Combinatorics and the Mandelbrot Set

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LOG(M)

Goal Given a quadratic polynomial of the form $f_c(z) = z^2 + c$ with c a complex number, we can iterate the function and look at the behavior that arises. For example, if we plot all values of c in the complex plane so that the orbit of 0 under iteration of f_c remains bounded, we get the famous Mandelbrot set. In this project, we will look at specific values of c that lie in the Mandelbrot set, and compute algebraic data associated to the polynomials f_c, using the combinatorics of the Mandelbrot set as a guide for looking for patterns in this data.

1. Mandelbrot Set and Julia Set

Fix $c \in \mathbb{C}$. Let $f_c : \mathbb{C} \to \mathbb{C}$ be the function $f_c(z) = z^2 + c$.

Mandelbrot Set

Definition.

$$M = \{c \in \mathbb{C} | f_c^k(0) \not\to \infty \text{ as } k \to \infty\}$$

The Mandelbrot set M consists of all (complex) c-values for which the corresponding orbit of 0 under iteration of f does not escape to infinity.

Remark 1. *The Mandelbrot set is connected.*

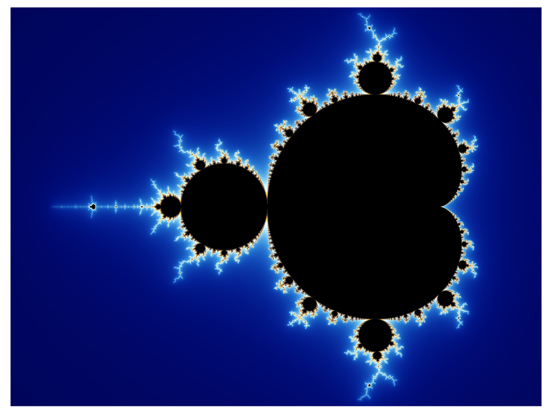

Figure 1: *The Mandelbrot set M*

Julia Sets

Definition. *We define the filled Julia set of f_c as*

$$J = \{z \in \mathbb{C} | f_c^k(z) \not\to \infty \text{ as } k \to \infty\}$$

Remark 2. *If $c \in M$, the filled Julia set is connected, and if $c \notin M$, the filled Julia set consists of disjointed points.*

Remark 3. *An equivalent definition of the Mandelbrot set is that it is the set of $c \in \mathbb{C}$ whose associated polynomials have filled Julia sets that are connected.*

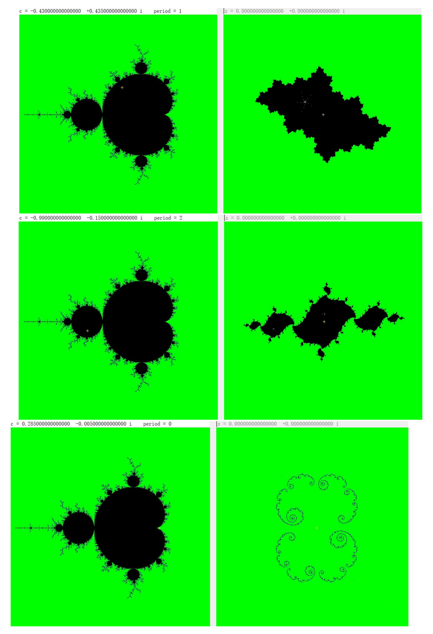

Figure 2: *Examples of filled Julia sets for different points in M*

2. Centers

Definition. *A center is a parameter $c \in \mathbb{C}$ where the orbit of 0 under f_c is periodic.*

Theorem 4. *Centers are contained in M.*

Proof. Since centers are periodic at 0, this implies that the orbit is also bounded, which shows that these points are contained in M. □

After our definition of the Mandelbrot set as points where the associated polynomial evaluated at 0 is bounded, it makes sense to ask more questions about the specifics of this orbit of z_0. It turns out that the orbit under iteration of this polynomial actually forms an $n-$cycle – that is, after n iterations of the function, we return to 0. Equivalently, if we let $f = z^2 + z_0$, then

$$f(f(...(f(0))...)) = f^{\circ n}(0) = 0.$$

Note that every attracting cycle must attract 0, as $f'(z^2 + c) = 2z \implies 0$ is a critical point of f. We also have that there is at most one attracting cycle for any quadratic polynomial, so it suffices to study the orbit of 0.

