Optimal Adoption of Complementary Technologies

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When a production process requires two extremely complementary inputs, conventional wisdom holds that a firm would always upgrade them simultaneously. We show, however, that if upgrading each input involves a fixed cost, the firm may upgrade them at different dates, "asynchronously." This insight helps us understand why productivity rises with the age of a plant, why investment in structures is more spiked than equipment investment, and why plants have spare capacity. The bigger point of the paper is that complementarity does not necessarily imply comovement—not even for a single decision maker. (*JEL* E22, O31, P11)

Firms often use inputs whose qualities are complements—e.g., computer and modem, equipment and structures, train and track, transmitter and receiver.[1] In the absence of adjustment costs, this complementarity should induce quality upgrades to be *synchronous:* When raising the quality of an input, the firm should upgrade its complements at the same time.

As long as they are convex, input-adjustment costs do not change this conclusion. When Tobin's q rises, for instance, a firm will step up its investment in all the complementary capital goods. The adjustment of these inputs is partial, but still synchronous, as Fumio Hayashi and Tohru Inoue's (1991) model implies.

For many inputs, however, the costs of quality adjustment are not convex. For example, replaced buildings and machines often cannot be sold off to another user, and are therefore scrapped even though they still function. The positive in-house value of such old capital represents an opportunity cost of replacing it. It acts as a fixed cost on upgrading capital, and it implies that optimal upgrading will be spaced out, and that investment will exhibit "spikes."

Such lumpiness in the costs of adjustment also implies that the optimal adjustment of inputs is often *asynchronous.* Even if the inputs are extreme complements—Leontief—the firm will tend to upgrade them at different dates. The firm will reduce the fixed costs of repeated upgrades by upgrading an input by a lot, and then waiting for the quality of the other input to catch up.

Upgrading an input "by a lot" creates excess capacity of that input. This explanation of the phenomenon differs entirely from the usual one that spare capacity arises because investment is irreversible and demand is uncertain, and relies, instead, on lumpy costs of implementing new technology. It is, we believe, the natural explanation for why, for example, office buildings and manufacturing plants are improved infrequently and yet are filled, as they age, with better and better equipment. It also explains why, as a plant ages, labor productivity grows—the plant does not get better with age, but the equipment that it houses gets better over time.

Upgrading an input by a lot also creates bigger spikes in investment. Data confirm this as well. For instance, Michael Gort et al. (1999) find that for surviving buildings there is evidence of major reconstruction or renovation after about the 45th year, yet only slowly rising maintenance and repair until then. Since office equipment is replaced every few years, this

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[1] A referee suggested that that theory and data are complements in producing good economics.

means that the lumpiness in capital outlays on structures far exceeds that for equipment.

We fit our model to sectoral data on plant and equipment.[2] The model correctly predicts several cross-industry correlations: the negative correlation between the average age of inputs and the growth rate of labor productivity, and the positive correlation between the average ages of equipment and structures.

More importantly, our model explains a feature of the data that the standard model cannot. In the one-capital model, when the cost rises, upgrading is delayed so that the age of capital rises. Yet in our data, when the cost of structures rises, the age of structures *falls*. Our model explains this because it has optimal adjustment of spare capacity: Cheap structures have more spare capacity, so they are not replaced for a long time. When the cost of structures rises, up-front spending on them falls, and so does their spare capacity, and their average age.

I. The Model

This section describes the optimal adoption policy of a firm that faces exogenously given prices of its output and its two inputs, and an exogenously advancing technological frontier. The three sets of key assumptions are the following:

The Production Function.—A firm produces output $y(t)$ with a production function that is Leontief with inputs x and z being the services provided by two capital goods:

$$(1) \qquad y(t) = \min\{x(t), z(t)\}.$$

The third input—labor—is normalized at unity. So the firm's only worker uses two machines, and he does not share them with other workers. The quantity of each input is normalized at unity, and each input is indivisible. An example of such a process could be the output of a worker that uses a computer of quality x and a modem of quality z.

Technological Upgrading.—Because the firm must use exactly one unit of each input, raising the quality of an input requires buying a better one and scrapping the old one. The scrap value of each machine is zero. Purchases of z are constrained by the exogenously growing frontier $Z(0)e^{gt}$. On the other hand, we assume that *any* quality of x is a feasible choice.[3]

Prices.—The price of output is unity. Let the input prices per unit quality be constant and equal to p_x and p_z.[4] This makes the solution of the firm's problem stationary.

A few comments about these assumptions. We have suppressed the multiplicative constants in the production function. Now this can be accomplished by rescaling either the numbers of inputs or their quality.[5] The replacement of old inputs by new ones occurs because proportions are fixed and because technological progress makes it possible to buy better inputs over time.

Before stating the general problem formally, we shall sketch two of its special cases.

A. *The Case Where $p_x = 0$*

When x is free, the model degenerates, effectively to a standard, one-capital vintage

[2] These two inputs sometimes appear to be extreme complements. In the silicon-chip industry, for example, the quality of a computer chip depends on two characteristics: the precision of the equipment, and the quality of the air in the production facility. No amount of equipment precision can make up for dirty air, and vice versa.

[3] This may be because the frontier of x had grown faster than that of z in the past, and because, perhaps, the two had grown at roughly the same rate since.

[4] The one-capital-input model is a special case of ours: If $p_x = 0$, then for all possible values of the other parameters, the model behaves exactly as if there were just one input, z.

[5] Unless we can measure x and z directly, the statement "input x is relatively cheap" is equivalent to saying that "input x is relatively unimportant in production." If, instead of (1) we were to write

$$y = \min\left\{\frac{x'}{a}, \frac{z'}{b}\right\},$$

and if we were to assume that p'_x and p'_z are the prices per unit of quality of x' and z', respectively, this setup would be equivalent to (1), so long as we defined $p_x = ap'_x$, and $p_z = bp'_z$. If a were small, input x' would be relatively unimportant in production. Under the assumed form in (1), this would show up as a low value of p_x.

model. The upgrading of x is indeterminate: The only property of $x(t)$ that the firm cares about is that it never fall below $z(t)$. The firm's upgrading of z will behave as if the production function was

$$y = z,$$

so that our model contains the one-capital model as a special case.

The reader may wonder when our model differs the most from the standard one. The analysis of the general case suggests that the difference is largest when $p_z = 0$, and this is the case we turn to next.

B. *The Case Where* $p_z = 0$

This case will highlight the following point: Asynchronous adjustment and spare capacity *must* arise in the general case if input z is cheap. Moreover, it is the *expensive* input that adjusts less frequently, and that, as a result, experiences excess capacity. The intuition carries over to the general case, but it is the clearest in the case when $p_z = 0$.

If z is free, it will cost the firm nothing to set $z = Z$ whenever necessary, so that output is $y(t) = \min\{x(t), Z(0)e^{gt}\}$. Suppose that $x(0) = 0$ and $Z(0) = 1$. The firm has to decide when to buy new x and how much to buy. Suppose that the firm chooses to upgrade x every T periods.[6] The first purchase of x must occur at $t = 0$, because otherwise no output will be produced. Let the quality of x purchased at time 0 be equal to x_0. