

Revised-Path Dependence

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Abstract

In this paper, we define a class of *revised-path dependent processes* and characterize some of their basic properties. A process exhibits revised-path dependence if the current outcome causes the value of a past outcome to be revised. This revision could be an actual change to that outcome or a reinterpretation. We characterize a revised-path dependent process called the *accumulation process*: in each period a randomly chosen past outcome is changed to match the current outcome. We show that this process converges to all outcomes being identical. We then construct a general class of models that includes the Bernoulli process, the Polya process, and the accumulation process as special cases. In processes where path revision is possible, apart from knife edge cases, the conventional path dependence prediction of the possibility of any equilibrium distribution is rejected. Instead, all of these processes converge to either homogeneous equilibria or to an equal probability distribution over types. Further, if random draws advantage one outcome over the other, the “anything can happen” result disappears entirely.

In a path dependent process, the value or type of the current outcome depends on the path or the set of previous outcomes (David 1985, Arthur, 1984). That outcome can be either an exogenous variable produced by some generating function or an endogenous choice by a strategic actor, such as choices over government policies. In both cases, the history of past outcomes influences or has sway over current and future outcomes.

Path dependent outcomes are ubiquitous in social processes. Many decisions or actions in some way depend on prior events. In contrast, path dependent equilibria, in which the long run state of a system depends on the events along the way, occur less frequently (Page 2006). Evidence exists to support claims of path dependence in choices over technologies, standards, institutional features, social behaviors, norms, laws, and city locations.¹ In each of these domains, equilibrium outcomes depend on the path of previous actions.

The canonical model of path dependence is the Polya process. Classically, one imagines an urn filled with balls of two colors. One ball is drawn. In the Polya models used to characterize path dependence, that ball is returned to the urn, and in addition, a second ball of the same color is added to the urn.² In future draws, that color ball becomes slightly more likely to be drawn. As the number of balls in

¹See for example, (Hacker 2002, Pierson 2000, 2004, Cowen and Gunby 1996, Crouch and Farrell 2004, and North 1990).

²The Polya process is generalizable to describe sampling with replacement (just returning the ball to the urn, so each draw is independent), sampling without replacement (not putting the ball in the urn, so that color becomes *less* likely to be drawn in the future), or adding any number of balls of the same color to the urn, to amplify outcome-dependence.

the urn grows large, the distribution of the colors of balls within the urn converges, so that each period of drawing the ball and replacing it has an insignificant effect on the overall distribution of ball colors. Each draw is an outcome, but the distribution of ball colors in the urn is (eventually, once the process converges) the equilibrium distribution. In the Polya process, any proportion of outcomes can be a long run equilibrium, and, in addition, any proportion occurs with equal probability. In other words, anything can happen and is equally likely to happen.

Once a Polya process converges it has great predictive power, but at the outset, one can make no predictions about the process's eventual equilibrium. For example, suppose that the Court's judgments are guided by the principles of *stare decisis* (Hathaway 2001, Schauer 2011). With sufficient precedent, the legal process converges. At this point, if the Court has decided in favor of business interests eighty percent of the time, we might then expect it to favor business interests in a current case with similar probability. However, at the outset of the process, say, the launch of the legal system, we cannot predict the equilibrium distribution: whether the Court will be biased in favor of business interests or not. Whenever choices over technologies, beliefs, or policies follows a Polya process, we have little hope of predicting the future when we initiate the process.

If these possible outcomes don't differ in their efficiency or fairness, then the existence of path dependence has less relevance. It matters

little whether we drive on the left or the right provided that we all drive on the same side of the road. However, in many cases, outcomes are not equally efficient. Evidence suggests that configurations that gain early leads can become locked-in (Arthur 1984).³

In this paper, we analyze the potential for path dependent equilibria within a broader class of path dependent processes. A process produces path dependent equilibria if the long run distribution over outcomes depends on the path of outcomes (Page 2006). We highlight conditions which limit the set of equilibria, in contrast to the “anything goes” predictions from the Polya process. A process can be path dependent, i.e. produce outcomes in each period that depend on past outcomes, but converge to a unique equilibrium. There may be many roads, but they may all lead to Rome.

We highlight in particular the effect of path revision on equilibrium distributions. With path revision, current outcomes can change the state—the perceived outcome history—to revise the “as if” values of past outcomes in light of recent events. Path revision is intuitive within social science. History, as many have noted, tends to be written by the winners.

Return for a moment to our earlier example of a court that bases current decisions on prior opinions. When rendering a judgment, the Court might reinterpret a past decision and yet claim to adhere to precedent. In doing so, it does not change the past decision, but it is

³See Liebowitz and Margolis 1990, 2002 for critiques of many canonical examples of inefficient lock-in.

“as if” the past decision has changed. These reinterpretations can leap across political institutions: Brandwein (2011) presents evidence that the courts did not abandon the rights of blacks through their use of state action doctrine during the American Reconstruction. Yet, those rulings were interpreted by the president and Congress in such a way as to abandon the rights of blacks.

The class of processes that we consider allow for prior actions to be revisited in other ways as well. Often, past decisions and actions can be reversed or undone, though often at high cost. Consider a scenario where a consumer purchases product A, but then exchanges that product for product B. If such phenomena as store inventory, product popularity ratings, or price are a product of history, the fact that the consumer changed her mind does not matter. Although the exchange does not change the fact that the consumer originally purchased A, from the perspective of the popularity ratings, or store inventory, the history appears “as if” the consumer had chosen B initially.

Consider also the choice over computer operating systems. A person who previously chose DOS may switch to a Mac if a number of his or her friends have recently chosen Macs. One might ask, why not just model the purchase of a Mac as a new outcome? One could, but that would ignore the fact that this person had previously owned a DOS machine. If we think of the current state of the world as being the percentage of Mac owners and the percentage of DOS owners, adding a new Mac owner has less of an impact on that state than having an

existing DOS owner switch to Mac. A new user adds a single Mac. A switch both adds a Mac and takes away a DOS user. Revising the path has a larger effect than adding a new outcome.

Finally, these processes include cases where ongoing processes respond to the current state. Suppose that an organization has chosen a hierarchical procedure for allocating raises and that this procedure has been in operation for several years. Now suppose that the organization in the current period introduces a decentralized market mechanism to allocate offices following a move to a new building. This mechanism might work so well that the organization decides to apply it to their raise allocation task. The past doesn't change here, but to an outside observer, the organization might look "as if" it had always used the market mechanism for both tasks.

We call this general class of processes *revised-path dependent* to capture that (a) they're path dependent processes and (b) past outcomes can be altered, or revised, to align with current outcomes. The concept of revised-path dependence would merit attention purely for the sake of mathematical completion. Scholars have analyzed what happens when the past influences the current outcome. We might then naturally ask, what happens if in addition, the current outcome causes past outcomes to be revised? By considering this opposite assumption, we might well shed light on the standard assumption and its implications. That said, we don't see this investigation as merely an exercise in logical completion. To the contrary, the motivation for

studying these processes originated when analyzing experimental data and a realization that, in fact, many paths do get revised.

The experimental study that spurred this investigation revealed evidence that subjects revised play in one game when a second game was added to their ensemble of strategic choices (Bednar et al 2011a). In these experiments, subjects first played one game, call this game A, and then added a second game, game B, that was played with a different player. The subjects then played both games simultaneously for multiple periods. The point of the experiments was to test if subjects would copy behaviors used in game A when they played game B. In other words, the experiments were designed to test if subjects' behaviors would exhibit institutional path dependence, which in fact did occur. The experimenters also found that subjects' behavior in game B influenced the continuation play in game A, the original game. For example, as we detail later in this paper, selfish behavior in the new game would often bleed over into the original game. In other words, adding a new game sometimes revised play in the old game.

In this paper, we provide basic definitions and unpack the logic of revised-path dependence. We begin with some examples, including what we see as the canonical revised-path dependent process, the *accumulation process*, where past outcomes switch to match the current outcome. We then construct a class of models that allows for both revised-path and traditional path dependence. In these models, some of the outcomes are random, independent of the state. As one would

expect, this infusion of randomness creates both greater contingency and biases the equilibrium toward equally likely outcomes. This general class includes as special cases the Bernoulli process, the Polya process, and the accumulation process.

Our main results are the following: First, we show that the long run equilibria of the accumulation process consist of all outcomes being of the same type, where the type depends on the order in which outcomes occur. Equilibria are path dependent as in the Polya process, but it is no longer the case that any distribution of types is possible. Second, we show that if we do not include any random outcomes, then the long run equilibrium of any process that includes path revisions also consists of all outcomes being the same type. Thus, even a small amount of path revision implies that outcomes lie at the extremes, problematizing claims that anything can happen. Finally, when we include the possibility of random outcomes, we find that it is possible to resurrect the result that anything can happen, but that this result only holds in knife-edge cases. And if one outcome is advantaged over another, by creating a biased Bernoulli process, then the “anything can happen” result goes away entirely. Outcomes will either be of one type or be at a specific interior equilibrium.

The remainder of this paper is organized into four sections. In the next section, we provide a brief overview of the experimental data that demonstrates path revision in behavior. We then characterize path revision and define the accumulation process. Building upon

the intuitions from our specific model, we construct a general model that includes the accumulation process, the Polya process, and the Bernoulli process as special cases. In the context of that model, we prove results about the long run equilibria of each of those processes. In the final section, we discuss the difficulties of empirical discrimination between these processes as path dependent processes.

The Experimental Data

To provide some context for the theoretical results that follow, we begin by presenting evidence of revised-path dependence from behavioral experiments (Bednar et al 2011a). In these experiments, the path revision arises when people change an ongoing behavior to align with new behavior. These data demonstrate existence and also provide a new context, human behavior, within which to contemplate the implications of path and revised-path dependence.

The data shown describe outcomes from experiments in which one of three games, the Prisoner's Dilemma (PD), a Strong Alternation game (SA), or a Self Interest game (SI), was first played as a control.⁴ In the Prisoner's Dilemma, the efficient equilibrium involves both players choosing to cooperate. In the strong alternation game, the efficient equilibrium involved alternating, and in the self interest

⁴In Strong Alternation, the payoff-maximizing play involves agents trading off getting a high payoff and getting a lower payoff. In Self-Interest, payoff-maximizing play is a dominant strategy; players do not need to take one another's actions into account when determining their own best play.

Table 1: Game Forms Used in Path Dependence Experiments

		C	S
Prisoner's Dilemma: (PD)	C	7, 7	2,10
	S	10,2	4,4
		C	S
Strong Alternation: (SA)	C	7, 7	4,14
	S	14,4	5,5
		C	S
Self Interest: (SI)	C	7, 7	2,9
	S	9,2	10,10

game, the efficient equilibrium required always playing selfish.

Table 2 shows differences in play between a control and a treatment. In the control, players played a single game for approximately 200 rounds with an anonymous opponent. In the treatment, players first played the control game for one hundred rounds, and then a second game was added, which they played with a distinct opponent. Players played both games for an additional one hundred rounds. Table 2 reports the prevalence of simple strategies: SS corresponds to both players choosing selfish, CC correspond to both cooperating, and ALT corresponds to the players alternating where one player cooperates and the other acts selfishly. The first set of three columns shows the percentage of the outcomes SS, CC, and ALT in the control game. The next set of three columns shows the percentage of the outcomes for the initial game only, but after the second game has been added. In the final three columns we show statistical p-values for the differences

in play between the control and the treatment.

We focus attention on the SA game. When played alone, 86% of the time the players alternated (ALT), the payoff-maximizing behavior. Only six percent of outcomes were SS. When the self interest game, SI, is added as the second game, over twenty-seven percent of the outcomes in SA are SS and only thirty-five percent of outcomes are classified as alternating. The increase in SS and decline in the payoff-maximizing ALT are both significant at the five percent level. Thus, the presence of the SI game led to a significant increase in selfish behavior in the other games.⁵ Not only did the human subjects revise their behavior in response to a new behavior, but even more interestingly, the behavioral revision occurred in a game where the subjects had established a payoff-optimizing routine. In these cases, players' net payoffs are reduced. Revised-path dependence not only alters outcomes, but it has the power to reduce payoffs.

Revised-Path Dependence

We now present formal definitions. To distinguish revised-path dependence from path dependence, we first present a simple definition of the latter. In period t , denote the outcome by x_t . We assume that x_t belongs to a finite set of outcomes X that have real values. We next define a family of functions $\{A_t\}_{t=1}^{\infty}$. A_t maps a sequence of outcomes

⁵Bednar et al (2011b) investigate the behavioral spillovers between games, and find a strong influence of SI, a game with a dominant strategy, on other games played simultaneously.

Table 2: Cognitive Load Effect on Historical Game: Treatment vs. Control

Control	% Simple Strategies			Ensemble	% Simple Strategies			Control vs. Treatment		
	SS	CC	ALT		SS	CC	ALT	SS	CC	ALT
PD	14.50	59.50	19.17	(PD,(PD,SA))	26.75	57.92	6.33	0.417	0.948	0.272
				(PD,(PD,SI))	59.00	34.75	1.00	0.019	0.289	0.009
SA	6.17	4.00	86.00	(SA,(SA,PD))	14.83	14.25	65.17	0.312	0.141	0.252
				(SA,(SA,SI))	27.17	28.67	35.50	0.022	0.003	0.005
SI	100.00	0.00	0.00	(SI,(SI,PD))	98.50	0.25	0.00			
				(SI,(SI,SA))	99.50	0.00	0.08			

in X of length t into the real numbers. $A_t(x_1, x_2, \dots, x_t) \in R$. This family of functions produces an aggregate statistic. In the cases that we consider, A_t will be the average value of the outcomes, but it could be any statistic.

An *outcome function* G maps A_t into a probability distribution over outcomes in the next period. Thus, formally, the outcome function maps the reals into the set of probability distributions $G : R \rightarrow \Delta(X)$. In other words, the outcome in period $t + 1$ is given by $G(A_t(x_1, x_2, \dots, x_t))$. Implicitly, G is defined over a sequence of outcomes of length t . In a slight abuse of notation that simplifies the presentation, we hereafter suppress the A_t and just write G as a function of the sequence of outcomes. To present the definitions, we often rely on sequences of finite length. Let $\{x_t\}_{t=1}^N$ denote a finite sequence of outcomes of length N . By convention, a process is path dependent if there exists some N and two distinct paths of outcomes of length N such that G produces different probability distributions over outcomes under these two paths. Note that we allow for the possibility that G is degenerate.

*A process is **path dependent** if $x_t = G(x_1, x_2, \dots, x_{t-1})$ and if there exist $\{x_t^*\}_{t=1}^N \neq \{x_t\}_{t=1}^N$ s.t. $G(x_1, x_2, \dots, x_{t-1}) \neq G(x_1^*, x_2^*, \dots, x_{t-1}^*)$.*

Given this definition, the Polya process will be path dependent, as the probability of an outcome at time t depends on the previous set of outcomes. This definition does not distinguish between set dependence (what Page 2006 calls phat dependence) and path dependence, but it

suffices for our purposes.

To define revised-path dependence, we must allow for the possibility that the values of past outcomes can be revised. Formally, we define a family of *revision functions*, $\{H_t\}_{t=1}^\infty$. A revision function takes the past sequence of outcomes of length $(t - 1)$ along with the current outcome, x_t , and maps them into a new sequence of outcomes of length $(t - 1)$: $H_t : X^{(t-1)} \times X \rightarrow X^{(t-1)}$. If no revision exists, then H is just a projection onto the the first $(t - 1)$ dimensions, e.g. $H(x_1, x_2, \dots, x_{t-1}, x_t) = (x_1, x_2, \dots, x_{t-1})$. If path revision occurs, then the values of one or some of the x_j 's for $j < t$ will change. To keep track of these changes, we let the variable $x_{j,t}$ denote the value at time t of the outcome that originally occurred at time j . If the value of an outcome is never revised then all of the $x_{j,t}$'s will be the same. When a revision occurs at time t , we have that $x_{j,t} \neq x_{j,t-1}$. If we take all such variables, we obtain an upper triangular matrix of subsequences $X = \{\{x_{j,t}\}_{j=1}^t\}_{t=1}^N$.

*A process exhibits **path-revision** if $H_t : X^{(t-1)} \times X \rightarrow X^{(t-1)}$ is not a projection onto X^{t-1} for all t .*

We can then define the *equilibrium* of a revised-path dependent process; it equals the limit of the mean value of the x_{it} as t goes to infinity.

$$\lim_{t \rightarrow \infty} \frac{\sum_{i=1}^t x_{it}}{t}$$

Note that this limit is not the mean value of the outcome in period t but the mean value of all of the revised outcomes. Thus, a revised-path equilibrium could converge to all revised outcomes being of one type even if with positive probability the outcomes generated by the process are of some other type with positive probability. All that would need to occur would be for those outcomes to be later revised.

The Accumulation Process

We now define the canonical revised-path dependent process. We call it the *accumulation process* because revisions accumulate. Like the Polya process, this process considers only two possible values for each outcome. Also, like Polya, it can be generalized. In the process, an urn begins with approximately equal numbers of blue (outcome 1) and yellow (outcome 0) balls. In each period, a ball is drawn from the urn. That ball is then replaced in the urn and a new ball is added that matches the color of the chosen ball. In addition, an existing ball has its color switched to match the color of the ball chosen.

Definition: In the *accumulation process* let $x_{t,t} \in \{0, 1\}$ with equal probability for all t . Let S equal the length of initial history. If $t > S$, let $D = \{j : x_{j,t-1} \neq x_{t,t}\}$. Choose $k \in D$, and set $x_{k,t} = x_{t,t}$. For all other $i < t, i \neq k$, let $x_{i,t} = x_{i,t-1}$.

Claim 1 *In any long run equilibrium of the accumulation process all outcomes will be of the same type.*

pf. Let $V_t = \sum_{i=1}^t x_{i,t}$. In period $T + 1$, if $x_{t,t} = 1$ then $V_{T+1} = V_T + 2$. The old total increases by two because a new ball is added *and* an existing ball of value zero is transformed to value one. Similarly, if $x_{t,t} = 0$ then $V_{T+1} = V_T - 1$. Therefore, it is possible to write the expected value of the ratio of V_{T+1} to $T + 1$ as follows:

$$E \left[\frac{V_{T+1}}{T+1} \right] = \left[\frac{V_T}{T} \right] \left[\frac{V_T + 2}{T+1} \right] + \left[\frac{T - V_T}{T} \right] \left[\frac{V_T - 1}{T+1} \right]$$

This expression can be shown to reduce to the following:

$$E \left[\frac{V_{T+1}}{T+1} \right] = \frac{V_T}{T} + \frac{2V_T - T}{T(T+1)}$$

The result follows from this expression. The ration $\frac{V_T}{T}$ is either monotonically increasing or decreasing in T depending upon whether $2V_T$ is greater than or less than T . Not only does the accumulation process converge to all outcomes being of the same type, it converges relatively quickly. The rate of convergence depends on the length of the initial process in which balls are drawn randomly.

A General Model With Binary Outcomes

We now construct a general model that allows for both revised-path and traditional path dependent processes. As with the description of the Polya process in the introduction, to describe these processes, we use standard urn models where different colors of balls represent

different types of outcomes and the proportion of balls of each color correspond to the probability of an outcome of that type occurring. Our general class relies on three parameters, p , q , and S .

The parameter p allows us to introduce randomness. It denotes the probability that a new ball is selected from the urn, and so $(1 - p)$ denotes the probability that the new ball is chosen randomly, that is, from an equally weighted distribution. (We relax this equal probability assumption later.) If the ball was chosen from the urn, it is replaced and another ball of the same color is added. If the ball was randomly chosen, it is simply added to the urn. Thus, in either case, whether chosen randomly or from the urn, the urn now contains one more ball, but its color depends on whether the ball was drawn from the urn or randomly.

The second parameter q denotes the probability that a ball of the opposite color within the urn (provided such a ball exists) is switched to match the color of the ball chosen. We refer to this as the *probability of path revision*. For higher values of q the current outcome is more likely to change the state of a past outcome. Finally, we introduce a parameter S to denote the number of initial periods in which all outcomes are determined by a Bernoulli process. In the proofs that follow, we consider cases where $S = 1$, which implies only a single random draw prior to path dependence coming into play. Note that S must be at least one. The first outcome cannot depend on the past, as there is no past on which it can depend. Larger S would imply

that the forces that produce path dependence are not in operation for some initial number of periods. Together, p , q , and S create what we call a *revised-path dependent processes*.

Definition: In a *revised-path dependent process* $F(p, q, S)$, the outcome in period t is characterized as follows:

Step 1: Choose $x_{t,t}$ according to the following rules:

- (i) if $t \leq S$, then $x_{t,t} \in \{0, 1\}$ with equal probability.
- (ii) if $t > S$, with probability $1 - p$, $x_{t,t} \in \{0, 1\}$ with equal probability, and with probability $(1 - p)$ $x_{t,t} = 1$ with probability $\frac{\sum_{i=1}^{t-1} x_{i,t-1}}{t-1}$ and 0 otherwise.

Step 2: Let $D = \{j : x_{j,t-1} \neq x_{t,t}\}$. Set $x_{i,t}$ for $i < t$ according to the following rules:

- (i) Set $x_{i,t} = x_{i,t-1}$ for all $i < t$
- (ii) if $t > S$, then with probability q , choose $k \in D$, and set $x_{k,t} = x_{t,t}$.

Given this model setup, if $p = 0$ and $q = 0$, we have a Bernoulli process. Balls are chosen randomly and balls in the urn are not changed. If $p = 1$ and $q = 0$, we have the standard Polya process that we described in the introduction. If $p = 1$ and $q = 1$ we have the accumulation process that we defined in the previous section.

Given this framework, we can prove several results. We first prove a result for the special case $p = 1$, where the current outcome depends entirely on the proportions of the two colors in the urn.

Claim 2 *In any equilibrium for a process in $F(1, q, S)$ where $q \in (0, 1]$ all outcomes will be of the same type.*

pf. Using the notation from the previous claim:

$$E_{F(1,q,S)} \left[\frac{V_{T+1}}{T+1} \right] = \left[\frac{V_T}{T} \right] \left[\frac{V_T + 2q + (1-q)}{T+1} \right] + \left[\frac{T - V_T}{T} \right] \left[\frac{V_T - q}{T+1} \right]$$

Rearranging terms gives

$$E_{F(1,q,S)} \left[\frac{V_{T+1}}{T+1} \right] = \frac{V_T}{T} + q \frac{2V_T - T}{T(T+1)}$$

As in the previous claim, the result follows immediately. Unless $2V_T = T$, the rasion $\frac{V_T}{T}$ will be either monotonically increasing or decreasing.

The previous result states that if there exists any path-revision, then all outcomes converge to the same type if the process does not have any random outcomes. The intuition for this result is as follows. Notice that this subclass of processes combines two dynamics, the accumulation process and the Polya process. The former tugs the distribution to the end points. The latter keeps the distribution at its current proportions but only weakly so. The equilibria in Polya are only weakly stable. If you disturb the status quo proportion, the Polya process produces no force to return to the status quo. Metaphorically, you can think of an equilibrium of the Polya process like a ball resting on a flat surface, say a table. If you move the ball a little to the left or right, it will remain in its new location. The accumulation process,

in effect, pulls down on both ends of the flat table. In this way, the drift to the extremes created by the accumulation process becomes the dominant dynamic.

This result has implications for empirical claims of path dependence. If one models a dynamic as a Polya process, one would expect a priori that any distribution of types is possible. The previous claim shows that if even a small amount of path revision exists, then that result goes away; equilibria are significantly constrained. We discuss this implication at greater length in the next section.

We next consider the case where p is less than one. This allows for randomness and resurrects the conventional Polya possibility that anything can happen. However, as the claim shows, such processes are knife-edge in the class of all processes.

