing the Birman-Series Set on the Punctured Torus

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Introduction

Goals

- Understand the distribution of simple, closed geodesics on hyperbolic surfaces
- Create pictures to demonstrate phenomena in hyperbolic geometry

Example 1 (The Punctured Torus)**.** *The once-punctured torus $T_{1,1}$ can be obtained by gluing opposite sides of an ideal quadrilateral in the hyperbolic plane. This allows us to equip $T_{1,1}$ with a hyperbolic metric and identify the universal cover of $T_{1,1}$ with the hyperbolic plane $\mathbb{H}^2$.*

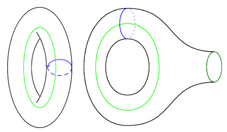


Figure 1: *The once-punctured torus has a fundamental group generated by two curves (shown above)*

Definition. *1. A path between a, b is a* **geodesic** *if it locally minimizes distance.*

2. A path is **simple** *provided that it is non self-intersecting.*

3. The **Birman-Series set** *is the union of all simple, closed geodesics on a finite area hyperbolic surface.*

Theorem 1

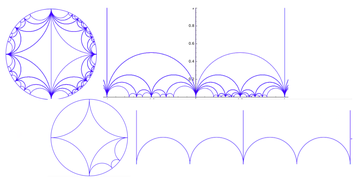
1. (Birman-Series 1985) The Birman-Series set has Hausdorff dimension 1; in particular, it is nowhere dense.
2. (Buser-Parlier 2008) If X is finite-area orientable hyperbolic surface, then there exists a constant $r > 0$ (depending only on the topology of the surface) and a hyperboilc ball of radius r embedded in X disjoint from the Birman-Series set.

Fundamental Domains

Example 2 (Definition of Γ and Γ^+)**.** *Let $\Gamma = \langle z \mapsto -\overline{z}, z \mapsto \frac{\overline{z}}{2\overline{z}-1}, z \mapsto -\overline{z}+2 \rangle$ and let Γ^+ be the index 2 subgroup of Γ consisting of orientation preserving transformations. Each generator of Γ is a reflection about a geodesic and the union of these geodesics bound an ideal triangle $T \subset \mathbb{H}^2$ satisfying:*

- $\bigcup_{\gamma \in \Gamma} \gamma(T) = \mathbb{H}^2$
- $\mathrm{Int}(T) \cap \gamma(\mathrm{Int}(T)) \neq \emptyset \implies \gamma = \mathrm{Id}$.

We say that $\mathbb{H}^2$ is tessellated *by the images of T under Γ and that T is a* fundamental domain *for the action of Γ on $\mathbb{H}^2$. Γ is an example of what is called a* **Fuchsian Group**.



The tessellation of an ideal triangle by Γ in $\mathbb{D}$ (top left), in $\mathbb{H}$ (top right). Any one triangle defines a fundamental domain.

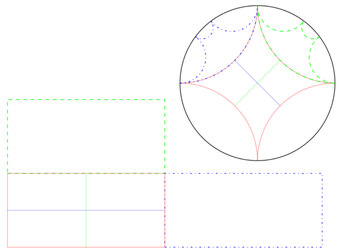
Definition. *An action of $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ is* **discrete** *if given a sequence $\{\gamma_n\} \subset \Gamma$ satisfying $z \in \mathbb{H}$ such that $\gamma_n(z) \to z$ as $n \to \infty$, there exists N such that $\gamma_m = \mathrm{Id}$ for all $m > N$. A* ***Fuchsian group*** *is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$.*

Theorem. *Fix generators a, b of $\pi_1(T_{1,1}) \simeq F_2$ as in Figure 1. Given $\lambda \in (1,\infty)$, let $Q_\lambda \subset \mathbb{H}^2$ be the ideal quadrilateral with vertices $0, 1, \lambda$, and ∞. Define $\rho_\lambda : F_2 \to \mathrm{PSL}(2,\mathbb{R})$ by setting $\rho(a)$ to be the hyperbolic translation along the geodesic orthogonal to the geodesics $[0,\infty]$ and $[1,\lambda]$ such that $\rho(a)(0) = 1$ and $\rho(a)(\infty) = \lambda$. Similarly define $\rho(b)$ with $\rho(b)(\infty) = 0$ and $\rho(b)(\lambda) = 1$. If $\lambda, \mu \in (1,\infty)$, then*

- *ρ_λ is injective,*
- *$\Gamma_\lambda = \rho_\lambda(F_2)$ is Fuchsian, and*
- *Γ_λ is conjugate to Γ_μ in $\mathrm{PSL}(2,\mathbb{R}) \iff \Gamma_\lambda \setminus \mathbb{H}^2$ is isometric to $\Gamma_\mu \setminus \mathbb{H}^2 \iff \lambda = \mu$.*

Analogy to Euclidean Space

In both pictures above, we have a red square. The two full colored lines represent the action of $\rho_\lambda(a)$ and $\rho_\lambda(b)$. If the blue line corresponds to $\rho_\lambda(a)$, then, in the flat case (left), the square's left side is transformed by means of translation to its right side, and each other point in the interior is transformed accordingly to the right. In the hyperbolic case (right), the square's upper-left side is transformed by means of Γ_λ to the bottom-right, and each other point in the interior is transformed accordingly by the transformation group.