In this project, we wrote scripts to compute centers using two methods. The initial approach was to compute the polynomials $f^{\circ n}(0)$ in general, which required writing a Python script to generate these polynomials.

Solving these polynomials will in theory produce all possible centers if we consider the appropriate corresponding polynomial, but note that for any n-cycle we wish to find, we will have to compute the zeroes of a 2^n-degree polynomial. Therefore we instead chose use a heuristic based on a modified version of Newton's method to efficiently compute these centers rather than generating these polynomials and solving them naively.

Example 1. *After mapping these points on the Mandelbrot set, an interesting relation seems to appear: the centers of these polynomials are in direct correspondence with the bulbs in the Mandelbrot set.*

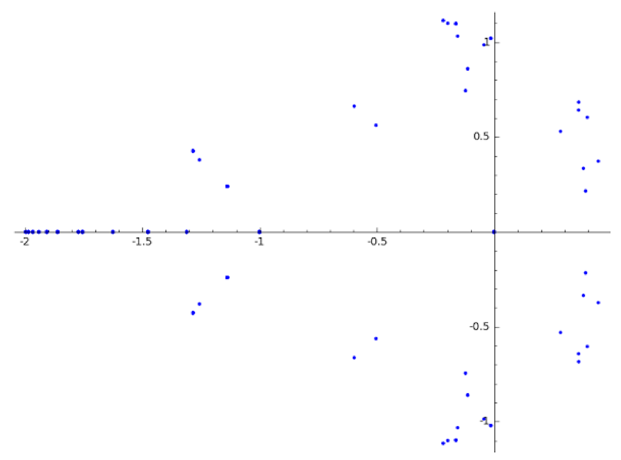

Figure 3: *Plotting the centers of M*

We also discussed the different patterns that could emerge from these polynomials, and we can see some examples taken from FractalStream below.

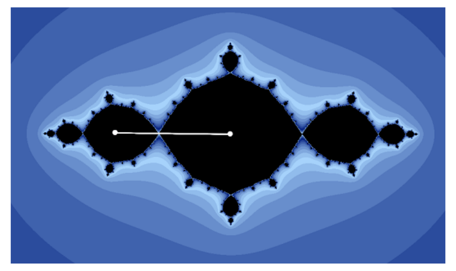

Figure 4: *This is the basilica pattern, with a cycle length of 2.*

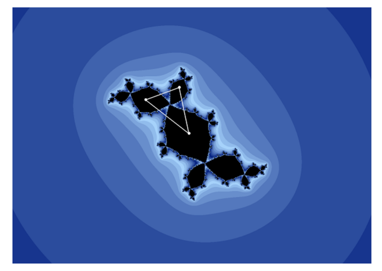

Figure 5: *The rabbit pattern, with a cycle length of 3.*

3. Internal and External Angles

Internal Angles:

If we consider the main cardioid of the Mandelbrot Set, we see that there are lots of bulbs connected to it at different points, called root points, along its boundary. The behavior of the attracting cycle in filled Julia sets whose constant comes from these bulbs can be described by internal angles.

If we pick a rational number p/q between 0 and 1, the angle $2\pi(p/q)$ is the proportion p/q around the circle. We call this angle the rational angle p/q. We can map the unit circle to the boundary of the main cardioid, and we see that the point at the rational angle p/q maps to the root point of a bulb, and points in that bulb correspond to filled Julia sets with a cycle of period q, and all bulbs connected to the main cardioid with period q have root points mapped to by a rational angle of the form s/q.

This idea can be generalized to any other bulb: a bulb with period r has bulbs of period $q * r$ connected at root points at that bulb's internal angles of the form p/q.

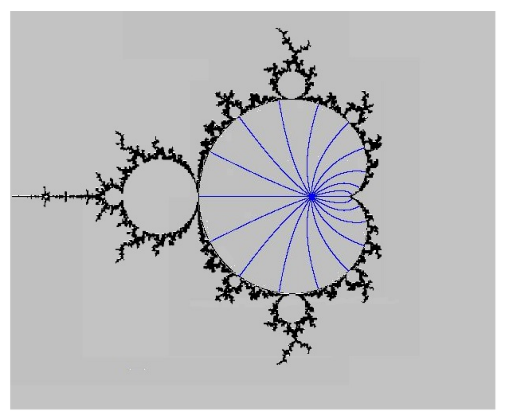

Figure 6: *Internal angles of the main cardioid for period 10 components*

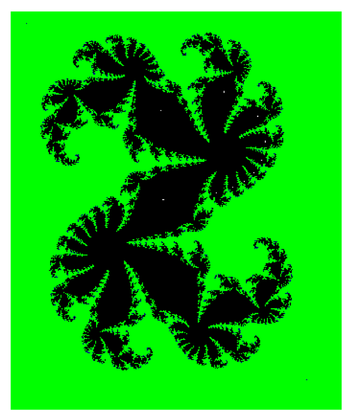

Figure 7: *An example of a Julia set with a cycle of degree 5*

External Angles and Rays:

It is possible to map the area outside the unit circle to the area outside the Mandelbrot set (by an analytic function), and this maps rays from the origin at rational angles in interesting ways.

In particular, if we choose rational angles of the form $k/(2^n - 1)$ for some n, we see that rays at these angles map to pairs of external rays that meet at the root points of bulbs whose filled Julia sets have cycles of period n. Since any two external rays do not intersect, this allows us to view components in terms of their relative position, with some bulbs being contained within the "wake" drawn by the external rays of other bulbs. This provides a useful combinatorial way of understanding the arrangement of the Mandelbrot set.

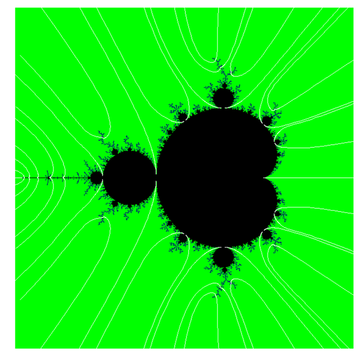

Figure 8: *External rays on the Mandelbrot set*

The external rays pair up in a way that can be understood combinatorially even without reference to the Mandelbrot set itself, allowing us to learn about the structure of the set simply by looking at the way these external angles pair up.

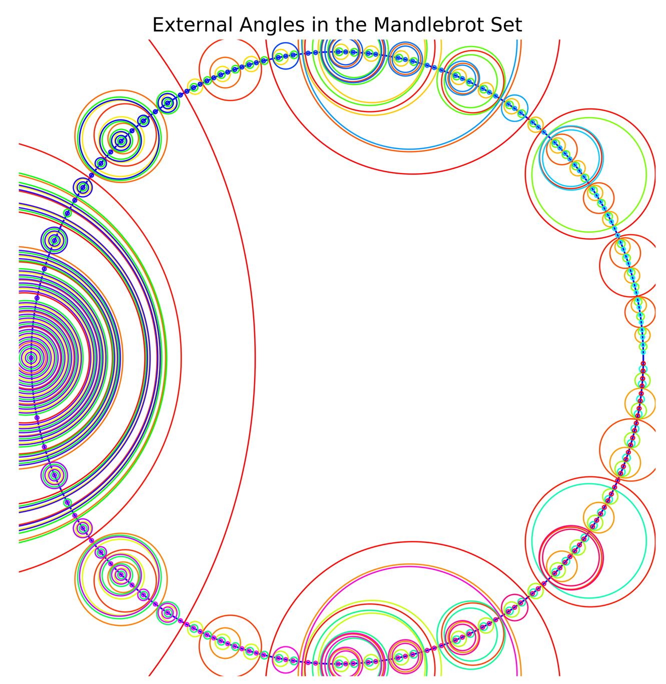

Figure 9: *A combinatorial model of external ray pairings*

4. Further Research and Questions

External angles provide a way of expressing the components of the Mandelbrot set, and the centers they contain, in terms of their relative position. This allows us to look for patterns in information about centers in a new way, with respect to our diagram of where the centers fall in the combinatorics of the set.

In order to understand this information more clearly, it is important to find a way to associate a center with the external rays it falls within. There are some patterns with this that we have observed, encoded in the way that the attracting cycles in filled Julia sets associated with a given center move between different components of the Julia set. There already exists an algorithm called the spider algorithm that takes an external angle and finds the center associated with it. Finding a way to go in the other direction would be a helpful development.

Question 1. *Find efficient ways to compute external angles given a center.*

Additionally, any further information and patterns we can determine about the patterns in the combinatorial diagram for external angles would allow us to further refine our ideas about how the Mandelbrot set fits together, and how we can understand the information encoded in its centers.

Question 2. *Continue to find more interesting patterns in the combinatorial model of the external angles of the Mandelbrot set!*