Claim 3 *The equilibrium for a process in $F(p, q, S)$ depends on the value of $p + 2pq$ as follows:*

(i) *if $p + 2pq > 1$, in equilibrium with probability one, all $x_{j,t}$ are of the same type.*

(ii) *if $p + 2pq < 1$, in equilibrium each outcome exists in equal proportions.*

(iii) *if $p + 2pq = 1$, any proportion of each outcome can be an equilibrium.*

pf. With probability p the outcome depends on the V_T , and with probability $(1 - p)$ the outcome is equally likely to be a 0 or a 1. Therefore, it follows that

$$E_{F(p,q,S)} \left[\frac{V_{T+1}}{T+1} \right] = p E_{F(1,q,S)} \left[\frac{V_{T+1}}{T+1} \right] + (1-p) E_{F(0,q,S)} \left[\frac{V_{T+1}}{T+1} \right]$$

Given the definition of the process,

$$E_{F(0,q,S)} \left[\frac{V_{T+1}}{T+1} \right] = \frac{1}{2} \left[\frac{V_T + 2q + (1-q)}{T+1} \right] + \frac{1}{2} \left[\frac{V_T - q}{T+1} \right]$$

Canceling terms gives

$$E_{F(0,q,S)} \left[\frac{V_{T+1}}{T+1} \right] = \frac{V_T + \frac{1}{2}}{T+1}$$

This can be rewritten as

$$E_{F(0,q,S)} \left[\frac{V_{T+1}}{T+1} \right] = \frac{V_T}{T} + \frac{1}{2} \frac{T - 2V_T}{T(T+1)}$$

Plugging in the value for $E_{F(1,q,S)} \left[\frac{V_{T+1}}{T+1} \right]$ from the previous claim gives

$$E_{F(p,q,S)} \left[\frac{V_{T+1}}{T+1} \right] = \frac{V_T}{T} + pq \frac{2V_T - T}{T(T+1)} + (1-p) \frac{1}{2} \frac{T - 2V_T}{T(T+1)}$$

It follows that

$$E_{F(p,q,S)} \left[\frac{V_{T+1}}{T+1} \right] = \frac{V_T}{T}$$

if and only if $2pq = 1 - p$, which can be rewritten as $p + 2pq = 1$.

To complete the proof, it suffices to recognize that if $2pq + p > 1$ then the process amplifies whichever outcome is in greater proportion and if $2pq + p < 1$, the process converges to each outcome being equally likely.

The result would appear to undermine the conventional logic in the path dependence literature that anything can happen. Note that the Polya process, the Bernoulli process, and the accumulation process are all special cases of this general result. One way to read the claim is that if historical processes include a little bit of randomness, dependence, or both, then we should not expect any equilibrium to be possible. We should expect extreme outcomes or equally likely outcomes. The model implies that to get extreme path dependence, the process, be it the Polya process or any other member of the family of processes in which $p + 2pq = 1$, must achieve a delicate balance between heading to the middle or to the extremes. These cases are knife edge.

One question that might be asked is whether path revision could preclude the “anything can happen” result. Surprisingly, the answer here is ‘no’. Any amount of path revision q can be offset by choosing a p , the extent of path dependence of outcomes, to satisfy the constraint $p + 2pq = 1$. In the most extreme case, where $q = 1$, p must equal $\frac{1}{3}$ in order to recover the possibility of any distribution. To see how this process can stabilize at any proportion of outcomes, suppose that 80% of outcomes were of type 1. Two-thirds of the time the outcome would be random. This results in an expected addition of 0.5 to the

value of V_T . One-third of the time, the outcome would depend on the proportion $\frac{V_T}{T}$ which equals 0.8. Thus, with probability 0.8, V_T would increase by two, and with probability 0.2 it would decrease by one. Combining all of these expected contributions yields

$$V_{T+1} = V_T + \frac{2}{3}[0.5] + \frac{1}{3}[0.8(2) + 0.2(-1)] = V_T + 0.8$$

So, the expected proportion remains unchanged.

One counterintuitive corollary of this claim is that all of the outcomes can converge to one type even if most of the outcomes are chosen by an equal probability Bernoulli Process.

Corollary 1 *For any $p > \frac{1}{3}$, there exists a probability of path revision sufficiently high such that all outcomes are eventually of the same type.*

This corollary implies that path revision can drive the equilibrium to one in which all outcomes become the same type even though when those outcomes first occur they tend to be random. It's worth working through an example to see the logic. Suppose that $p = \frac{1}{2}$, so that half of all draws are Bernoulli with equal likelihood and that $q = 1$ so that in each period a previous outcome gets revised. Thus, each period half of the time the draw is a random draw. What we'll show is that over time, all of the outcomes will get revised to be of the same type.

To see how this can happen, first note that an equal distribution isn't an equilibrium because whichever outcome gets drawn will become more likely in the next period due to revision. So, let's suppose

that we have an equilibrium in which some proportion $r \in (\frac{1}{2}, 1)$ of the balls are of one type, or color. For the purposes of this example, let red be the more likely color. In the next period, with probability one-half, we draw a ball from the urn. This means that we'll add a red ball with probability r . And, with probability one half, we draw a random ball, which will be red with probability one-half. Combining these probabilities, we get that the expected probability of drawing a red ball equals $\frac{2r+1}{4}$. It follows that the probability of drawing a ball of the other color equals $\frac{3-2r}{4}$.

