

CONCEPT QUESTIONS — LECTURE 2 — 01/06/17

Subjects: model systems, extensive and intensive variables

Question #1: Let “E” stand for extensive, and “I” for intensive. For a gas, the variables V , T , and p are

- | | |
|------------|------------|
| A) E, E, E | C) I, E, I |
| B) E, I, E | D) E, I, I |

Answer: D. The volume is the size of the system, so when we double the number of gas molecules in the system, the corresponding volume V should double as well. The volume is therefore extensive. The system stays locally the same, though, so the density of the gas (and hence the pressure p) should be unchanged. Similarly, the temperature T of the gas should remain the same in order for the local physics to be the same. p and T are therefore both intensive.

Question #2: The number density of a gas is $\rho = N/V$. The variable ρ is

- A) Extensive
- B) Intensive
- C) Neither

Answer: B. Both N and V are extensive. As the system size doubles, both N and V double, but their ratio will stay the same:

$$N \rightarrow 2N, \quad V \rightarrow 2V, \quad \frac{N}{V} \rightarrow \frac{2N}{2V} = \frac{N}{V}.$$

The number density ρ is therefore intensive, since it stays the same when we double the system size.

Question #3: For a capacitor with charge q , capacitance C , and a voltage difference of V , these variables are

- | | |
|------------|------------|
| A) I, I, E | C) E, I, E |
| B) E, I, I | D) E, E, I |

Answer: D. When we talk about doubling the system size, we mean that the area A of the plates is being doubled, not the spacing h between them. In order for the local electric field to be the same after doubling A , we need the charge density q/A to be constant, so q must double. q is therefore extensive.

The electric field magnitude for a capacitor is related to the voltage difference by

$$|\vec{E}| = \frac{V}{\epsilon h} .$$

Since h is fixed and we require the electric field to remain the same (in order for local physics to remain unchanged), the voltage must also stay the same. V is therefore intensive.

The equation of state for the capacitor is given by

$$q = CV .$$

q is extensive, and so the left-hand side of this equation doubles when the system size doubles. V is intensive, and we need both sides of the equation to scale the same way, and thus we need C to double as well. C is therefore extensive. We can also see this from the parallel-plate capacitance relation

$$C = \frac{\epsilon A}{h} ,$$

which forces C to scale with A , thereby forcing C to be extensive.

CONCEPT QUESTIONS — LECTURE 3 — 01/09/17

Subjects: thermodynamic equilibrium, the 0th law of thermodynamics, isotherms and empirical temperature

Question #1: Consider the two following made-up equations of state:

$$1.) \quad p(V + k_B T) = N k_B T$$

$$2.) \quad pV = N^2 k_B T$$

Which of these equations of state are okay?

- A) ok, ok C) not, ok
B) ok, not D) not, not

Answer: D. The first equation has a $V + k_B T$ term, which is bad for two reasons:

- (1) The units of the two terms aren't the same. V has units of volume, while $k_B T$ has units of energy.
- (2) V is an extensive variable, while $k_B T$ is an intensive variable. This is bad; we can't add two things that scale differently as we scale the system size.

The first equation of state is therefore not allowed.

The second equation is bad because the two sides of the equation scale differently. If we double the system size, p and T remain the same, while V and N both double. This means that pV doubles, while $N^2 k_B T$ quadruples! That is,

$$pV = \text{extensive} , \quad N^2 k_B T = (\text{extensive})^2 .$$

The two quantities therefore cannot be equal, and thus the second equation of state is not allowed.

CONCEPT QUESTIONS — LECTURE 4 — 01/11/17

Subjects: isotherms, the first law of thermodynamics, partial derivatives, reversibility

Question #1: An *adiabatic* process is one in which

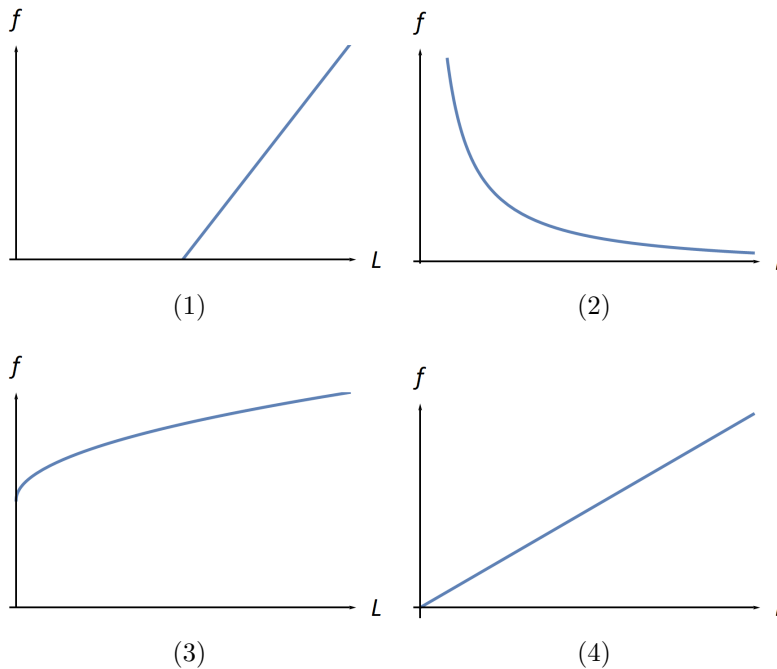
- A) No work is done C) The process is very slow
 B) No heat flows D) Both A and B

Answer: B. By definition, an adiabatic process is one where no heat flows. These processes are not necessarily slow, and it is possible for work to be done for these processes.

Question #2: Consider a spring with the Hooke's law equation of state

$$f = k(L - L_0) ,$$

where f is the force applied to the spring, k is the spring constant, L_0 is the relaxed length of the spring, and L is the displacement of the spring. Then, consider the following isotherms:



Which of these isotherms are consistent with the equation of state?

- A) 1, 3, 4 C) 1, 2
 B) 1, 4 D) 4

Answer: B. Both k and L_0 are constant at a fixed temperature, and so the equation of state therefore says that f is linear with L when the temperature is constant. An isotherm for the spring must therefore be a straight line in the (f, L) plane.

Question #3: Consider the following partial derivatives:

$$(1) \left. \frac{\partial T}{\partial p} \right|_T \quad (2) \left. \frac{\partial T}{\partial p} \right|_V \quad (3) \left. \frac{\partial p}{\partial p} \right|_T \quad (4) \left. \frac{\partial V}{\partial p} \right|_T$$

For each of these derivatives, which of the following describes them?

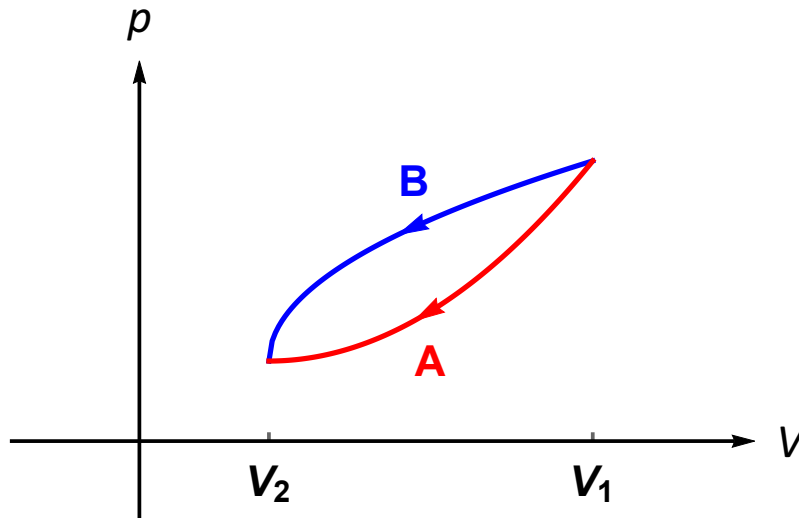
- A) Zero
- B) Non-zero but constant
- C) Not constant

Answer: 1A, 2C, 3B, 4C.

- (1) If the temperature is held constant, it cannot vary. Its derivative (with respect to anything) must therefore be zero.
- (2) This will, in general, not be a constant; it will be a function of the pressure.
- (3) The temperature is held fixed, so the pressure can vary. Its derivative with respect to itself, however, is simply 1, which is constant.
- (4) Again, this is (in general) not constant, and will instead be a function of pressure.

Subjects: work in reversible processes, heat capacities

Question #1: A gas is reversibly compressed from volume V_1 to V_2 along two different paths in the (p, V) plane, as depicted below.



Let W_A denote the work done along path A, and W_B denote the work done along path B. Which of the following statements is true?

- A) $W_A > W_B$
- B) $W_A < W_B$
- C) $W_A = W_B$

Answer: B. The work done on a gas throughout a reversible process is given by

$$W = - \int p dV .$$

Both paths have $p > 0$ and $\Delta V = V_2 - V_1 < 0$, so both paths have a positive work W that is given by the area under their corresponding curve in the (p, V) plane. Since path B lies above path A, the area under its curve is greater, and so $W_B > W_A$.

Question #2: Suppose I compress a gas from an initial volume V_1 to a final volume $V_2 < V_1$. Do I do more work when I compress the gas isothermally or adiabatically?

- A) Isothermally
- B) Adiabatically
- C) I do the same work in both cases
- D) Not enough information to decide

Answer: B. For an adiabatic process, the pressure is related to the volume by

$$p_{\text{adiabatic}} \propto \frac{1}{V^\gamma}, \quad \gamma > 1,$$

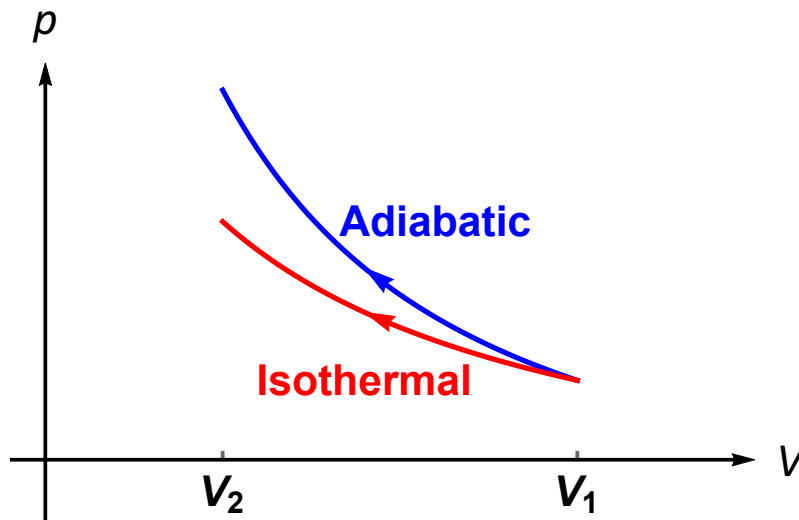
while the pressure for an isothermal process is related to the volume by

$$p_{\text{isothermal}} \propto \frac{1}{V}.$$

As V gets small, V becomes larger than V^γ for any $\gamma > 1$, and thus

$$\frac{1}{V^\gamma} > \frac{1}{V} \implies p_{\text{adiabatic}} > p_{\text{isothermal}}.$$

If we draw the two processes in the (p, V) plane, we should therefore find something like the following graph:



The area under the adiabatic curve is therefore negative and larger in magnitude than the area under the isothermal curve, and thus (by the arguments made in Question #1) the work done adiabatically is greater than the work done isothermally.

Question #3: The heat capacity at constant volume for a reversible process is given by $C_V = \left. \frac{dU}{dT} \right|_V$. Which of the following is C_V ?

- A) Extensive
- B) Intensive
- C) Neither

Answer: A. The energy U is extensive (since doubling the number of particles will double the total energy) while the temperature T is intensive (in order for the system to locally look the same). This means that dU is extensive and dT is intensive, and thus the ratio $\frac{dU}{dT}$ is extensive.

Question #4: Consider a canister of gas at a fixed volume, well-insulated from the surrounding air. We place this canister on a hot plate and heat the gas from T_1 to T_2 in one of two ways:

- (A) Slowly turn up the temperature on the hot plate, so that $T_{\text{gas}} \approx T_{\text{hot plate}}$.
- (B) Suddenly turn the hot plate temperature up to T_2 , and wait for the gas to equilibrate to T_2 .

For which method is the heat put into the gas larger for?

- A) Method A
- B) Method B
- C) Equal in both cases
- D) Not enough information to decide

Answer: C. The first thing to note is that both methods start at (T_1, V) and end at (T_2, V) . Since they both start and end at the same points, the change in energy ΔU is the same along both paths. That is, U is a state function and therefore changes in energy are path-independent.

Method A heats up the gas reversibly, so that $dW = -p dV$. The volume of the gas is fixed, and so no work is done, and thus the heat Q_A put into the system is simply $Q_A = \Delta U$, the total change of energy. Method B is not reversible, but the work done is still zero. The system remains at constant volume, and no obvious mechanical work was done when rapidly heating the gas, and so we conclude that $dW = 0$. Therefore it is also true that Q_B , the heat put into the system in method B, is equal to ΔU , the total change in energy. We therefore find that

$$Q_A = \Delta U = Q_B ,$$

and so the heat for both methods is the same.

CONCEPT QUESTIONS — LECTURE 6 — 01/18/17

Subjects: heat capacities, second law of thermodynamics, cyclic processes

Question #1: Consider a canister of gas at a fixed volume, well-insulated from the surrounding air. As discussed on last week's concept questions, if we place this canister on a hot plate and heat the gas from T_1 to T_2 in one of two ways:

- (A) Slowly turn up the temperature on the hot plate, so that $T_{\text{gas}} \approx T_{\text{hot plate}}$, or
- (B) Suddenly turn the hot plate temperature up to T_2 , and wait for the gas to equilibrate to T_2 ,

then the heat put into the gas is the same for both methods, so $Q_A = Q_B$. Let's now consider a third method:

- (C) Pick up the gas canister at temperature T_1 , shake it, then put it back on the hot plate (that has now been set to T_2) and wait for the gas to equilibrate.

How does Q_C , the heat put into the gas via method C, compare to Q_B and Q_A ?

- A) $Q_C > Q_B = Q_A$
- B) $Q_C = Q_B = Q_A$
- C) $Q_C < Q_B = Q_A$
- D) Not enough information to decide

Answer: C. The net change in energy the same for all paths, since the initial and final states are the same, and so we can conclude that

$$\Delta U = \Delta U_A = \Delta U_B = \Delta U_C .$$

As discussed last week, though, methods A and B both involve no mechanical work, and so the change in energy must come entirely from heat going into the gas. That is,

$$W_A = W_B = 0 , \quad Q_A = Q_B = \Delta U .$$

Method C is different, though. By shaking the gas, we are doing work on the system that excites the gas molecules. The work W_C for method C is therefore non-zero and positive, and so the first law tells us that

$$\Delta U = Q_C + W_C > Q_C .$$

If we compare the heats for method C and methods A and B, we therefore find that

$$Q_C < \Delta U = Q_B = Q_A ,$$

and so $Q_C < Q_B = Q_A$.

Question #2: Consider a piston in contact with a system at temperature T . Let this system expand reversibly and isothermally, so that it stays at constant temperature. And, suppose that we measure the energy of the system before and after the expansion and find no change in energy ($\Delta U = 0$). The system has expanded, so it does work on the piston and has a negative work $W < 0$, while it has absorbed some heat $Q > 0$. By conservation of energy, we find that

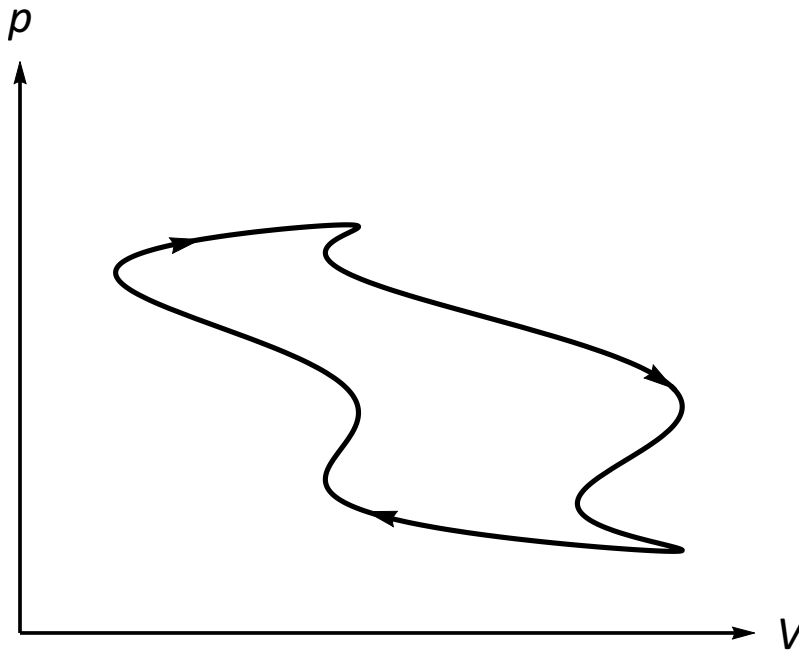
$$\Delta U = 0 = Q + W \implies -W = Q .$$

So, all of the heat into the system has been converted directly into work. Is this allowed by the second law?

- A) Yes
- B) No
- C) Not enough information to tell

Answer: A. We've changed heat completely into work, but that was not the *only* result. The system expanded, and thus ended up in a different state than it started! The second law is therefore not violated.

Question #3: Consider the following reversible, cyclic process in the (p, V) plane for a gas system. What is the sign of the work W for this cycle?



- A) $W > 0$
- B) $W < 0$
- C) $W = 0$

Answer: B. The work for a reversible, cyclic process for a gas is

$$W = - \oint p dV ,$$

where we have used the integral sign \oint instead of \int to indicate that we are integrating over a closed cycle.

The net area under the curve shown above is positive, since the lines on top are going to the right and the lines on the bottom are going to the left. The integral $\oint p dV$ is therefore positive for the cycle, and so $W < 0$.

CONCEPT QUESTIONS — LECTURE 7 — 01/20/17

Subjects: Carnot cycles, the equivalence of the Clausius and Kelvin formulations of the second law

Question #1: Consider a system (e.g. an engine) in contact with a much larger system. The two systems exchange an amount of heat Q . What condition must the heat capacity C_V of the large system satisfy so that it acts as a good reservoir?

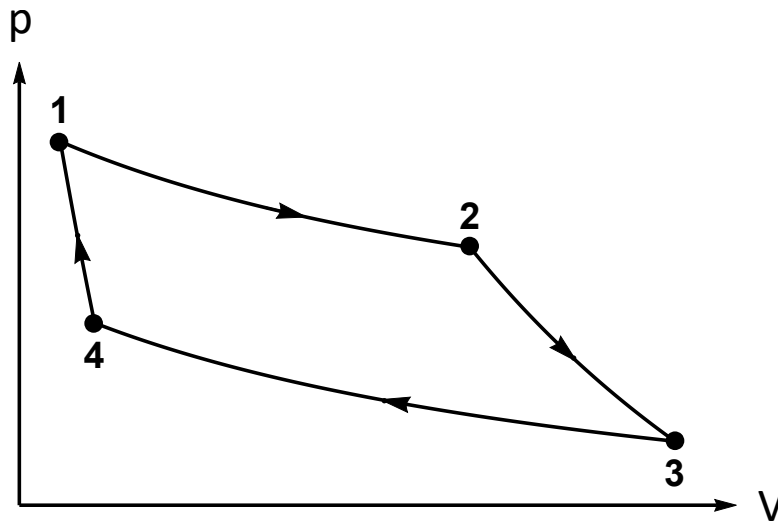
- A) C_V must be small
- B) C_V must be large
- C) It doesn't matter

Answer: B. We want the temperature of the reservoir to change very little when the heat is exchanged. That is, we want

$$\frac{dT}{dQ} \text{ small} \implies \frac{dQ}{dT} \text{ large} \implies C_V \text{ large,}$$

assuming we hold the volume of the reservoir fixed.

Question #2: Consider a gas undergoing a Carnot cycle, as depicted below:



Let W_{23} be the work done on the gas going from 2 to 3, and W_{34} be the work done on the gas going from 3 to 4. Which of the following is true?

- A) $W_{23} > 0, W_{34} > 0$
- B) $W_{23} < 0, W_{34} < 0$
- C) $W_{23} > 0, W_{34} < 0$
- D) $W_{23} < 0, W_{34} > 0$

Answer: D. The work for a reversible process for a gas is

$$W = - \int p dV ,$$

which is simply $-(\text{area under curve})$ in the (p, V) plane. The path going from 2 to 3 has $p > 0$ and $\Delta V > 0$, and so the area under the curve is positive, and thus $W_{23} < 0$. Similarly, the path from 3 to 4 has a positive pressure but a negative change in volume, and so $W_{34} > 0$.

Question #3: Again, consider the Carnot cycle plotted above. Which is true of the total work W done on the system?

A) $W > 0$

B) $W < 0$

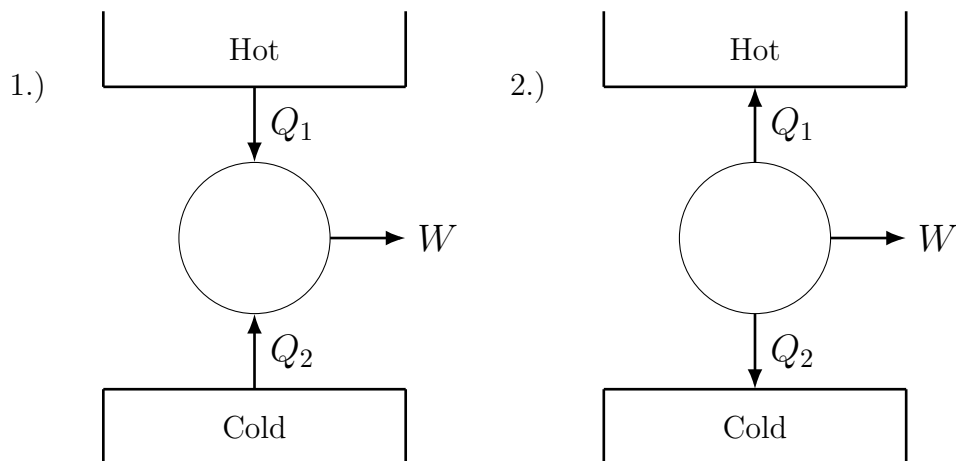
C) $W = 0$

Answer: B. The work for a reversible, cyclic process for a gas is

$$W = - \oint p dV ,$$

where we have used the integral sign \oint instead of \int to indicate that we are integrating over a closed cycle. The net area under the curve shown above is positive, since the lines on top are going to the right and the lines on the bottom are going to the left. The integral $\oint p dV$ is therefore positive for the cycle, and so $W < 0$.

Question #4: As discussed in class, the only way to get work out a Carnot engine is to have heat flow from the hot reservoir to the cold reservoir. Consider the two Carnot engine diagrams below:



For each of these Carnot engine diagrams, which of the following do they violate?

- A) Kelvin form of the second law
- B) Clausius form of the second law
- C) The first law

Answer: 1A, 2C. In the first diagram, heat is coming into the center from both reservoirs. This means that heat is being converted entirely into work (e.g. $W = Q_1 + Q_2$) with no other change in the gas, which violates the Kelvin form of the second law.

The total change of energy for a cycle is $\Delta U = 0$. In the second drawing, though, the gas is both doing work and losing heat, and so (by the first law) $\Delta U = -Q_1 - Q_2 - W < 0$. These two statements contradict each other, and so the first law is violated.

CONCEPT QUESTIONS — LECTURE 8 — 01/23/17

Subjects: Carnot theorem and efficiency, thermodynamic temperature, fridges and heat pumps

Question #1: Consider two different reversible engines A and B operating between the same two reservoirs. And, let's say that $W_A > W_B$, where $W_{A,B}$ is the work done in one cycle of engine A,B. What can we conclude about their efficiencies η_A and η_B ?

- A) $\eta_A > \eta_B$ C) $\eta_A = \eta_B$
B) $\eta_A < \eta_B$ D) Not enough information to tell

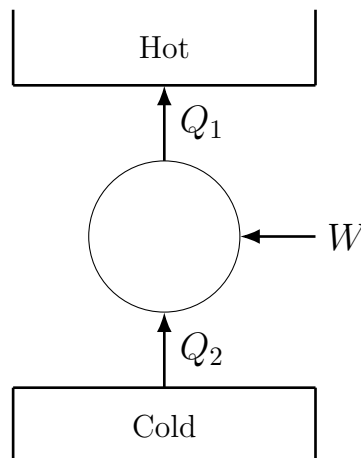
Answer: C. Any two reversible engines operating between the same heat reservoirs have the same efficiency!

To see this, you can first assume that $\eta_A > \eta_B$. Then, repeat the proof done in class, where you reverse engine B (the less efficient one) and pipe the work done from engine A into B. The net result is a perfect refrigerator where heat flows from the cold reservoir to the hot reservoir without work being done, which violates the Clausius form of the second law. This is a contradiction, and so our assumption that engine A had a greater efficiency than B is wrong. Similarly, we can repeat this and show that engine B cannot have an efficiency greater than engine A.

Question #2: If you open a fridge door, does this cool the room?

- A) Yes C) Depends on how cold the fridge is
B) No D) Depends on the temperature of the room

Answer: B. The Carnot cycle diagram for a refrigerator is the following:



The net change in energy for a cycle is $\Delta U = 0$, and so the first law tells us that

$$\Delta U = 0 = -Q_1 + Q_2 + W \implies Q_1 = Q_2 + W .$$

And, the Clausius form of the second law says that there are no perfect fridges, and so the fridge requires us to put in a non-zero amount of work W into the system. So, since $W > 0$, we find that

$$Q_1 = Q_2 + W > Q_2 \implies Q_1 > Q_2 .$$

So, the heat Q_1 put into the room is greater than the heat Q_2 removed from the fridge. Opening the fridge door will therefore not cool the room.

Subjects: fridges, the Clausius inequality

Question #1: What is the difference between an ordinary refrigerator and an air conditioner?

- | | |
|---|--|
| A) Air conditioners are reversible, fridges are not | C) The fridge is in the room, while the air conditioner is in the wall |
| B) Fridges are reversible, air conditioners are not | D) Air conditioners produce water via condensation |

Answer: C. Thermodynamically, these two systems are the same, as they use work to move heat from a cold reservoir to a hot reservoir. For the fridge, the hot reservoir is the room the fridge is in, while the cold reservoir is the inside of the fridge. For the air conditioner, the hot reservoir is the outside, while the cold reservoir is the room. So, the difference is that the air conditioner is in the wall (and heats up the outside) instead of the fridge, which is in the room (and heats up the room).

Simply stated, you can't have an air conditioner sitting inside the room you want to cool!

Question #2: Which is a better heater?

- A) A very large resistor
- B) A very efficient air conditioner

Answer: B. If we flip the air conditioner around such that it heats up the room in order to cool down the outside, it becomes a very efficient heater. Conservation of energy tells us that the heat generated by the air conditioner is

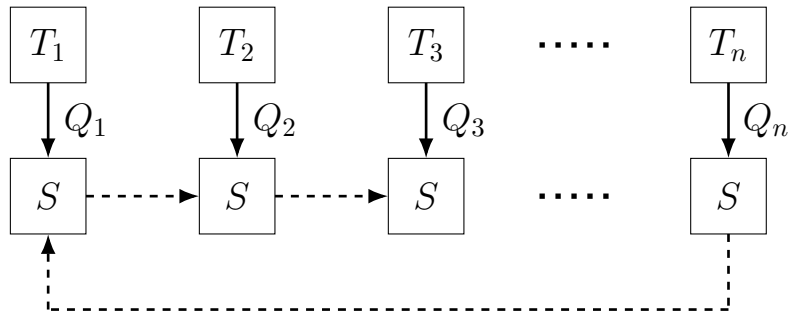
$$Q_1^{\text{A/C}} = W + Q_2 ,$$

where W is the electric work supplied to the air conditioner and Q_2 is the heat drawn from the cold reservoir (e.g. the outside). For the resistor, though, it does not draw any kind of heat from the outside, and is instead only heated up through the electrical work we do on the resistor. That is,

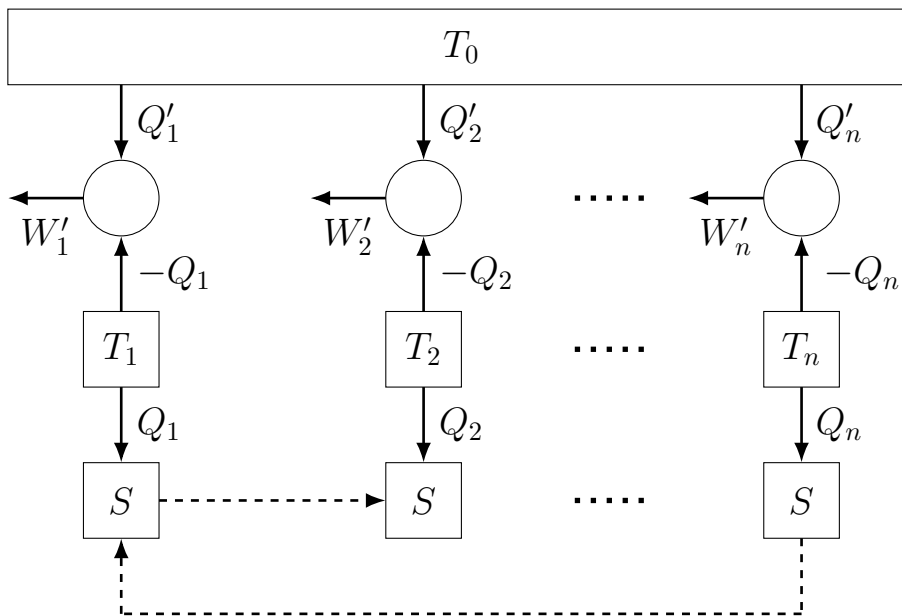
$$Q_1^{\text{resistor}} = W .$$

We can therefore conclude that $Q_1^{\text{A/C}} > Q_1^{\text{resistor}}$, and so the (flipped) air conditioner makes a better heater.

Question #3: In class, we generalized the Clausius inequality by considering a cycle process where a system S exchanges heat with n different heat reservoirs with temperatures T_1, T_2, \dots, T_n , as follows:



We then imagined hooking each of the reservoirs up to a new reservoir at temperature T_0 and placed a Carnot cycle in between T_0 and each T_i , $0 \leq i \leq n$, tuned such that no net heat flows into or out of each T_i reservoir:



Now, we will define the new system S' to be everything *except* for the T_0 reservoir. And, we will define Q as the heat going into S' and W as the net work done on S' for a whole cycle. How are Q and W related?

- A) $Q = W$ C) $Q + W < 0$
 B) $Q + W = 0$ D) $Q + W > 0$

Answer: B. The process drawn above is cyclic. And, for any cyclic process, the net change in energy is zero. Therefore the net change of energy ΔU in S' is zero for a complete cycle,

and so the first law tells us that

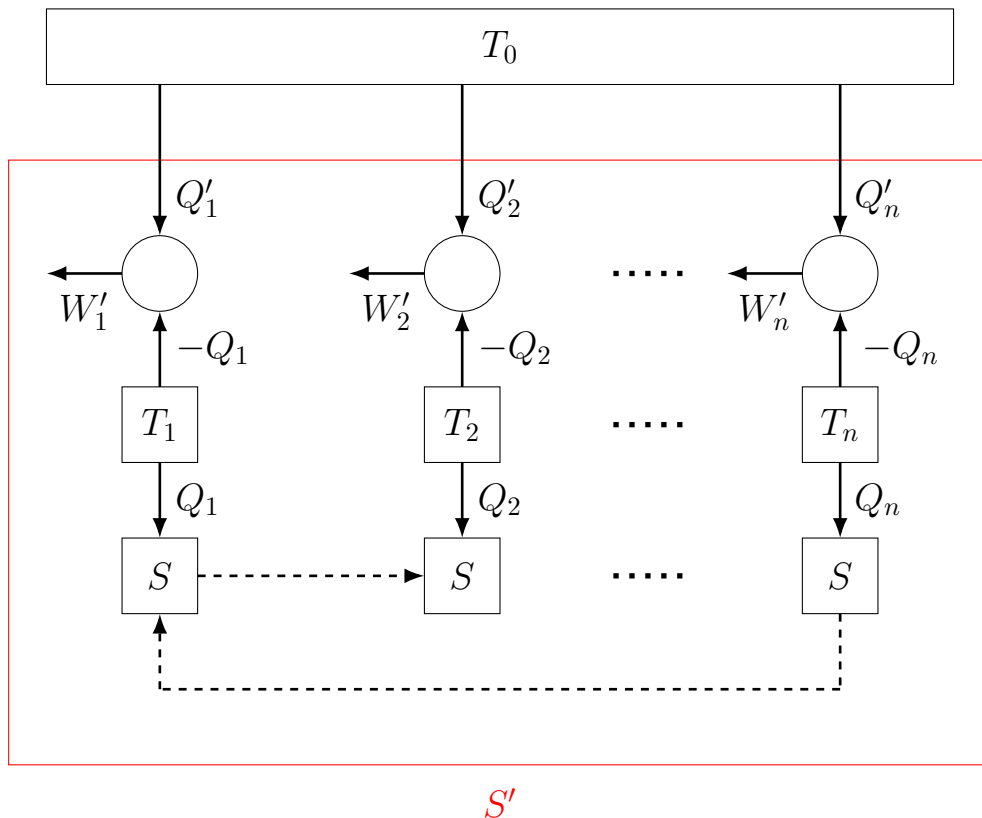
$$0 = \Delta U = Q + W \implies Q + W = 0 .$$

Note that we have used the usual sign conventions where Q and W are positive when they are going into the system S' .

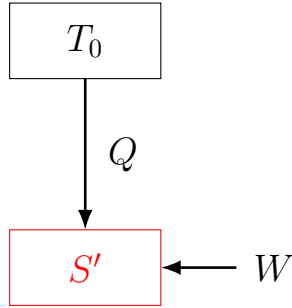
Question #4: Consider the same system described in Question #3. Which of the following is required in order for the second law of thermodynamics to be satisfied?

- A) $W = 0$
- B) $W \leq 0$
- C) $W \geq 0$

Answer: C. The system S' , defined to be everything except the T_0 reservoir, is depicted in the figure below:



From this diagram, we can clearly see that all of the heat going into S' comes entirely from the T_0 reservoir, but none of the work done on S' comes from the T_0 reservoir. We can therefore redraw this system as follows:



Note that, due to our sign conventions where work is positive when done on the system S' , we have drawn the arrow going into S' .

Now, we know from the second law that we cannot convert heat entirely into work out. And, we know from Question #3 that $W = -Q$. If the heat Q going into S' was positive, we would find that W would be negative, which would mean net work out! This would violate the second law, since this process would be converting the heat Q entirely into the net work out. Thus, we must have $Q \leq 0$. Then, using $W = -Q$, we thus find that $W \geq 0$.

CONCEPT QUESTIONS — LECTURE 10 — 01/27/17

Subjects: entropy, entropy and the first law, entropy and heat capacities, entropy for irreversible processes

Question #1: In class, we derived the Clausius inequality

$$\oint \frac{dQ}{T} \leq 0 ,$$

for any closed cycle. In this equation, what do we interpret the temperature T as?

- A) The temperature of the reservoirs giving heat
- B) The temperature of the working substance
- C) The average of A and B

Answer: A. The derivation discussed in class used T as the reservoir temperature. For reversible processes, the reservoir temperature is the same as the working substance temperature at each stage in the cycle. However, for irreversible processes, the working substance temperature may not be uniform or even well-defined! The temperature of the reservoirs is *always* defined, though, and they are what the Clausius inequality refers to.

Question #2: Which of the following is entropy?

- A) Extensive
- B) Intensive
- C) Neither

Answer: A. The change in entropy for a process going from a point A to a point B reversibly is defined by

$$S(B) = S(A) + \int_A^B \frac{dQ}{T} .$$

The heat Q is extensive, since doubling the system size doubles the amount of energy (and thus heat flow) in the system. The temperature T is intensive, since we want the physics to locally look the same. The ratio $\frac{dQ}{T}$ is therefore extensive, and so the entropy is extensive.

Question #3: Consider a gas at constant volume V heated from temperature T_A to T_B . What is the change in entropy ΔS ?

- A) $C_V(T_B - T_A)$
- B) $C_V \log\left(\frac{T_B}{T_A}\right)$
- C) $\frac{C_V}{T_A}$
- D) $\int_A^B \frac{C_V}{T} dT$

Answer: D. The heat capacity (at constant volume) is related to the entropy by

$$C_V = T \left. \frac{dS}{dT} \right|_V .$$

We can rearrange this equation to show that

$$dS \Big|_V = \frac{C_V}{T} dT \Big|_V .$$

We can then integrate this expression to show that, when the volume is held constant,

$$\Delta S = S(B) - S(A) = \int_A^B dS = \int_A^B \frac{C_V}{T} dT ,$$

which is exactly option D. Note that we have not assumed that C_V is constant. If we did, our expression would be equal to option B, but in general we have to go with option D.

CONCEPT QUESTIONS — LECTURE 11 — 01/30/17

Subjects: ΔS in reversible and irreversible processes, free expansion, bounds on entropy

Question #1: Consider a gas at constant volume. We will heat Q to the gas in one of two ways:

- (1) Slowly and reversibly
- (2) Quickly and irreversibly

How does the change in entropy ΔS for method 2 compare to ΔS in method 1?

- A) ΔS is larger for method 2 than method 1
- B) ΔS is smaller for method 2 than method 1
- C) ΔS is the same in both cases

Answer: C. The volume is held constant for both methods. Method (1) is reversible, and so $dW = -pdV$, which immediately tells us that no work is done, so $W = 0$. Method (2) is not reversible, but the work done is still zero. The system remains at constant volume, and no obvious mechanical work is done in heating the gas, and so we must have $W = 0$.

Both methods start with the same state, a gas at volume V with energy U_A . Since W is zero for both methods and the same heat Q is added in both cases, both methods give the same final state energy $U_B = U_A + Q$. Thus, the initial and final states are the same for both methods. This tells us that the change in entropy is the same for both methods, since entropy is a state function that only depends on the initial/final states. Therefore we find that ΔS is the same in both cases.

Question #2: Consider a reservoir at temperature T that loses heat $Q_{\text{out}} > 0$ reversibly to the environment. What is the change in entropy $\Delta S_{\text{reservoir}}$ of the reservoir?

- A) 0
- B) Q_{out}/T
- C) $-Q_{\text{out}}/T$

Answer: C. The temperature T of the reservoir is constant, and we know that $dS = \frac{dQ}{T}$ for reversible processes. The total change in entropy is therefore given by

$$\Delta S = \int \frac{dQ}{T} = \frac{1}{T} \int dQ = -\frac{Q_{\text{out}}}{T} .$$

Note that, in deriving this, we had to account for the fact that the usual sign convention is that heat in is positive, but Q_{out} is the heat going *out* of the system, and so $\int dQ = -Q_{\text{out}}$.

Question #3: Consider a thermally isolated gas initially at temperature T_A and volume V_A . We will consider three ways for this gas to expand to a volume $V_B > V_A$:

- (1) Slowly and reversibly
- (2) Quickly and irreversibly
- (3) Imaginary reversible process with the same endpoints as method (2).

Note that when method (2) is done sufficiently quickly, there is nothing for the gas to push against, and thus the total work $W = 0$ for method (2). Additionally, methods (1) and (2) are thermally isolated, but the imaginary process (3) is not necessarily thermally isolated! The only condition on method (3) is that it is reversible and has the same endpoints as method (2).

For each of these three processes, we want to understand the signs of the change in energy ΔU , the heat Q , the work W , and the change in entropy ΔS . That is, for each of the quantities

$$\Delta U, \quad Q, \quad W, \quad \Delta S,$$

determine whether they are

- A) positive,
- B) negative, or
- C) zero,

for each of the three methods (1), (2), and (3).

Answer: The answers are shown in the table below.

	Method (1)	Method (2)	Method (3)
ΔU	–	0	0
Q	0	0	+
W	–	0	–
ΔS	0	+	+

First, we know that method (2) happens sufficiently quickly such that $W_2 = 0$. The problem also specifies that methods (1) and (2) are thermally isolated, meaning no heat flows in these processes, and thus $Q_1 = Q_2 = 0$. And, since method (1) is reversible, the work is given by

$$dW = -pdV .$$

The gas expands ($dV > 0$), and so the work $W_1 < 0$.

The entropy for the reversible process (1) is given by

$$\Delta S = \int \frac{dQ_{\text{rev}}}{T} .$$

Since there is no heat exchanged in method (1), we can therefore conclude that $\boxed{\Delta S_1 = 0}$. We cannot say this for method (2), though, since method (2) is irreversible and does not have the same final state energy as method (1). We must therefore go on to consider the imaginary reversible process of method (3) that has the same endpoints as method (2), in order to compute ΔS_2 .

In order for method (3) to have the same endpoints as method (2), it must have the same change in energy. So, we conclude that $\boxed{\Delta U_3 = 0}$. And, since it's a reversible process ($dW = -pdV$) where the gas expands ($dV > 0$), we find that $\boxed{W_3 < 0}$. To determine Q_3 , we can use the first law to relate ΔU_3 , Q_3 , and W_3 by

$$\Delta U_3 = Q_3 + W_3 \implies Q_3 = \Delta U_3 - W_3 = -W_3 > 0 ,$$

since $\Delta U_3 = 0$ and $W_3 < 0$. We therefore find that $\boxed{Q_3 > 0}$.

Now that we know that the heat into the gas is positive for method 3, we can use the entropy formula

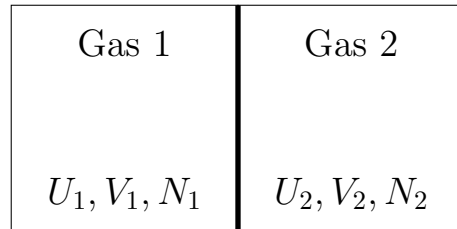
$$\Delta S = \int \frac{dQ_{\text{rev}}}{T}$$

to calculate the change in entropy in method (3). This is equal to the change in entropy for method (2), since they have the same endpoints. Since $dQ_{\text{rev}} > 0$ for method (3), we thus immediately conclude that $\boxed{\Delta S_2 = \Delta S_3 > 0}$.

CONCEPT QUESTIONS — LECTURE 12 — 02/01/17

Subjects: thermally isolated systems, entropy maximization, available work and free energy

Question #1: Consider an isolated container with a partition holding two gases:



The first gas has energy U_1 , volume V_1 , and number of gas molecules N_1 , while the second gas has energy U_2 , volume V_2 , and number of gas molecules N_2 . Consider the following four different ways that we can specify and constrain this system:

- (A) Container is insulating and the partition is fixed, so $U_1, V_1, N_1, U_2, N_2,$ and V_2 are all held constant.
- (B) The container is thermally conducting and the partition is fixed. This means that the total energy $U = U_1 + U_2$ is held constant, as are $N_1, N_2, V_1,$ and V_2 .
- (C) The container is thermally conducting and the partition is movable. This means that the total energy $U = U_1 + U_2$, the total volume $V = V_1 + V_2, N_1,$ and N_2 are all held constant.
- (D) The container is thermally conducting and the partition is removed. The only fixed quantities in this case are the total energy $U = U_1 + U_2$, the total volume $V = V_1 + V_2,$ and the total number $N = N_1 + N_2$.

Which of these cases has the highest entropy at equilibrium?

Answer: D. An isolated system in equilibrium always maximizes the entropy subject to the imposed constraints. System (D) is the least constrained of the four cases, and so there are fewer constraints to worry about when we maximize its entropy. It therefore has the most entropy.

Question #2: The Helmholtz free energy is defined by $F = U - TS$. Is the free energy F

- A) extensive,
- B) intensive, or
- C) neither?

Answer: A. The energy U of the system is extensive, since doubling the system will double the total energy. We also know that the entropy S is extensive, since it is a measure of heat flow (which doubles when the system size doubles). And, the temperature T is intensive, since we want the physics of the system to locally look the same when doubling it.

So, this means that the free energy is

$$F = (\text{extensive}) - (\text{intensive}) \times (\text{extensive}) = (\text{extensive}) .$$

CONCEPT QUESTIONS — LECTURE 13 — 02/03/17

Subjects: thermodynamic potentials, natural variables, Maxwell relations

Question #1: The Helmholtz free energy F is given by $F = U - TS$. Which of the following is a small change in free energy dF given by?

- A) $SdT - pdV$ C) $-SdT + Vdp$
 B) $-SdT - pdV$ D) $SdT + Vdp$

Answer: B. The first law of thermodynamics tells us that a small change in energy dU is given by $dU = TdS - pdV$. With this in mind, we can use $F = U - TS$ to compute dF as follows:

$$\begin{aligned} dF &= dU - TdS - SdT \\ &= (TdS - pdV) - TdS - SdT \\ &= -SdT - pdV . \end{aligned}$$

We therefore find that $dF = -SdT - pdV$, which is answer B.

Another way to derive this is to note that $F = U - TS$ indicates that F is a Legendre transform of U that swaps T and S . This Legendre transform will also introduce a sign flip for the term with T and S . We therefore find that

$$dU = TdS - pdV \xrightarrow{\text{Legendre transform of } T \text{ and } S} dF = -SdT - pdV .$$

Once again, we find answer B.

Question #2: The Gibbs free energy G is given by $G = U - TS + pV$. What are the natural variables of G ?

- A) V, S C) p, S
 B) V, T D) p, T

Answer: D. The natural variables of U are S and V , since $dU = TdS - pdV$. (The terms with differentials indicate that they are the natural variables.) Since $G = U - TS + pV$, this means that G is a Legendre transform of U that swaps both pairs of variables, i.e. $T \leftrightarrow S$ and $p \leftrightarrow V$. Both natural variables swap to their conjugates, and we therefore find that the natural variables of G are p and T .

Alternatively, we could just compute a small change dG in the Gibbs free energy. Doing so, we find that

$$dG = -SdT + Vdp .$$

The variables with differentials are p and T , and so these are the natural variables.

Question #3: In class, we discussed that the Maxwell relation derived from the thermodynamic potential U is

$$\left. \frac{\partial T}{\partial V} \right|_S = - \left. \frac{\partial p}{\partial S} \right|_V .$$

What is the Maxwell relation derived from F ?

A) $\left. \frac{\partial S}{\partial V} \right|_T = - \left. \frac{\partial p}{\partial T} \right|_V$

B) $\left. \frac{\partial V}{\partial T} \right|_S = \left. \frac{\partial S}{\partial p} \right|_V$

C) $\left. \frac{\partial S}{\partial V} \right|_T = \left. \frac{\partial p}{\partial T} \right|_V$

D) $\left. \frac{\partial S}{\partial V} \right|_p = \left. \frac{\partial p}{\partial T} \right|_S$

Answer: C. As discussed in Question #1, $F = U - TS$ is a Legendre transform of U that swaps T and S . This Legendre transform also introduces a sign flip (i.e. an extra factor of -1) in the term with T and S . So, we can take the Maxwell relation for U , flip T and S , and put an extra minus sign in front of S . We therefore find that

$$\left. \frac{\partial T}{\partial V} \right|_S = - \left. \frac{\partial p}{\partial S} \right|_V \xrightarrow{\text{Legendre transform of } T \text{ and } S} \left. \frac{\partial S}{\partial V} \right|_T = \left. \frac{\partial p}{\partial T} \right|_V .$$

This relation is the one given in C.

CONCEPT QUESTIONS — LECTURE 14 — 02/06/17

Subjects: Applications of thermodynamics: C_p and C_V , sea currents

Question #1: In class, we defined the expansivity β_p (at constant pressure) and the compressibility κ_T (at constant temperature) as

$$\beta_p = \frac{1}{V} \left. \frac{dV}{dT} \right|_p, \quad \kappa_T = -\frac{1}{V} \left. \frac{dV}{dp} \right|_T.$$

Which of the following describes β_p , κ_T ?

- | | |
|-------------------------|-------------------------|
| A) extensive, extensive | C) intensive, extensive |
| B) extensive, intensive | D) intensive, intensive |

Answer: D. Both quantities have factors of the volume V in both the numerator and denominator. V is extensive, but since V appears in both the numerator and denominator, this cancels, and we don't have to worry about it. Then, since the pressure p and the temperature T are both intensive, we conclude that β_p and κ_T are both intensive.

Question #2: An ocean current carries a large mass of water down 1 km. We want to know the change in temperature ΔT for this process. Which of the following derivatives do we need to know to find ΔT ?

- | | |
|-------------------------------------|-------------------------------------|
| A) $\left. \frac{dT}{dp} \right _V$ | C) $\left. \frac{dT}{dp} \right _S$ |
| B) $\left. \frac{dT}{dS} \right _p$ | D) $\left. \frac{dp}{dV} \right _T$ |

Answer: C. We want to know the change in temperature, and so we need a derivative of T (which rules out answer D). As the water goes down 1 km, its pressure is increasing, so we want to know how the temperature changes as a function of pressure (which rules out answer B).

This leaves us with only two possible options: A and C. Both are derivatives of the form $\frac{dT}{dp}$, but A holds V constant while C holds S constant. The volume is clearly not constant; the ocean current is not held in a container of fixed size, and so it's free to expand or contract throughout this process. The ocean current does carry a large mass of water, though. And, when the system size is big enough, it has a small surface area-to-volume ratio, and so the amount of heat absorbed by the water is negligible. We can therefore neglect the heat in this process, which means that the entropy is (approximately) constant. We therefore conclude that answer C will give us ΔT for this process.

CONCEPT QUESTIONS — LECTURE 15 — 02/08/17

Subjects: Applications of thermodynamics: ideal gas, adiabatic expansion

Question #1: For a gas whose energy depends only on temperature (i.e. $U = U(T)$), we derived in class that the pressure is given by

$$p = T \left. \frac{dS}{dV} \right|_T .$$

Our goal is to find the equation of state for the gas, which will give the pressure in terms of volume and temperature (i.e. $p = p(V, T)$). What should I use to transform the above equation to find $p(V, T)$?

- | | |
|------------------------|-----------------------|
| A) Chain rule | C) Reciprocal theorem |
| B) Reciprocity theorem | D) Maxwell relations |

Answer: D. We want to get rid of S entirely in the above equation in favor of V and T . The only way to swap out variables like this is through the use of Maxwell relations.

Question #2: We found that, for a gas whose temperature does not change during free expansion, and that also satisfies the Boyle experiment, the equation of state is

$$pV = aT ,$$

for some constant a that does not depend on p , V , or T . Is the constant a

- A) extensive,
 B) intensive, or
 C) neither?

Answer: A. We can rearrange the above equation as

$$a = \frac{p}{T} V .$$

Both p and T are intensive, while V is extensive, and so a is extensive.

Question #3: Consider an ideal gas expanding reversibly and adiabatically. Which of the following derivatives do we need to know to find p versus V for this process?

- | | |
|-------------------------------------|-------------------------------------|
| A) $\left. \frac{dp}{dV} \right _T$ | C) $\left. \frac{dp}{dV} \right _S$ |
| B) $\left. \frac{dp}{dS} \right _V$ | D) $\left. \frac{dS}{dV} \right _T$ |

Answer: C. To find p versus V , we need a derivative of the form $\frac{dp}{dV}$, which rules out B and D. An adiabatic process is one in which no heat is exchanged, so $dQ = 0$. And, since it is also a reversible process, we know that $dQ = TdS$, and thus $dS = 0$ for the process. That is, the entropy S is held constant. Thus, the answer is C.

Question #4: In class, we derived that, for an ideal gas undergoing adiabatic, reversible expansion/compression, the pressure and the volume are related by

$$pV^\gamma = \text{constant} ,$$

where $\gamma = \frac{C_p}{C_V}$ is assumed to be constant. If the ideal gas expands adiabatically and reversibly, what happens to its temperature?

- A) Increases C) Stays the same
B) Decreases D) Increases, then decreases

Answer: B. There are two ways to see that the temperature decreases:

Mathematically: If we use the ideal gas law, we can rewrite the adiabatic relation as

$$TV^{\gamma-1} = \text{constant} .$$

And, we showed in class that $C_p > C_V$, and thus $\gamma > 1$. So, $\gamma - 1 > 0$, and so if we increase V , T must decrease (in order for the above expression to be constant).

Physically: $Q = 0$ for an adiabatic process, and $W = -\int pdV < 0$ for a reversible expansion (since $dV > 0$). The total change in energy ΔU of the gas is therefore negative, since

$$\Delta U = Q + W = W < 0 .$$

For an ideal gas, we know that U depends only on temperature. And, we know that the heat capacity $C_V > 0$, which means that

$$C_V = \left. \frac{dU}{dT} \right|_V > 0 .$$

Therefore, if U decreases, T must decrease as well. And, we showed that $\Delta U < 0$, and therefore $\Delta T < 0$ for this ideal gas expansion.

CONCEPT QUESTIONS — LECTURE 16 — 02/10/17

Subjects: Adiabatic expansion, counting microstates and macrostates, multiplicity

Question #1: In class, we showed that when a gas undergoes adiabatic expansion with $\beta_p > 0$, the gas cools down as it expands. That is,

$$\left. \frac{dT}{dV} \right|_S < 0, \quad \text{if } \beta_p > 0.$$

Conversely, this means that if a gas is adiabatically compressed, it will heat up.

Now, consider a typical gas station pump, where gas is funneled from a large tank into a small nozzle and then pumped into your car. Does the nozzle of the gas pump heat up as the gas is compressed into the nozzle?

- A) Yes B) No

Answer: B. Gas station pumps are not adiabatic processes. They typically have some mechanism that forces the gas to come out at a constant pressure (instead of at a constant S), which means that the above relation does not hold.

Question #2: Consider a system of N spins that can each point either up or down. How many possible microstates are there?

- A) N C) N^2
 B) $2N$ D) 2^N

Answer: D. Each of the N spins has two possible states, and so the total number of possible microstates is given by

$$\# \text{ of microstates} = \underbrace{2}_{\# \text{ choices for spin 1}} \times \underbrace{2}_{\# \text{ choices for spin 2}} \times \dots \times \underbrace{2}_{\# \text{ choices for spin } N} = 2^N.$$

We therefore find 2^N microstates.

Question #3: The multiplicity $g(N, s)$ is defined as the number of microstates of N spins with a spin excess of s . What is the sum

$$\sum_{s=-N/2}^{s=+N/2} g(N, s)$$

equal to?

- A) N C) N^2
B) $2N$ D) 2^N

Answer: D. The multiplicity $g(N, s)$ is the number of microstates for a particular value of N and a particular value of s . The sum is over all possible values of s , and so the sum is simply the total number of possible microstates for N spins. We therefore get the same answer as from Question 2.

CONCEPT QUESTIONS — LECTURE 17 — 02/13/17

Subjects: multiplicity $g(N, s)$, large N behavior of $g(N, s)$

Question #1: Let $g(N, s)$ be the multiplicity, the total number of microstates for N spins with spin excess s . The total number N and the spin excess s are related to N_{\uparrow} , the number of spins pointing up, and N_{\downarrow} , the number of spins pointing down, by

$$N = N_{\uparrow} + N_{\downarrow}, \quad s = \frac{N_{\uparrow} - N_{\downarrow}}{2}.$$

We can also rewrite this as

$$N_{\uparrow} = \frac{N}{2} + s, \quad N_{\downarrow} = \frac{N}{2} - s.$$

Compute the multiplicity $g(N, s)$ for the following values of N and s :

$$(N, s) = \left\{ (2, 0), (2, 1), (2, -1), (3, 1/2), (3, 3/2) \right\}.$$

Answer:

$$\begin{aligned} g(2, 0) &= 2, \\ g(2, 1) &= 1, \\ g(2, -1) &= 1, \\ g(3, 1/2) &= 3, \\ g(3, 3/2) &= 1. \end{aligned}$$

For $N = 2$, the spins can either both point up, both point down, or one can point up and one can point down. There are therefore two microstates with $s = 0$, namely $(\uparrow\downarrow)$ or $(\downarrow\uparrow)$, and so

$$g(2, 0) = 2.$$

The state $(\uparrow\uparrow)$ has $s = 1$, while the state $(\downarrow\downarrow)$ has $s = -1$. These are the only microstates when $N = 2$ with $s = \pm 1$, and so we find that

$$g(2, 1) = g(2, -1) = 1.$$

When $N = 3$, having two spins up and one spin down results in a spin excess of $s = \frac{2-1}{2} = \frac{1}{2}$, while having all spins up results in a spin excess of $s = \frac{3}{2}$. There are three ways to have two spins up and one spin down, namely $(\uparrow\uparrow\downarrow)$, $(\uparrow\downarrow\uparrow)$, and $(\downarrow\uparrow\uparrow)$. These microstates are all of the $s = 1/2$ microstates, and so we find that

$$g(3, 1/2) = 3.$$

For the $s = 3/2$ case, though, there is only one way to have a state with all spins pointing up: $(\uparrow\uparrow\uparrow)$. So, since there's only one corresponding microstate, we find that

$$g(3, 3/2) = 1.$$

Question #2: As derived in class, the multiplicity $g(N, s)$ can be written as

$$g(N, s) = \frac{N!}{N_{\uparrow}! N_{\downarrow}!} .$$

What value of s maximizes $g(N, s)$?

- A) $N/2$ C) 0
B) $-N/2$ D) N

Answer: C. We know from symmetry that $g(N, s) = g(N, -s)$, since the number of microstates with a spin excess in the $+\hat{z}$ direction should equal the number of microstates with a spin excess in the $-\hat{z}$ direction. So, this tells us that at $s = 0$, $g(N, s)$ either has a local maximum or a local minimum.

We now need to rule out the other answers. Answer D cannot be correct, since s can never go above $s = N/2$. To rule out answer A, we plug in $s = N/2$ and find that

$$g(N, s = N/2) = \frac{N!}{N!0!} = 1 ,$$

so $s = N/2$ is clearly a minimum. Similarly, $s = -N/2$ will also correspond to a minimum for $g(N, s)$. The only remaining option is $s = 0$.

CONCEPT QUESTIONS — LECTURE 19 — 02/17/17

Subjects: spin systems in the large N limit, Stirling's approximation, the Gaussian approximation, probability

Question #1: Using Stirling's approximation and the Gaussian approximation, we derived the number of microstates $g(N, s)$ for a system of N spins with spin excess s to be

$$\frac{g(N, s)}{g(N, 0)} = \exp\left(-\frac{2s^2}{N}\right), \quad \text{where } g(N, 0) = 2^N \sqrt{\frac{2}{\pi N}}.$$

What single variable can we view the ratio $\frac{g(N, s)}{g(N, 0)}$ to be a function of?

- A) $s\sqrt{N}$ C) s/\sqrt{N}
 B) s D) s/N

Answer: C. We can rewrite the ratio as

$$\frac{g(N, s)}{g(N, 0)} = \exp\left(-2\left(\frac{s}{\sqrt{N}}\right)^2\right),$$

which is clearly a function only of s/\sqrt{N} .

Question #2: Consider a system with g microstates, each of them labeled by j . And, let's assume that all of the microstates are equally likely, so $P(j)$ is the same for all microstates. What is $P(j)$?

- A) g C) $1/\sqrt{g}$
 B) 1 D) $1/g$

Answer: D. All of the states are equally likely, and we need the sum of the probabilities to equal 1, so $\sum_j P(j) = 1$. There are g total states, and we therefore find that $P(j) = \frac{1}{g}$, so that

$$\sum_j P(j) = \sum_j \frac{1}{g} = g \times \frac{1}{g} = 1.$$

As an example, consider flipping a coin. There are two possible outcomes (heads or tails), and so $g = 2$. And, the probability of getting heads is $1/2 = 1/g$, as is the probability of getting tails. We can also consider a six-sided dice. There are six possible rolls, each of which is equally likely with a probability of $1/6$.

Question #3: Consider flipping two different coins. What are the probabilities of measuring the following outcomes?

- (1) The first coin is heads, the second coin is tails.
- (2) One of the coins is heads, one of the coins is tails.

- A) $1/4$ C) $1/2$
B) $1/3$ D) $3/4$

Answer: 1A, 2C. When flipping two coins, there are four possible results:

HH , HT , TH , TT ,

where H stands for heads and T for tails. All of these states are equally likely, with a probability of $1/4$ each. Only one result gives a heads for the first coin and a tails for the second coin, so $P_1 = 1/4$. There are two outcomes with one heads and one tails, though, since both HT and TH satisfy this. The total probability P_2 for this is therefore $P_2 = 1/4 + 1/4 = 1/2$.

Question #4: Consider two independent systems (whose states don't affect one another) with g_1 and g_2 total microstates, respectively. What is the number of microstates of the combined system?

- A) $g_1 + g_2$ C) $\max(g_1, g_2)$
B) $\sqrt{g_1 g_2}$ D) $g_1 g_2$

Answer: D. Each microstate of the combined system corresponds to choosing one of the microstates of the first system and one of the microstates of the second system. There are g_1 ways to choose which of the microstates the first system is in, while there are g_2 ways to choose which of the microstates the second system is in. The total number of microstates is therefore $g_1 g_2$.

For example, consider flipping a coin and rolling a dice. There are two possible outcomes for the coin flip (heads or tails) and six possible dice rolls. We will denote the combined system microstates by (coin flip result, dice roll result). We can list out all of the combined system microstates as follows:

$(H, 1)$, $(H, 2)$, $(H, 3)$, $(H, 4)$, $(H, 5)$, $(H, 6)$,
 $(T, 1)$, $(T, 2)$, $(T, 3)$, $(T, 4)$, $(T, 5)$, $(T, 6)$.

Clearly, there are $12 = 2 \times 6$ total microstates, as our earlier formula predicted.

CONCEPT QUESTIONS — LECTURE 20 — 02/20/17

Subjects: probability, spin systems

Question #1: Suppose we flip five coins. What is (1) $P(\text{HTHTT})$ and (2) $P(\text{exactly 2 heads})$?

- A) $1/32$ C) $5/16$
 B) $1/2$ D) 1

Answer: 1A, 2C. There are five coins, each with two possible coin flip results, so the total number of microstates is

$$g = 2^5 = 32 .$$

Each outcome is equally likely, with a probability of $1/g = 1/32$. This means that the probability of the configuration HTHTT is $P(\text{HTHTT})$.

For the second case, we can compute the probability by noting that

$$P(\text{exactly 2 heads}) = \frac{\text{number of states with 2 heads}}{\text{total number of states}} = \frac{\text{number of states with 2 heads}}{32} .$$

To compute the number of states with two heads, we recall our spin microstate counting formula:

$$g(N, N_{\uparrow}) = \frac{N!}{N_{\uparrow}!N_{\downarrow}!} ,$$

which counts the number of states in a spin system with N spins that have N_{\uparrow} spins pointing up and N_{\downarrow} spins pointing down. The same formula works for counting coin flips, since each coin can be thought of as a spin (since there are exactly two possible spin directions and two possible coin flips). So, we find that the number of ways to flip N coins and get N_H heads is

$$g(N, N_H) = \frac{N!}{N_H!N_T!} .$$

If we plug in $N = 5$, $N_H = 2$, and $N_T = 3$, we find that

$$g(N = 5, N_H = 2) = \frac{5!}{2!3!} = 10 = \text{number of ways to flip 5 coins and get 2 heads} .$$

We therefore find that the probability of getting two heads is

$$P(\text{exactly 2 heads}) = \frac{\text{number of states with 2 heads}}{\text{total number of states}} = \frac{10}{32} = \frac{5}{16} .$$

Question #2: Consider a spin system with N spins, where $N \gg 1$. Using Stirling's approximation and the Gaussian approximation, the probability of finding the system to have a spin excess s is

$$P(N, s) \approx \sqrt{\frac{2}{\pi N}} \exp\left(-\frac{2s^2}{N}\right) .$$

If we were to compute the average value $\langle s^2 \rangle$, what would we expect it to be proportional to?

- A) N^0 C) N
 B) $N^{1/2}$ D) N^2

Answer: C. As discussed in class, the width of the Gaussian distribution is roughly \sqrt{N} , which means that we expect $|s| \sim \sqrt{N}$. We can therefore conclude that we expect $|s^2| \sim N$.

Question #3: For the spin system described in Question #2, we used the Gaussian approximation in class to compute that

$$\langle s \rangle = 0, \quad \langle s^2 \rangle = \frac{N}{4}.$$

What is the $\langle N_{\uparrow}^2 \rangle$?

- A) $N^2/4$ C) $N/4$
 B) N^2 D) $N^2/4 + N/4$

Answer: D. Recall that the number of spins pointing up N_{\uparrow} can be expressed in terms of N and s as

$$N_{\uparrow} = \frac{N}{2} + s,$$

and so we can compute that the square of this quantity is

$$N_{\uparrow}^2 = \frac{N^2}{4} + sN + s^2.$$

The average is therefore

$$\langle N_{\uparrow}^2 \rangle = \left\langle \frac{N^2}{4} + sN + s^2 \right\rangle.$$

Averages are linear, and so we can just take the average of each individual term in the sum:

$$\langle N_{\uparrow}^2 \rangle = \left\langle \frac{N^2}{4} \right\rangle + \langle sN \rangle + \langle s^2 \rangle.$$

The number of spins N is constant, and so we can rewrite this as

$$\langle N_{\uparrow}^2 \rangle = \frac{N^2}{4} + N\langle s \rangle + \langle s^2 \rangle.$$

Plugging in $\langle s \rangle = 0$ and $\langle s^2 \rangle = \frac{N}{4}$, we therefore find that

$$\langle N_{\uparrow}^2 \rangle = \frac{N^2}{4} + \frac{N}{4}.$$

Concept Questions – Lecture 21 – 02/22/17

Subjects: probability, averages, and variance

Question #1: Suppose we have some variable x that can take two values: $x = b + a$, or $x = b - a$. Both of these possibilities have a probability of $P = 1/2$. What is the variance of x ?

- A) $a^2 + b^2$ C) a^2
B) b^2 D) $b^2 - a^2$

Answer: C Since both states are equally likely with probability $1/2$, the average of x is given by

$$\langle x \rangle = \frac{1}{2}(b + a) + \frac{1}{2}(b - a) = b . \quad (1)$$

The variance of x is therefore given by

$$\begin{aligned} \text{var}(x) &= \langle (x - \langle x \rangle)^2 \rangle \\ &= \langle (x - b)^2 \rangle \\ &= \frac{1}{2}(b + a - b)^2 + \frac{1}{2}(b - a - b)^2 \\ &= a^2 . \end{aligned} \quad (2)$$

Concept Questions – Lecture 22 – 02/24/17

Subjects: fundamental assumption of stat mech, spin systems with a magnetic field, systems in thermal contact

Question #1: Consider a system of spins with no magnetic field applied. If there are $N = 5$ spins, what are the following probabilities?

- 1.) $P(\uparrow\downarrow\uparrow\downarrow\uparrow)$, the probability of the spins being in the $(\uparrow\downarrow\uparrow\downarrow\uparrow)$ state.
- 2.) $P(s = 1/2)$, the probability of having a spin excess of $s = 1/2$.

- A) 1/32 C) 5/16
B) 1/2 D) 5/32

Answer: 1A, 2C. For $N = 5$ spins, there are $g = 2^N = 32$ possible microstates. There is an equal probability of being in any of these 32 states, and so the probability of being in the $(\uparrow\downarrow\uparrow\downarrow\uparrow)$ state is

$$P(\uparrow\downarrow\uparrow\downarrow\uparrow) = \frac{1}{32} . \quad (1)$$

For the second part of the problem, we first recall that the spin excess is given by

$$s = \frac{N_{\uparrow} - N_{\downarrow}}{2} . \quad (2)$$

In order to have $s = 1/2$ when there are $N = 5$ total spins, we are forced to have three spins pointing up and two spins pointing down, e.g. $N_{\uparrow} = 3$ and $N_{\downarrow} = 2$. The corresponding number of states with $s = 1/2$ is therefore obtained by figuring out how many different ways there are to pick three of the spins to be pointing up. So,

$$\# \text{ of states with } s = 1/2 \text{ is } \binom{N}{N_{\uparrow}} = \binom{5}{3} = \frac{5!}{3!2!} = 10 . \quad (3)$$

The probability of being in a state with $s = 1/2$ is therefore given by

$$P(s = 1/2) = \frac{\# \text{ of states with } s = 1/2}{\text{total } \# \text{ of states}} = \frac{10}{32} = \frac{5}{16} . \quad (4)$$

Question #2: Consider a system of spins with a non-zero magnetic field, denoted by B , in the $+\hat{z}$ direction. The total energy of the system is related to the spin excess s by

$$U = -2mBs , \quad (5)$$

where m is the magnetic moment of each spin.

Let's now consider a particular example with $N = 5$ spins where we know that the total energy of the system is $U = -mB$. Given these facts, what are the following probabilities?

- 1.) $P(\uparrow\downarrow\uparrow\downarrow\uparrow)$, the probability of the spins being in the $(\uparrow\downarrow\uparrow\downarrow\uparrow)$ state.
- 2.) $P(s = 1/2)$, the probability of having a spin excess of $s = 1/2$.
- 3.) $P(\uparrow\downarrow\downarrow\uparrow\downarrow)$, the probability of the spins being in the $(\uparrow\downarrow\downarrow\uparrow\downarrow)$ state.

- | | | |
|---------|------|---------|
| A) 1/32 | C) 1 | E) 1/10 |
| B) 5/16 | D) 0 | |

Answer: 1E, 2C, 3D. We know the energy is $U = -mB$ in this particular example. And, equation (5) tells us that U is related to the spin excess s by $U = -2mBs$. The spin excess is therefore fixed at $s = 1/2$ in our example.

For the first part of the question, we first note that (as discussed in the previous question) there are $\binom{5}{3} = 10$ states that have $s = 1/2$. These states have three spins pointing up and two spins pointing down. The $(\uparrow\downarrow\uparrow\downarrow\uparrow)$ state is therefore one of these states. And, all 10 of these states are equally probable, and so the probability of being in any one of them is simply $1/10$. We therefore conclude that

$$P(\uparrow\downarrow\uparrow\downarrow\uparrow) = \frac{1}{\# \text{ of states with } s = 1/2} = \frac{1}{10} . \quad (6)$$

The spin excess is forced to be $s = 1/2$ when $U = -mB$. Another way of saying this is that we have a probability of 1 to have $s = 1/2$. So,

$$P(s = 1/2) = 1 . \quad (7)$$

Finally, we need to look at the probability of being in the $(\uparrow\downarrow\downarrow\uparrow\downarrow)$ state. Note, however, that this state does *not* have a spin excess of $s = 1/2$! It has three spins pointing down and two up, and so it has $s = -1/2$. When $U = -mB$, we are therefore not allowed to be in this state. So,

$$P(\uparrow\downarrow\downarrow\uparrow\downarrow) = 0 . \quad (8)$$

Concept Questions – Lecture 23 – 03/06/17

Subjects: strongly peaked distributions, two systems in thermal contact, entropy and $\log(g)$

Question #1: Consider a system of a large number $N = 10^{22}$ of spins. How small must the the spin excess s be before the probability $P(N, s)$ becomes appreciable?

- A) 10^{-22} C) 10^{11}
B) 10^0 D) 10^{22}

Answer: C. Since we have a large number N of spins and we're considering relatively small spin values for the excess s , the Gaussian approximation is valid. The probability of having a spin excess of s is therefore approximated by

$$P(N, s) \approx \sqrt{\frac{2}{\pi N}} \exp\left(-\frac{2s^2}{N}\right). \quad (1)$$

In order for this probability to be non-negligible, we need it not to be exponentially small. That is, we cannot have $s^2 \gg N$, otherwise the exponential would be approximately zero. We therefore need $s^2 \lesssim N$, and thus

$$s = 10^{11} \quad (2)$$

is when the probability starts to become non-negligible.

Concept Questions – Lecture 24 – 03/08/17

Subjects: entropy and randomness, the ideal gas

Question #1: We have learned from quantum mechanics that a particle with mass m inside a 3D box with side lengths L has energy levels

$$E_{\vec{n}} = \frac{\hbar^2}{2m} \left(\frac{\vec{n}\pi}{L} \right)^2, \quad \vec{n} = (n_x, n_y, n_z), \quad (1)$$

where the quantum numbers n_x , n_y , and n_z define the quantum state the particle is in, and each of these numbers can take values of 1, 2, \dots . Let's now define the wave vector \vec{k} by

$$\vec{k} = \frac{\pi}{L} \vec{n}, \quad (2)$$

so that the energy takes the simpler form $E_{\vec{k}} = \frac{\hbar^2}{2m} k^2$. How many states are there in a volume of size $\Delta^3 \vec{k}$ in \vec{k} -space?

- A) $\frac{L}{\pi} \Delta^3 \vec{k}$ C) $\Delta^3 \vec{k}$
B) $\frac{L^3}{\pi^3} \Delta^3 \vec{k}$ D) $L^3 \Delta^3 \vec{k}$

Answer: B. $\Delta^3 \vec{k}$ has units of $1/L^3$, and so we need a factor of L^3 in front of it in order to obtain a dimensionless number of states. This immediately tells us that answers A and C are wrong. The correct answer is B, since we need the factors of π that come from the definition of \vec{k} .

Another way to see this is to break the system up into its components. We know that the number of states in a length Δk in 1D is given by

$$N_{1D} = \frac{L}{\pi} \Delta k. \quad (3)$$

The number of states in 3D will simply be the number of states in each of the x , y , and z directions multiplied together:

$$N_{3D} = \left(\frac{L}{\pi} \Delta k_x \right) \left(\frac{L}{\pi} \Delta k_y \right) \left(\frac{L}{\pi} \Delta k_z \right) = \frac{L^3}{\pi^3} \Delta k_x \Delta k_y \Delta k_z = \frac{L^3}{\pi^3} \Delta^3 \vec{k}. \quad (4)$$

Concept Questions – Lecture 25 – 03/10/17

Subjects: canonical ensemble, partition function

Question #1: Consider the composite system shown below, where a system S is connected to another system R . R is very large compared to S , so we can think of R as a reservoir.

The total energy of the combined system $U = U_R + U_S$ is fixed. Let $g_S(U_S)$ be the number of microstates in S with energy U_S , and let $g_R(U_R)$ be the number of microstates in R with energy U_R . How many states of the combined system have an energy U_S in S ?

- A) $g_S(U_S)$ C) $g_R(U - U_S)g_S(U_S)$
B) $g_R(U - U_S)$ D) $\sum_{U_S} g_R(U - U_S)g_S(U_S)$

Answer: C. The small system S has energy U_S , and has a corresponding number of microstates $g_S(U_S)$. The reservoir R has energy $U_R = U - U_S$, and the corresponding number of microstates is $g_R(U - U_S)$. The total number of microstates of $R + S$ is the product of these, since we have to account for every possible microstate of R in conjunction with every possible microstate of S . So,

$$\# \text{ of microstates with energy } U_S \text{ in } S = g_R(U - U_S)g_S(U_S) . \quad (1)$$

Question #2: Let's label the microstates of S by the index j . We'll also denote the energy of each of these microstates by ε_j . How many states of the combined $R + S$ system have S in microstate j ?

- A) $g_S(\varepsilon_j)$ C) $g_R(U - \varepsilon_j)g_S(\varepsilon_j)$
B) $g_R(U - \varepsilon_j)$ D) $\sum_j g_R(U - \varepsilon_j)g_S(\varepsilon_j)$

Answer: B. Since the system S has energy ε_j , the reservoir R must have energy $U - \varepsilon_j$. There are therefore $g_R(U - \varepsilon_j)$ microstates of R that have energy $U - \varepsilon_j$. To find the total number of microstates, we now need to multiply by the number of microstates of S . But, we already know that S is in a particular microstate j ! That is, we have fixed *which* microstate of S we are in, but there is still freedom to pick which microstate of R we are in. The total number of microstates is therefore given by

$$\# \text{ of microstates with } S \text{ in microstate } j = g_R(U - \varepsilon_j) \times 1 = g_R(U - \varepsilon_j) . \quad (2)$$

Question #3: Consider a spin in a magnetic field B pointing in the $+\hat{z}$ direction. The energy of the spin-up state is $\varepsilon_{\uparrow} = -mB$, while the energy of the spin-down state is $\varepsilon_{\downarrow} = +mB$. The partition function Z of the system is therefore given by

$$Z = e^{-\varepsilon_{\uparrow}/\tau} + e^{-\varepsilon_{\downarrow}/\tau} = e^{mB/\tau} + e^{-mB/\tau} = 2 \cosh\left(\frac{mB}{\tau}\right). \quad (3)$$

What is $P(\uparrow)$, the probability of the spin being in the up state?

- A) $e^{mB/\tau}$ C) $\frac{e^{mB/\tau}}{2 \cosh(mB/\tau)}$
B) $e^{-mB/\tau}$ D) $\frac{e^{-mB/\tau}}{2 \cosh(mB/\tau)}$

Answer: C. The (\uparrow) state has energy $\varepsilon_{\uparrow} = -mB$, and so its corresponding probability is

$$P(\uparrow) = \frac{e^{-\varepsilon_{\uparrow}/\tau}}{Z} = \frac{e^{-(-mB)/\tau}}{Z} = \frac{e^{mB/\tau}}{2 \cosh(mB/\tau)}. \quad (4)$$

Concept Questions – Lecture 26 – 03/13/17

Subjects: canonical ensemble, partition function for a single spin, free energy for a single spin

Question #1: Consider a spin in a magnetic field B pointing in the $+\hat{z}$ direction. The energy of the spin-up state is $\varepsilon_{\uparrow} = -mB$, while the energy of the spin-down state is $\varepsilon_{\downarrow} = +mB$. What is $P(\downarrow)$, the probability of the spin being in the up state?

- A) $e^{mB/\tau}$ C) $\frac{e^{mB/\tau}}{2 \cosh(mB/\tau)}$
B) $e^{-mB/\tau}$ D) $\frac{e^{-mB/\tau}}{2 \cosh(mB/\tau)}$

Answer: D. As shown in the notes (and on the prior concept questions summary), the partition function for a single spin is

$$Z = 2 \cosh(mB/\tau) . \tag{1}$$

The (\downarrow) state has energy $\varepsilon_{\downarrow} = +mB$, and so its corresponding probability is

$$P(\uparrow) = \frac{e^{-\varepsilon_{\downarrow}/\tau}}{Z} = \frac{e^{-mB/\tau}}{Z} = \frac{e^{-mB/\tau}}{2 \cosh(mB/\tau)} . \tag{2}$$

Concept Questions – Lecture 27 – 03/15/17

Subjects: canonical ensemble, N spin system

Question #1: Consider a spin in a magnetic field B pointing in the $+\hat{z}$ direction. The energy of the spin-up state is $\varepsilon_{\uparrow} = -mB$, while the energy of the spin-down state is $\varepsilon_{\downarrow} = +mB$. The partition function for a single spin is

$$Z(1) = e^{-mB/\tau} + e^{mB/\tau} = 2 \cosh(mB/\tau) . \quad (1)$$

What is the partition function $Z(2)$ for a system with two spins?

- A) $e^{-2mB/\tau} + e^{-mB/\tau} + e^{mB/\tau} + e^{2mB/\tau}$ C) $e^{-2mB/\tau} + e^{2mB/\tau}$
B) $2(e^{-mB/\tau} + e^{mB/\tau})$ D) $e^{-2mB/\tau} + 2 + e^{2mB/\tau}$

Answer: D. When we have two spins, the system has four possible microstates:

$$(\uparrow\uparrow) , (\uparrow\downarrow) , (\downarrow\uparrow) , (\downarrow\downarrow) .$$

The energy of each state is the sum of the individual spin energies. We can easily compute the energies as follows:

$$\begin{aligned} \varepsilon_{\uparrow\uparrow} &= \varepsilon_{\uparrow} + \varepsilon_{\uparrow} = -2mB , \\ \varepsilon_{\uparrow\downarrow} &= \varepsilon_{\uparrow} + \varepsilon_{\downarrow} = 0 , \\ \varepsilon_{\downarrow\uparrow} &= \varepsilon_{\downarrow} + \varepsilon_{\uparrow} = 0 , \\ \varepsilon_{\downarrow\downarrow} &= \varepsilon_{\downarrow} + \varepsilon_{\downarrow} = +2mB . \end{aligned} \quad (2)$$

The partition function for the two spin system is therefore given by

$$\begin{aligned} Z &= \sum_{\text{states}} e^{-\varepsilon/\tau} \\ &= e^{-\varepsilon_{\uparrow\uparrow}/\tau} + e^{-\varepsilon_{\uparrow\downarrow}/\tau} + e^{-\varepsilon_{\downarrow\uparrow}/\tau} + e^{-\varepsilon_{\downarrow\downarrow}/\tau} \\ &= e^{2mB/\tau} + e^0 + e^0 + e^{-2mB/\tau} \\ &= e^{-2mB/\tau} + 2 + e^{2mB/\tau} . \end{aligned} \quad (3)$$

Concept Questions – Lecture 28 – 03/17/17

Subjects: equivalence of ensembles in the thermodynamic limit, microcanonical ensemble versus canonical ensemble

Question #1: In class, we discussed that the microcanonical ensemble is one in which the system is isolated and has fixed energy U , while the canonical ensemble is one in which the system is connected to a reservoir with fixed temperature τ . For the N spin system, consider the variance of the magnetization s , defined by

$$\text{var}(s) = \langle (s - \langle s \rangle)^2 \rangle . \quad (1)$$

How does this variance depend on N in the following ensembles?

- 1.) Microcanonical ensemble
- 2.) Canonical ensemble

- | | |
|-------------------------|---------------|
| A) 0 | C) \sqrt{N} |
| B) $\frac{1}{\sqrt{N}}$ | D) N |

Answer: 1A, 2D. In the microcanonical ensemble, the total energy U of the system is fixed. For the spin system, the energy is related to the magnetization via

$$U = -2mBs , \quad (2)$$

meaning that the energy depends directly on the number of spins aligned with the magnetic field. If U is fixed in the microcanonical ensemble, then clearly s must be fixed as well. So, s always takes the same value, and so it can never vary from its average value. Thus, we expect $\text{var}(s) = 0$ in the microcanonical ensemble.

In the canonical ensemble, though, the energy is not fixed. The spin system is connected to a reservoir at temperature τ that can supply thermal energy to the spins, allowing for fluctuations in the magnetization. From our previous studies of the spin system, we know that these fluctuations go as

$$\text{var}(s) \sim N , \quad (3)$$

so that the standard deviation scales as \sqrt{N} (e.g. the width of the Gaussian probability distribution).

Concept Questions – Lecture 29 – 03/20/17

Subjects: ideal gas partition function, meaning of n_Q , identical particles and the Gibbs factor $1/N!$, rubber band demo

Question #1: In class, we derived that the partition function for a single gas particle in a 3D box is

$$Z(1) = n_Q V, \quad n_Q = \left(\frac{m\tau}{2\pi\hbar^2} \right)^{3/2}, \quad (1)$$

where V is the volume of the box, m is the mass of the particle, and τ is the temperature of the box. What are the dimensions of n_Q ?

- A) dimensionless
B) $\frac{1}{\text{length}}$
C) $\frac{1}{(\text{length})^3}$
D) $\frac{1}{\text{energy}}$

Answer: C. The partition function is defined as

$$Z = \sum_j e^{-\varepsilon_j/\tau}, \quad (2)$$

where j labels the microstates and ε_j is the energy of the corresponding microstate. This quantity is dimensionless, since both ε_j and τ have units of energy, and so Z has to be dimensionless. In order for equation (1) to hold, n_Q must have the opposite units as the volume V , and so $n_Q \sim 1/V = 1/(\text{length})^3$.

Question #2: If we now have N non-interacting gas particles in the box, and if we ignore the fact that all the particles are identical, what is the partition function $Z(N)$ for the whole system?

- A) $NZ(1)$
B) $Z(1)^N$
C) $\log N Z(1)$
D) $\frac{Z(1)}{N}$

Answer: B. The gas particles are non-interacting, and so the partition function should *factorize*, just like it did for the spin system example. That is,

$$Z(N) = \underbrace{Z(1)}_{Z \text{ for particle 1}} \times \underbrace{Z(1)}_{Z \text{ for particle 2}} \times \dots \times \underbrace{Z(1)}_{Z \text{ for particle } N} = Z(1)^N. \quad (3)$$

Question #3: If we take the naïve result from Question #2 and compute the free energy, we would find that

$$F = -\tau \log Z(N) = -N\tau \log(n_q V) . \quad (4)$$

What's wrong with this formula for F ?

- | | |
|-----------------------------|------------------------|
| A) It should be $\propto V$ | C) It's superextensive |
| B) It's subextensive | D) Nothing is wrong |

Answer: C. That is, it grows too quickly when we increase the system size. For example, if we were to double the system size and let $N \rightarrow 2N$, $V \rightarrow 2V$, we would expect that $F \rightarrow 2F$, since it should be extensive. However, what we actually find is that

$$\begin{aligned} F &\rightarrow -(2N)\tau \log(2n_q V) \\ &= 2(-N\tau \log(n_q V) - N\tau \log 2) \\ &= 2F - 2N\tau \log 2 . \end{aligned} \quad (5)$$

F therefore grows scales faster than we expect it to, due to this additional $2N\tau \log 2$ term. So, our formula is superextensive, which can't be right!

The resolution to this issue is that we forgot to account for the fact that all of the gas particles are *identical*. Our formula for $Z(N)$ has overcounted the number of microstates, since some of them are actually the same state. We have to add in the Gibbs factor of $1/N!$, leaving us with

$$Z(N) = \frac{Z(1)^N}{N!} = \frac{(n_q V)^N}{N!} , \quad (6)$$

for the ideal gas. If we compute the free energy now, we find that (using the Stirling approximation)

$$F = N\tau \left(\log \left(\frac{N}{n_q V} \right) - 1 \right) , \quad (7)$$

which is extensive, as expected.

Question #4: In class, we performed a rubber band demo, where a rubber band was hanging down and stretched out by a weight attached to its end. When the rubber band heats up, which of the following would we expect to happen to the weight?

- | | | |
|------------------|--------------------|------------------|
| A) It will go up | B) It will go down | C) It won't move |
|------------------|--------------------|------------------|

Answer: A. When the rubber band compresses, it actually increases its entropy, which the rubber band wants to do. Another way to see this is from the homework, where actually derived the spring constant for a simple rubber band model to be

$$k = \frac{\tau}{Na^2}, \quad (8)$$

where τ is the temperature of the rubber band, N is the number of polymers that make up the band, and a is the length of each polymer in the chain. By increasing τ , we increase the spring constant k , thus causing the restorative force to increase and the spring to contract.

Concept Questions – Lecture 30 – 03/22/17

Subjects: monatomic ideal gas, entropy of mixing

Question #1: Consider two boxes of equal volume $V_1 = V_2 = V$. Each box contains the same number of (ideal gas) particles $N_1 = N_2 = N$. If we remove the partition between the boxes and allow mixing, how does the entropy σ of the system change in each of the following cases?

- 1.) Particle types 1 and 2 are the same.
- 2.) Particle types 1 and 2 are different.

- | | |
|---|--|
| A) σ is unchanged | C) $\sigma_{\text{new}} = \sigma_{\text{old}} + N \log 2$ |
| B) $\sigma_{\text{new}} = 2\sigma_{\text{old}}$ | D) $\sigma_{\text{new}} = \sigma_{\text{old}} + 2N \log 2$ |

Answer: 1A, 2D. Let's first consider case 1. Before we open the partition, we have two containers, each with a volume V and a particle density of N/V . The species of particles are the same, and so after we remove the partition we have a single container of volume $2V$, but still with the same particle density of N/V . The system therefore looks exactly the same as it did before, and so the entropy should be unchanged.

Now, let's consider case 2. Before we remove the partition, we have two containers, each with a volume V and a particle density N/V . After removing the partition, we have a single container of volume $2V$ that contains *two different species of gas*, each with a density of $N/2V$. That is, the density of each gas drops by a factor of $1/2$ in the case where the gases are not the same. The entropy must therefore change!

We know from our previous work that the entropy goes roughly as

$$\frac{\sigma}{N} \sim \log V . \quad (1)$$

The volume that each gas sees is doubled, and so σ increases by $N \log 2$. But, since there are actually $2N$ total particles, the entropy actually increases by $2N \log 2$. This is precisely option D.

Note: We've argued the answers on more conceptual grounds. Another way to do this problem is to actually use the ideal gas entropy formula and compute the entropy before and after the partition is removed. Make sure that you can do this, and that you get the expected results!

Concept Questions – Lecture 31 – 03/24/17

Subjects: photon gas, blackbody radiation

Question #1: For a photon mode of frequency ω , the associated energy of the mode is

$$\varepsilon_s = s\hbar\omega , \quad (1)$$

where s is the number of photons in the mode. If the photon gas is at temperature τ , what is the probability of there being s photons in this mode?

- A) $e^{-s\hbar\omega/\tau}$ C) $e^{-s\hbar\omega/\tau} (1 - e^{-\hbar\omega/\tau})$
B) $\frac{e^{-s\hbar\omega/\tau}}{1 - e^{-\hbar\omega/\tau}}$ D) $se^{-s\hbar\omega/\tau} (e^{-\hbar\omega/\tau})$

Answer: C. The partition function for this mode is obtained by summing over all possible microstates. In this case, the microstates are described by how many photons are in the mode. Thus,

$$Z = \sum_{s=0}^{\infty} e^{-\varepsilon_s/\tau} = \sum_{s=0}^{\infty} e^{-s\hbar\omega/\tau} = \frac{1}{1 - e^{-\hbar\omega/\tau}} , \quad (2)$$

where the last equality comes from simply evaluating the geometric sum. The probability of having s photons in the mode is therefore given by

$$P(s) = \frac{e^{-\varepsilon_s/\tau}}{Z} = e^{-s\hbar\omega/\tau} (1 - e^{-\hbar\omega/\tau}) . \quad (3)$$

Question #2: In class, we showed that the average total energy of a photon gas in 3D is given by the integral

$$U = \pi \int_0^{\infty} \frac{\hbar\pi c}{L} \frac{n^3}{e^{\hbar\pi nc/L\tau} - 1} dn . \quad (4)$$

Without evaluating this integral, how does U depend on τ ?

- A) τ C) τ^3
B) τ^2 D) τ^4

Answer: D. This integral is hard to evaluate due to the complicated exponential term in the denominator. To make this simpler, we can make the change of variables

$$x = \frac{\hbar\pi n c}{L\tau}, \quad (5)$$

which is defined so that

$$\frac{1}{e^{\hbar\pi n c/L\tau} - 1} = \frac{1}{e^x - 1}. \quad (6)$$

To convert the rest of the integral, we have to express $n^3 dn$ in terms of $x^3 dx$. Our expression for x immediately tells us that $x \propto \frac{n}{\tau}$, and so

$$n^3 dn \propto (\tau x)^3 d(\tau x) = \tau^4 x^3 dx. \quad (7)$$

Putting this all together, the energy goes as

$$U \propto \tau^4 \int_0^\infty \frac{x^3}{e^x - 1} dx. \quad (8)$$

The integral is now completely dimensionless, with no τ dependence. We therefore conclude that $U \propto \tau^4$.

Concept Questions – Lecture 32 – 03/27/17

Subjects: blackbody spectral energy density, blackbody absorption and emission, the blackbody spectrum, imperfect absorbers

Question #1: We showed in class that, for a photon gas in 3D in a box of volume V and temperature τ , the average energy per unit volume is

$$\frac{U}{V} = \frac{\pi^2 \tau^4}{15 \hbar^3 c^3} . \quad (1)$$

That is, $U/V \propto \tau^4$ in three spatial dimensions. What does U/V go like in d spatial dimensions?

- | | |
|-----------------|---------------------|
| A) τ^d | C) τ^{2d-2} |
| B) τ^{d+1} | D) $\tau^{d/2+5/2}$ |

Answer: B. Right off the bat, we know that answer A is wrong, since it doesn't match our existing result when $d = 3$. The other three answers all give the right τ^4 dependence when $d = 3$, though. We need to work a little harder to determine which answer is right.

To do so, we recall that the average energy U , in any number of spatial dimensions, is

$$U \propto \sum_{\vec{n}} \frac{\hbar \omega_{\vec{n}}}{e^{\hbar \omega_{\vec{n}}} - 1} , \quad \omega_{\vec{n}} = \frac{\pi c}{L} |\vec{n}| , \quad (2)$$

and \vec{n} is a d -dimensional vector of integer quantum numbers, with one quantum number for each spatial direction. For example, $\vec{n} = (n_x, n_y, n_z)$ in three dimensions. Assuming the size of the box becomes very large, we can approximate the sum as an integral, and so

$$U \propto \int \frac{\hbar \omega_{\vec{n}}}{e^{\hbar \omega_{\vec{n}}} - 1} d^d \vec{n} = \int \frac{\hbar \pi |\vec{n}| c / L}{e^{\hbar \pi |\vec{n}| c / L \tau} - 1} d^d \vec{n} . \quad (3)$$

To make this integral, we can make the change of variables

$$\vec{x} = \frac{\hbar \pi c}{L \tau} \vec{n} , \quad (4)$$

so that the exponent in the denominator becomes far simpler. We are simply left with

$$U \propto \int \frac{|\vec{n}|}{e^{|\vec{x}|} - 1} d^d \vec{n} . \quad (5)$$

Our change of variables formula tells us that $\vec{n} \propto \tau \vec{x}$, and thus we find that

$$|\vec{n}| \propto \tau |\vec{x}|, \quad d^d \vec{n} \propto \tau^d d^d \vec{x}. \quad (6)$$

If we put this together with the energy expression, we therefore find that

$$U \propto \tau^{d+1} \int \frac{|\vec{x}|}{e^{|\vec{x}|} - 1} d^d \vec{x}. \quad (7)$$

The integral in this expression is dimensionless and does not depend on τ , and thus we conclude that $U \propto \tau^{d+1}$.

Question #2: Suppose we have a piece of black iron and a piece of polished steel, both at temperature τ . Which metal emits more energy per unit area?

- A) The iron. B) The steel. C) They both emit the same.

Answer: A. The black iron is closer to a blackbody than the steel, meaning that it absorbs a larger fraction of the incident photons than the iron does. And, as discussed in class, a blackbody must absorb and emit the same number of photons per unit area for every photon frequency! Therefore, since the black iron absorbs more photons, it must also emit more photons.

Concept Questions – Lecture 33 – 03/29/17

Subjects: blackbody frequency distributions, the cosmic microwave background, phonons

Question #1: Order, from coldest to hottest, an object W that is "white hot", an object R that is "red hot", and an object B that is "blue hot".

A) W, B, R

C) R, B, W

B) R, W, B

D) B, W, R

Answer: B. If an object is "red hot", that means it is emitting much more red light than any other color of visible light. This means that the peak frequency of its blackbody spectrum must be well below the frequency of red light. And, photon energy is directly proportional to frequency, and so R is emitting energy lower than most visible light emitters. Conversely, if something is "blue hot", it is emitting much more blue light than any other color of visible light. The peak frequency of its blackbody spectrum is therefore well above the frequency of blue light, and so B emits a larger amount of energy than most visible light emitters. We therefore conclude that B is hotter than R .

Now, what about W ? It is "white hot", which means that it's emitting pretty much all frequencies of visible light equally. This means that the peak frequency of its spectrum is somewhere in the middle of the visible spectrum frequency. This means that the energy it is emitting is somewhere between the red and the blue emitters. So, W must have a temperature somewhere between R and B . Thus, we conclude that the temperatures of the objects are ordered such that

$$T_R < T_W < T_B . \quad (1)$$

Question #2: Consider a 3D crystal made up of N atoms. We define the density of states $\mathcal{D}(\omega)$ for the phonon modes of the crystal such that

$$\mathcal{D}(\omega)d\omega = \# \text{ of modes } r \text{ with } \omega < \omega_r < \omega + d\omega . \quad (2)$$

Given this, what is $\int_0^\infty \mathcal{D}(\omega)d\omega$?

A) 1

C) $3N$

B) N

D) It depends on the crystal structure

Answer: C. The crystal can be thought of as being made up of N harmonic oscillators. And, we know that each harmonic oscillator has three normal modes, one for each spatial direction. The crystal therefore has $3N$ phonon modes total. Since $\mathcal{D}(\omega)d\omega$ counts how many of these phonon modes lie between ω and $\omega + d\omega$, integrating this quantity will simply count the total number of phonon modes. Therefore,

$$\int_0^\infty \mathcal{D}(\omega)d\omega = \text{total \# of phonon modes} = 3N . \quad (3)$$

Concept Questions – Lecture 34 – 03/31/17

Subjects: phonons; density of states, the Einstein model, the Debye model

Question #1: Consider a three-dimensional crystal at temperature τ made up of N atoms. In the $\tau \rightarrow \infty$ limit, what do we expect to happen to the heat capacity C_V of the crystal?

- A) $C_V \rightarrow N$ C) $C_V \rightarrow 3N$
B) $C_V \rightarrow \frac{3}{2}N$ D) $C_V \rightarrow \frac{3}{2}N\tau$

Answer: C. The atoms in the crystal lattice behave like three-dimensional harmonic oscillators. And, we know that three-dimensional harmonic oscillators have three normal modes of oscillations. The crystal therefore behaves like a collection of $3N$ normal harmonic oscillator modes. On homework #7, problem 3(a), we showed that such a system has a heat capacity of $C_V \approx 3N$ in the classical (e.g. high temperature) limit.

Another way to see this is to use the equipartition theorem. First, we note that the energy of a one-dimensional harmonic oscillator is given by the sum of the kinetic and potential energies:

$$U_{\text{1D H.O.}} = \langle KE \rangle + \langle PE \rangle = \left\langle \frac{p^2}{2m} \right\rangle + \left\langle \frac{kx^2}{2} \right\rangle . \quad (1)$$

That is, the 1D harmonic oscillator has two degrees of freedom: a translational degree of freedom (associated with movement, and thus the kinetic energy) and an oscillatory degree of freedom (associated with the spring stretching and contracting, and thus the potential energy). The equipartition theorem says that, in the classical (high-temperature) limit, a system has an average energy of $\frac{1}{2}\tau$ associated with each degree of freedom. So, the 1D harmonic oscillator therefore has an average energy of τ , due to its two degrees of freedom. So,

$$U_{\text{1D H.O.}} \rightarrow \frac{1}{2}\tau + \frac{1}{2}\tau = \tau , \quad (2)$$

in the large τ limit. The average total energy of the crystal, which behaves like $3N$ harmonic oscillators, is therefore given in the large τ limit as

$$U \rightarrow 3N\tau . \quad (3)$$

The heat capacity is therefore given by

$$C_V = \left. \frac{dU}{d\tau} \right|_{N,V} \rightarrow 3N . \quad (4)$$

Question #2: In the Debye model in 3D, we found the density of states to go as

$$\mathcal{D}(\omega) \sim \omega^2 . \quad (5)$$

How does $\mathcal{D}(\omega)$ depend on ω in d dimensions?

- | | |
|-------------------|--------------------|
| A) ω^2 | C) ω^d |
| B) ω^{d-1} | D) ω^{2d-4} |

Answer: B. In class, when we derived the density of states in the Debye model, we found that $\mathcal{D}(\omega)$ was roughly the surface area of a three-dimensional sphere of radius ω , and thus $\mathcal{D}(\omega) \sim \omega^2$. When generalizing this to general d , we will find that the density of states is the surface area of a d -dimensional sphere of radius ω . The surface area of such a hypersphere goes like $\sim \omega^{d-1}$, and thus we conclude that

$$\mathcal{D}(\omega) \sim \omega^{d-1} . \quad (6)$$

Concept Questions – Lecture 35 – 04/03/17

Subjects: phonons, particle exchange, chemical potential

Question #1: As discussed in class, we can approximate a system of N atoms in a crystal as a collection of $3N$ harmonic oscillators, known as phonons. If the crystal is at temperature τ , then its average energy is given by

$$U = \int_0^\infty \frac{\hbar\omega}{e^{\hbar\omega/\tau} - 1} \mathcal{D}(\omega) d\omega , \quad (1)$$

where $\mathcal{D}(\omega)$ is the density of states of the phonon modes. In the Debye model, this density of states is

$$\mathcal{D}(\omega) = \begin{cases} \frac{3V}{2\pi^2 v^3} \omega^2 , & \omega < \omega_D \\ 0 , & \omega > \omega_D \end{cases} \quad (2)$$

where V is the volume of the material, v is the velocity of the phonons, and ω_D is the maximum phonon frequency. The average energy of the system is therefore given by

$$U = \int_0^{\omega_D} \frac{\hbar\omega}{e^{\hbar\omega/\tau} - 1} \frac{3V}{2\pi^2 v^3} \omega^2 d\omega . \quad (3)$$

What is a good change of variables we can use to evaluate this integral?

- | | |
|----------------------------------|-----------------------------------|
| A) $x = \frac{\omega}{\omega_D}$ | C) $x = \frac{\hbar\omega}{\tau}$ |
| B) $x = \hbar\omega$ | D) $x = \omega^3$ |

Answer: C. The most complicated term in the integrand is the exponential $e^{\hbar\omega/\tau}$ term. If we set $x = \hbar\omega/\tau$, then this term simply becomes e^x , which gives us a good indication that this is a sensible change of variables to make. In fact, under this change of variables, the energy becomes

$$U = \frac{3V\tau^4}{2\pi^2 v^3 \hbar^3} \int_0^{\hbar\omega_D/\tau} \frac{x^3}{e^x - 1} dx , \quad (4)$$

which is clearly a simpler integral to evaluate.

Question #2: In a system of particles where the number of particles N can vary (e.g. N is not fixed), the chemical potential of the system is given by

$$\mu = -\tau \left(\frac{\partial \sigma}{\partial N} \right)_{U,V} . \quad (5)$$

What is dU in this system when N can vary?

- A) $dU = -pdV + \mu dN$ C) $dU = TdS - pdV + \mu dN$
 B) $dU = TdS - pdV - \mu dN$ D) $dU = TdS - pdV + \frac{\mu}{T}dN$

Answer: C. We can rewrite the chemical potential as

$$\mu = -T \left(\frac{dS}{dN} \right)_{U,V} , \quad (6)$$

which we can further rewrite as

$$0 = \left(TdS + \mu dN \right)_{U,V} . \quad (7)$$

Constant U and V corresponds to setting $dU = dV = 0$. The only one of the available answers that reproduces (7) when we set $dU = dV = 0$ is answer C. So, we conclude that

$$dU = TdS - pdV + \mu dN . \quad (8)$$

As a check on this result, we note that when the number of particles is fixed ($dN = 0$), we recover the usual thermodynamic energy $dU = TdS - pdV$, exactly as we would expect to.

Concept Questions – Lecture 36 – 04/05/17

Subjects: particle exchange, density vs. height example, particle reservoirs, grand sum

Question #1: The chemical potential μ of a system is defined by

$$\mu = -\tau \left(\frac{\partial \sigma}{\partial N} \right)_{U,V} = \left(\frac{\partial U}{\partial N} \right)_{\sigma,V} . \quad (1)$$

We can also express μ in terms of the free energy F . What is this expression?

- A) $\mu = \tau \left(\frac{\partial F}{\partial N} \right)_{T,V}$ C) $\mu = -\tau \left(\frac{\partial F}{\partial N} \right)_{T,V}$
B) $\mu = \tau \left(\frac{\partial F}{\partial N} \right)_{U,V}$ D) None of the above

Answer: D. The free energy is related to the total energy by

$$F = U - \tau \sigma . \quad (2)$$

We know from the previous concept test that $dU = \tau d\sigma - pdV + \mu dN$. Plugging this in, we find that

$$dF = -\sigma d\tau - pdV + \mu dN . \quad (3)$$

Clearly, if we hold both τ and V constant, then we find that

$$(dF)_{\tau,V} = (\mu dN)_{\tau,V} , \quad (4)$$

which we can solve for μ to find that

$$\mu = \left(\frac{\partial F}{\partial N} \right)_{\tau,V} . \quad (5)$$

This is not in the listed options, and so option D is the correct answer.

Question #2: Consider a gas in a gravitational field with gravitational acceleration g pointing in the $-\hat{z}$ direction. The layer of N_1 gas molecules at height $z = 0$ and in a volume V_1 has a chemical potential of

$$\mu_1 = \tau \log \left(\frac{N_1}{n_Q V_1} \right) , \quad (6)$$

while a layer of N_2 gas molecules at height $z = h$ in a volume V_2 has a chemical potential of

$$\mu_2 = \tau \log \left(\frac{N_2}{n_Q V_2} \right) + mgh . \quad (7)$$

At equilibrium, $\mu_1 = \mu_2$. What is the ratio of the gas densities n_2/n_1 at equilibrium?

- | | |
|-------------------|---|
| A) mgh | C) $e^{-mgh/\tau}$ |
| B) $e^{mgh/\tau}$ | D) $\log \left(\frac{mgh}{\tau} \right)$ |

Answer: C. Setting the two expressions equal, we find that

$$\tau \log \left(\frac{N_1}{n_Q V_1} \right) = \tau \log \left(\frac{N_2}{n_Q V_2} \right) + mgh , \quad (8)$$

where $n_{1,2} = N_{1,2}/V_{1,2}$ is the density for each gas. We can divide by τ and exponentiate both sides of the equation. The result is

$$\frac{n_1}{n_Q} = \frac{n_2}{n_Q} e^{mgh/\tau} . \quad (9)$$

If we rearrange terms slightly, we find that

$$\frac{n_2}{n_1} = e^{-mgh/\tau} , \quad (10)$$

and thus C is the correct answer.

Question #3: Consider a system connected to a particle reservoir at fixed temperature τ and fixed chemical potential μ . The system has a microstate energies $\varepsilon_{j,N}$, where j indexes the microstates and N is the number of particles in the system. The grand sum \mathfrak{Z} of this system is defined by

$$\mathfrak{Z} = \sum_N \sum_j e^{-\varepsilon_{j,N}/\tau} e^{N\mu/\tau} = \sum_N Z(N) e^{N\mu/\tau} , \quad (11)$$

where $Z(N)$ is simply the partition function for the N -particle system. The probability for the system to be in microstate j and have N particles is

$$P(N, j) = \frac{e^{-\varepsilon_{j,N}/\tau} e^{N\mu/\tau}}{\mathfrak{Z}} . \quad (12)$$

The average number of particles $\langle N \rangle$ is therefore given by

$$\langle N \rangle = \sum_N \sum_j \frac{N e^{-\varepsilon_{j,N}/\tau} e^{N\mu/\tau}}{\mathfrak{Z}} . \quad (13)$$

Using the derivative trick, which of the following quantities can we rewrite $\langle N \rangle$ as?

A) $\frac{\partial \mathfrak{Z}}{\partial \mu}$	C) $\tau \frac{\partial \mathfrak{Z}}{\partial \mu}$
B) $\tau \frac{\partial \log \mathfrak{Z}}{\partial \mu}$	D) $\frac{1}{\tau} \frac{\partial \log \mathfrak{Z}}{\partial \mu}$

Answer: B. To see this, we first note that acting with $\frac{\partial}{\partial \mu}$ on $e^{N\mu/\tau}$ brings down a factor of N/τ :

$$\frac{\partial}{\partial \mu} (e^{N\mu/\tau}) = \frac{N}{\tau} e^{N\mu/\tau} . \tag{14}$$

Since the grand sum \mathfrak{Z} has this $e^{N\mu/\tau}$ term in it, acting with a derivative on \mathfrak{Z} will help us bring down a factor of N . So,

$$\frac{\partial \mathfrak{Z}}{\partial \mu} = \sum_N \sum_j \frac{N}{\tau} e^{-\varepsilon_{j,N}/\tau} e^{N\mu/\tau} . \tag{15}$$

If we compare this expression to our earlier expression for $\langle N \rangle$, we can see that $\langle N \rangle$ can be rewritten as

$$\langle N \rangle = \frac{\tau}{\mathfrak{Z}} \frac{\partial \mathfrak{Z}}{\partial \mu} = \tau \frac{\partial \log \mathfrak{Z}}{\partial \mu} . \tag{16}$$

Concept Questions – Lecture 37 – 04/07/17

Subjects: grand canonical ensemble, equivalence of ensembles

Question #1: Consider a system non-interacting gas molecules in a box. This box also contains a site that only a single molecule can bind to. The energy of the molecule in this bound state is $-\varepsilon_0$. Now, let f be the probability that this site is occupied. How do we expect f to scale with temperature and density?

- A) increases with τ , increases with n C) decreases with τ , increases with n
B) increases with τ , decreases with n D) decreases with τ , decreases with n

Answer: C. In the high temperature limit, the states with higher energies are more likely to be excited, and so the bound state becomes relatively less likely. So, we expect f to decrease as τ increases. When the density increases, the increase in molecule leads to a greater probability that at least one molecule is in the bound state. So, we expect f to increase as n increases.

Question #2: In class, we computed the probability that this bound state is occupied f to be

$$f = \frac{ne^{\varepsilon/\tau}}{n_Q + ne^{\varepsilon/\tau}} . \quad (1)$$

From this expression, we can immediately see that, when n is fixed,

$$\lim_{\tau \rightarrow \infty} f = 0 , \quad \lim_{\tau \rightarrow 0} f = 1 . \quad (2)$$

Still keeping n fixed, at what temperature would we expect $f \approx 1/2$?

- A) μ C) ε_0
B) $\frac{n}{n_Q}$ D) $\frac{\varepsilon_0^2}{\mu}$

Answer: C. The only energy scale at fixed density is the binding energy ε_0 , so we expect $\tau \sim \varepsilon_0$, which leads us to answer C. A good check on this is to note that when $\tau \sim \varepsilon_0$, we can estimate $e^{\varepsilon_0/\tau} \sim 1$, and so

$$f \sim \frac{n}{n_Q + n} , \quad (3)$$

which clearly goes to $\sim 1/2$ when $n_Q \sim n$.

Concept Questions – Lecture 38 – 04/10/17

Subjects: quantum gases, degenerate Fermi gas

Question #1: The average occupation numbers for fermions and bosons in an orbital of energy ε in a system with temperature τ and chemical potential μ is given by

$$f(\varepsilon) = \begin{cases} \frac{1}{e^{(\varepsilon-\mu)/\tau} + 1} & \text{for fermions,} \\ \frac{1}{e^{(\varepsilon-\mu)/\tau} - 1} & \text{for bosons.} \end{cases} \quad (1)$$

Which of the following are phonons?

- A) Bosons B) Fermions C) Neither

Answer: A. The quick and easy way to see this is to recall that we treated phonons in the same way that we treated photons. And, photons are by definition bosons, and thus phonons must be as well.

The proper way to get this answer is to recall that we derived that the average number s of phonons in a mode of frequency ω is

$$\langle s \rangle = \frac{1}{e^{\hbar\omega/\tau} - 1} . \quad (2)$$

The energy of this mode is $\varepsilon = \hbar\omega$, and $\langle s \rangle$ is simply an occupation number. We can therefore rewrite this expression as

$$f(\varepsilon) = \frac{1}{e^{\varepsilon/\tau} - 1} . \quad (3)$$

The factor of -1 in the denominator immediately tells us that this is a boson! If it were instead a $+1$, this would be fermionic.

Question #2: The chemical potential μ for a phonon system is what?

- A) Depends on the density of modes C) $-\infty$
B) 0 D) $+\infty$

Answer: B. If we compare the phonon occupation number (3) to the original occupation number expressions (1), we can see that the only way to match the phonon expression to the boson expression is to set

$$\mu = 0 . \quad (4)$$

That is, phonon systems are forced to have no chemical potential. This indicates that the number of phonons is not constant, e.g. phonons are not conserved.

Concept Questions – Lecture 39 – 04/12/17

Subjects: degenerate Fermi gas, Fermi energy, degeneracy pressure

Question #1: The density of states for a spin-1/2 particle of mass m inside of a 3D box of volume V is

$$\mathcal{D}_{1/2}(\varepsilon) = \frac{V}{2\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \varepsilon^{1/2} . \quad (1)$$

What is the density of states $\mathcal{D}_0(\varepsilon)$ for a spin-0 particle with mass m in a 3D box of the same volume V ?

- A) $2\mathcal{D}_{1/2}(\varepsilon)$ C) $\varepsilon^{1/2}\mathcal{D}_{1/2}(\varepsilon)$
B) $\frac{1}{2}\mathcal{D}_{1/2}(\varepsilon)$ D) $\varepsilon^{-1/2}\mathcal{D}_{1/2}(\varepsilon)$

Answer: B. The allowed energies for a particle in a 3D box are given by $\varepsilon_{\vec{n}}$, where $\vec{n} = (n_x, n_y, n_z)$. For a spin-0 particle, there is a single orbital for each value of \vec{n} . For a spin-1/2 particle, though, the particle can have spin up or spin down, and thus there are two orbitals for each value of \vec{n} . There are therefore twice as many orbitals for the spin-1/2 particle as compared to the spin-0 particle. The density of states is what counts how many orbitals there are, and thus we conclude that

$$\mathcal{D}_0(\varepsilon) = \frac{1}{2}\mathcal{D}_{1/2}(\varepsilon) . \quad (2)$$

Question #2: As derived in class, the Fermi energy of a system of N spin-1/2 particles of mass m in a 3D box of volume V is given by

$$\varepsilon_F = \frac{\hbar^2}{2m} \left(3\pi^2 \frac{N}{V} \right)^{2/3} . \quad (3)$$

The total energy of this system at zero temperature is

$$\lim_{\tau \rightarrow 0} U = \int_0^{\varepsilon_F} \varepsilon \mathcal{D}(\varepsilon) d\varepsilon = \frac{3}{5} N \varepsilon_F . \quad (4)$$

What is the pressure of the system at zero temperature?

- A) $\left(\frac{U}{V} \right)^2$ C) $\frac{2U}{3V}$
B) $\frac{1U}{3V}$ D) $\frac{U}{V}$

Answer: C. The pressure of a gas is given by

$$p = - \left(\frac{\partial F}{\partial V} \right)_{\tau} = - \frac{\partial}{\partial V} \left(U - \tau \sigma \right)_{\tau} = - \left(\frac{\partial U}{\partial V} \right)_{\tau} + \tau \left(\frac{\partial \sigma}{\partial V} \right)_{\tau} . \quad (5)$$

At zero temperature, the second term goes to zero when we set $\tau = 0$, and thus the pressure is simply

$$p = - \left(\frac{\partial U}{\partial V} \right)_{\tau} . \quad (6)$$

If we look at the earlier expressions for the total energy and the Fermi energy, we can see that (at $\tau = 0$)

$$U \propto \varepsilon_F \propto V^{-2/3} . \quad (7)$$

If we differentiate with respect to V , then we can easily see that

$$\frac{\partial U}{\partial V} \propto -\frac{2}{3} V^{-5/3} = -\frac{2}{3V} V^{-2/3} , \quad (8)$$

and thus we conclude that

$$\frac{\partial U}{\partial V} = -\frac{2U}{3V} , \quad (9)$$

at $\tau = 0$. The pressure at $\tau = 0$ is therefore simply given by

$$p = - \left(\frac{\partial U}{\partial V} \right)_{\tau} = \frac{2U}{3V} , \quad (10)$$

which is answer C. This is known as the *degeneracy pressure* of the Fermi gas. It arises purely from the fact that fermions can't occupy the same orbitals, thus leading to an effective pressure pushing the fermions away from the lowest energy orbital.

Concept Questions – Lecture 40 – 04/14/17

Subjects: degenerate Fermi gases (specific heat); degenerate Bose gases (behavior of chemical potential)

Question #1: Consider a gas of non-interacting fermions at low (but nonzero) temperature τ . Give an order of magnitude estimate of the number of gas particles that are excited from states below the chemical potential $\mu \approx \varepsilon_F$ to states above the chemical potential in the limit of small τ .

- A) τ/ε_F C) $\mathcal{D}(\varepsilon_F)\tau$
B) ε_F/τ D) $\mathcal{D}(\tau)\varepsilon_F$

Answer: C. We know from the shape of the Fermi function $f(\varepsilon) = 1/[\exp(\varepsilon - \mu)/\tau) + 1]$ that, roughly speaking, particles are only excited from states within about τ of the Fermi energy. On the other hand, a significant fraction of particles in those orbitals within τ of ε_F will be excited. So, what we need to do is figure out how many states there are within τ of ε_F . In general, there are $\int_{\varepsilon_F - \tau}^{\varepsilon_F} \mathcal{D}(\varepsilon)d\varepsilon$ such states. But for small τ , the integrand does vary much over the domain of the integration, so we can take it to be approximately constant, leading to an estimate of $\mathcal{D}(\varepsilon_F)\tau$ (to leading order for small τ and ignoring numerical factors of order unity).

Question #2: For a gas of non-interacting fermions at low temperature τ , how much does each particle that is excited from below ε_F to above ε_F increase its energy? Given an order of magnitude estimate.

- A) τ C) $N\tau$
B) ε_F D) $N^{1/2}\varepsilon_F$

Answer: A. Each excited particle moves from an orbital (i.e. a single particle state) with an energy of $\varepsilon_F - c\tau$ to one with an energy of about $\varepsilon_F + c'\tau$, where c and c' are constants of order 1. So, the particle's energy changes by $(c + c')\tau$, which is of order τ .

Question #3: For non-interacting bosons in a box, the lowest possible energy orbital has energy $\varepsilon = 0$, and the average number of particles in an orbital

of energy ε is $f(\varepsilon) = 1/[\exp(\varepsilon - \mu)/\tau) - 1]$. What range of values for μ is physically possible for this system?

- A) All real μ
- B) $\mu < 0$
- C) $\mu > 0$
- D) $\mu > \varepsilon$

Answer: B. If we plug the lowest possible energy, $\varepsilon = 0$, into the occupancy function $f(\varepsilon)$, we have

$$f(\varepsilon = 0) = \frac{1}{e^{-\mu/\tau} - 1} .$$

This function gives the average number of particles in the orbital, which must be a positive number. Thus, we need

$$e^{-\mu/\tau} - 1 > 0 \iff \mu < 0 .$$

Concept Questions – Lecture 41 – 04/17/17

Subjects: bosons and Bose-Einstein condensates

Question #1: In class, we discussed how a boson gas in 3D undergoes a phase transition to a Bose-Einstein condensate (BEC) at the Einstein temperature τ_E given by

$$\tau_E = \frac{2\pi\hbar^2}{m} \left(\frac{N}{2.6V} \right)^{2/3} . \quad (1)$$

At temperatures $\tau < \tau_E$, the system becomes a Bose-Einstein condensate, with an appreciable fraction of the total number of bosons sitting in the ground state with $\varepsilon = 0$. For $\tau < \tau_E$, how does the total energy U depend on N ?

- | | |
|----------|--------------|
| A) N^0 | C) N^2 |
| B) N^1 | D) $B^{3/2}$ |

Answer: A. When $\tau < \tau_E$, the system is a Bose-Einstein condensate, which means it has a number density of particles *larger* than $2.6n_Q$. That is,

$$\tau < \tau_E \iff \frac{N}{V} > 2.6n_Q . \quad (2)$$

The number of particles in the excited states N_e is bounded by $N_e < 2.6n_Q V$. This means that if we were to add more particles to the gas, all of the particles go into the ground state! The ground state has energy $\varepsilon = 0$, and so the total energy will not change by adding more particles. Therefore, we conclude that $U = \text{const}$ for $\tau < \tau_E$, and thus

$$U \sim N^0 . \quad (3)$$

Question #2: For $\tau < \tau_E$, how does the total energy U depend on τ ?

- | | | |
|-----------------|-----------------|-----------------|
| A) τ^0 | C) τ^1 | E) $\tau^{5/2}$ |
| B) $\tau^{1/2}$ | D) $\tau^{3/2}$ | |

Answer: E. The particles in the ground state do not contribute to the total energy of the system, and so the total energy is simply given by the usual density of states expression:

$$U = \int_0^\infty \varepsilon f(\varepsilon) \mathcal{D}(\varepsilon) d\varepsilon , \quad (4)$$

where the occupancy number $f(\varepsilon)$ and the density of states $\mathcal{D}(\varepsilon)$ are, for a system of bosons with mass m inside a 3D box with volume V ,

$$f(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/\tau} - 1}, \quad \mathcal{D}(\varepsilon) = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \varepsilon^{1/2}. \quad (5)$$

When $\tau < \tau_E$, the chemical potential is very small, and so we can approximate it as $\mu \approx 0$ in this integral. Doing so, we find that

$$U \propto \int_0^\infty \frac{\varepsilon^{3/2}}{e^{\varepsilon/\tau} - 1} d\varepsilon. \quad (6)$$

If we now make the change of variables $x = \varepsilon/\tau$, the energy becomes

$$U \propto \tau^{5/2} \int_0^\infty \frac{x^{3/2}}{e^x - 1} dx. \quad (7)$$

The integral in this expression is simply a constant number, and thus we find that $U \propto \tau^{5/2}$.