We may now visualize the union of all closed geodesics in the surface $\Gamma_\lambda \setminus \mathbb{H}^2$ by intersecting the axis of every hyperbolic element of Γ_λ with a single fundamental domain for the action of Γ_λ on $\mathbb{H}^2$.

Representing Simple Closed Geodesics

Given a group G and a set of generators $\mathcal{S}$, we say the *word length* of an element $g \in G$ is the minimal length word in $\mathcal{S} \cup \mathcal{S}^{-1}$ that is equal to G. As there are no relations in a free group, the length of any reduced word in F_2 is computed by counting the number of a's and b's.

Generating Closed Geodesics

To generate all closed geodesics in Γ_λ we first build a list of all words up to a fixed word length n in F_2. We then need to throw out any words that correspond to loops about a puncture. This is done by fixing a λ, taking a word w in F_2, and calculating the determinate of $\rho_\lambda(w)$. As long as $|\mathrm{tr}\rho_\lambda(w)| \geq 2$, then $\rho_\mu(w)$ will correspond to a closed geodesic in $\Gamma_\mu \setminus \mathbb{H}^2$ for every $\mu \in (1,\infty)$.

Generating Simple Closed Geodesics

In order to pick out the simple closed geodesics from the list of closed geodesics, we use a characterization of simpleness given by Buser.

Definition. *(Small Variation) A finite sequence of nonzero integers $N_1, ..., N_p$ is said to have* **small variation** *provided that sums of m consecutive elements (indices $\bmod p$) never differ by more than ± 1*

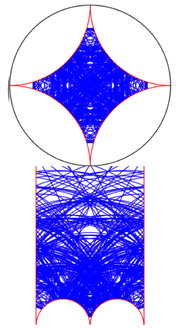
Theorem 2: (Buser 1988)

Every nontrivial simple, closed curve can, after suitably renaming generators, be represented by one of the following words

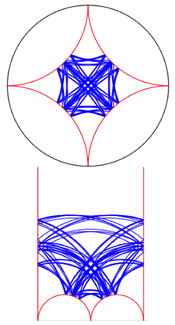
- a
- $aba^{-1}b^{-1}$
- $ab^{N_1}ab^{N_2}\cdots ab^{N_p}$, where the sequence $N_1, ..., N_p$ has small variation.

Conversely, each of these words is homotopic to a multiple of a simple closed curve.

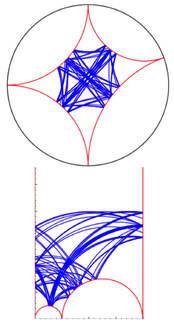
Since we had already generated a list of the closed geodesic curves of length up to n, to draw the simple, closed geodesics, we need to filter out the non-simple curves. To do this we took a word, cyclically reduced it, and checked if it met Buser's criteria.
The following pictures represent geodesics of word length up to $n = 6$.



The set of closed geodesics are dense in this fundamental domain Γ_2, which approximately seen here.



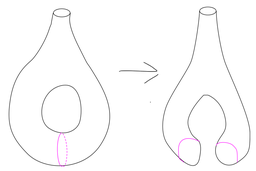
This is the Birman-Series set in the fundamental domain of Γ_2. Note the regions where there are no simple closed geodesics.

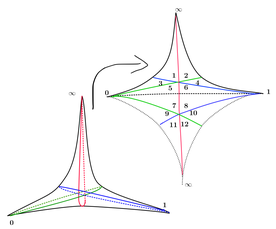


The sparseness and gaps in the Birman-Series set remain for all λ; pictured here is the fundamental domain of Γ_4.

Areas Devoid of Simple Closed Geodesics

$\Gamma^+ \setminus \mathbb{H}$ is homeomorphic to the thrice-punctured sphere $S_{0,3}$, pictured below. Shown on $S_{0,3}$ are the six simple complete geodesics. Shown next to the thrice-punctured sphere is the ideal quadrilateral obtained by cutting $S_{0,3}$ along the simple geodesic between 0 and 1. Observe that the complement of these six geodesics has 12 connected components.





Cutting the punctured torus along a simple closed geodesic α yields a thrice-punctured sphere (with boundary). Every geodesic intersecting α must intersect this copy of $S_{0,3}$ in simple arcs; hence, when α is short, the simple closed geodesics of the torus remain disjoint from the 12 regions (of definite diameter) described previously. This observation is essential to the proof of the Buser-Parlier result.

References

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