Recall that revision occurs with probability one. So, after we draw a ball of either color, we add a ball of that color and also change the color of an existing ball of the opposite color. Therefore, there exists one more ball in the urn than there was previously. Overall then, with probability $\frac{2r+1}{4}$, we have two more red balls and with probability $\frac{3-2r}{4}$ we have one less red ball. Therefore, the expected increase in the number of red balls equals $\frac{(4r+2)}{4} - \frac{(3-2r)}{4}$ which equals to $\frac{(6r-1)}{4}$. Thus, the ratio of the red balls will increase provided $\frac{(6r-1)}{4}$ exceeds r . This will be satisfied provided that $2r > 1$, which holds given our assumption that the urn contained mostly red balls. Therefore, even though half of the outcomes are random, those outcomes that are not red initially, eventually are revised to become red.

A second corollary relates the amount of path revision needed to produce homogeneity as a function of the amount of path dependence.

Corollary 2 *The probability of path-revision q required to produce all*

outcomes of the same type in the long run decreases in p , the amount of path dependence.

This corollary states that the more path dependence in a process, i.e. the less randomness, the less path-revision that needs to occur in order to produce a single type of outcome in the long run. Thus, if a process appears highly path dependent, then it only takes a little bit of revision to drive the system to homogeneity, but if the process only has a little bit of path dependence, the probability of path revision must be rather large for the process to result in a homogeneous distribution.

So far, we have assumed Bernoulli processes in which the probability of the two outcomes are equally likely. One might also consider cases in which one of the outcomes has an advantage. The next claim states a somewhat surprising result. When one of the outcomes has an advantage, this outcome is only guaranteed to be the equilibrium in the case where $p + 2pq = 1$, the knife edge case. If path-revision and dependence are stronger than that, the other outcome could come to dominate. If the forces are weaker, then the equilibrium is internal with the advantaged outcome more prominent. Note that none of the cases support the conventional path dependence claim that any equilibrium distribution is possible.

Claim 4 *Let $b > \frac{1}{2}$ equal the probability of getting an outcome of one from the Bernoulli process.*

(i) if $p + 2pq = 1$, in equilibrium all x_{jt} are of type one.

(ii) if $p + 2pq > 1$, in equilibrium with probability one, all x_{jt} are of the same type.

(iii) if $p + 2pq < 1$, the proportion of outcomes of type one equals $b + \frac{q(1-2b)}{(2q+p-1)}$.

Recall from claim 3 that the expected proportion of type one outcomes in time $T + 1$ can be written as follows:

$$E_{F(p,q,S)} \left[\frac{V_{T+1}}{T+1} \right] = pE_{F(1,q,S)} \left[\frac{V_{T+1}}{T+1} \right] + (1-p)E_{F(0,q,S)} \left[\frac{V_{T+1}}{T+1} \right]$$

With a biased Bernoulli process, $E_{F(1,q,S)}$ does not change. Thus, we need only consider the second term, $E_{F(0,q,S)}$. Define the Bernoulli process such that the probability of choosing outcome 1 is b , $Pr\{x_{t,t} = 1\} = b$. We can now solve for the second term as follows:

$$E_{F(0,q,S)} \left[\frac{V_{T+1}}{T+1} \right] = bq \left[\frac{V_T + 2}{T+1} \right] + b(1-q) \left[\frac{V_T + 1}{T+1} \right] + (1-b)q \left[\frac{V_T - 1}{T+1} \right] + (1-b)(1-q) \left[\frac{V_T}{T+1} \right]$$

Simplifying this expression we obtain the following:

$$E_{F(0,q,S)} \left[\frac{V_{T+1}}{T+1} \right] = b \left[\frac{V_T + 2q + (1-q)}{T+1} \right] + (1-b) \left[\frac{V_T - q}{T+1} \right]$$

Plugging this back into the general expectation expression, $E_{F(p,q,S)}$,

$$E_{F(p,q,S)} \left[\frac{V_{T+1}}{T+1} \right] = \frac{V_T}{T} + pq \frac{2V_T - T}{T(T+1)} + (1-p) \frac{bT + qT(2b-1) - V_T}{T(T+1)}$$

To simplify notation, let $\alpha = (2pq + p - 1)$. Substituting the

above equilibrium condition back into the equation for $E_{F(p,q,S)}$ and re-arranging, we have,

$$E_{F(p,q,S)} \left[\frac{V_{T+1}}{T+1} \right] = \frac{V_T}{T} + \left[\frac{1}{T(T+1)} \right] \left[\alpha T \left(\frac{V_T}{T} - b \right) + qT(2b-1) \right]$$

We now consider the three cases:

Case (i) $p + 2pq = 1$: In the expression above $\alpha = 0$. Rewriting the previous equation gives

$$E_{F(p,q,S)} \left[\frac{V_{T+1}}{T+1} \right] = \frac{V_T}{T} + \left[\frac{1}{T(T+1)} \right] [qT(2b-1)]$$

Given that $(2b-1) > 0$, it follows that in equilibrium all $x_{j,t}$ are of type one.

Case (ii) $p + 2pq > 1$: In equilibrium $E_{F(p,q,S)} \left[\frac{V_{T+1}}{T+1} \right] = \frac{V_T}{T} = R^*$. Therefore,

$$\alpha(R^* - b) + q(2b-1) = 0,$$

which can be rewritten as

$$R^* = b - \frac{q(2b-1)}{\alpha}.$$

Given $\alpha > 0$, a straightforward argument shows that R^* is unstable. To see this, let $\frac{V_T}{T} = R^* + \epsilon$. The expression

$$\alpha \left(\frac{V_T}{T} - b \right) + q(2b - 1)$$

can then be written as

$$\alpha(R^* + \epsilon - b) + q(2b - 1) = \alpha\epsilon > 0.$$

Therefore, a perturbation above R^* leads to an increase in the proportion of type one outcomes. Similarly, a perturbation below R^* leads to a decrease. The result follows.

Case (iii) $p + 2pq < 1$: From the previous case, we have that the equilibrium proportion of outcomes of type one equals $b + \frac{q(1-2b)}{(2q+p-1)}$. Using the same argument as above, it can be shown that this equilibrium is stable, completing the proof.

To summarize, if one outcome has an advantage, then in the knife edge case it takes over with probability one. So here we do not get the possibility that anything can happen. If the forces of path revision and path dependence are strong, then the advantaged outcome is more likely to be the equilibrium, but the other outcome could become the equilibrium if it got an early advantage and then leveraged the revision process. If the forces are weak, then there is an internal, mixed equilibrium in which the advantaged outcome occurs with greater frequency.

Discussion: The Empirical Content of Path Dependence

The theoretical results that we have presented in this paper raise several empirical issues. Recall from the introduction that path dependent outcomes need not imply that long run equilibria are path dependent (Page 2006). A system can have path dependent outcomes and it can also have path dependent equilibria. In this paper, we've focused on path dependent equilibria. We've shown that when path revision is possible, path dependent equilibria will either tend to equal numbers of each type or to homogeneous distributions.

These results stands in sharp contrast with the conventional interpretation of path dependent equilibria, where any distribution is equally likely. The results highlight the importance of distinguishing between the concept of path dependence and the notion that anything can happen. A process can be path dependent, but it might not produce any possible outcome. In fact, for the class of processes that we consider, if the forces of path revision and path dependence are strong (formally, if $p + 2pq$ exceeds one), the outcomes are indeed path dependent, but they can only be one of two extremes. So, one could argue that the process is even more path dependent than the Polya process, but one cannot say that anything can happen. In fact, only two things can happen, and both are extreme.

We base that conclusion on the mathematical fact that $p + 2pq = 1$

is a knife edge case; it is of lower dimension than the space of possible distributions. If p and q are exogenous and drawn from continuous, non-atomic distributions, lying on the line would be a zero probability event. One might challenge our inference by an argument that processes adjust so as to lie on that line, thus making p and q endogenous. For processes to converge to the line, when $p + 2pq > 1$, actors would have to introduce some randomness (lowering p) or revise paths less often. And, when the opposite inequality holds, they would have to decrease randomness and revise outcomes more often. Other than having a goal of creating the possibility of anything happening, we see no reason why either of these forces, let alone both, would arise. Furthermore, if we relax the assumption of equally likely random outcomes, i.e. if one of the outcomes is advantage, then our final claim states that the anything can happen result goes away entirely. In other words, the anything can happen result requires the random outcomes to be equally likely and that the parameters satisfy the linear constraint. Thus, we conclude that the anything can happen results may be less likely than previously thought.

In thinking about the dynamics of the process, the magnitude of $p + 2pq - 1$ is of central importance. If this value is near zero, or near its maximum of three, the resulting convergence times will be very fast. If on the other hand it takes on a value close to one, convergence could take a long time. Also, even though we would know that the equilibrium is homogenous, the equilibrium might not be a good

predictor for relevant time periods for political processes of interest. Thus, a process that's drifting toward all outcomes becoming of the same type could appear to be path dependent for even quite a long time series.

For empiricists attempting to identify path dependence in qualitative or quantitative data (eg Bennett and Elman 2006, Vergne and Durand 2010, Jackson and Kollman 2010), our findings suggest the importance of testing for path revisions in the data. Suppose, for example, that an empirical model assumed no path revision and estimated p to be 0.95. The prediction of such a model would be equal numbers of each type. But suppose that in the process generating the data that $q = 0.07$, so that 7% of the time, a ball in the urn would switch its color to match the current selection. Notice that $p + 2pq$ exceeds one in this case, so that the process would have an equilibrium of all balls having the same color. Therefore, the statistical finding of p less than one would not be in alignment with the empirical finding.

The empirical stakes concern issues much larger than multiple colored balls being drawn from urns. Beliefs, behaviors, laws, and institutions all exhibit some degree of path dependence. To a lesser extent, they also may exhibit path revision. In the case of ongoing behaviors such as shown in the experimental data, those revisions may occur frequently. In law, where precedent has a long tradition, revisions may be less likely, however, as our results show, a little revision is all that's needed. For the sort of structural path dependence considered

by Pierson (2004), revision entails changing past actions, so this too may be rare. However, as we show even a little revision can lead to extreme outcomes.

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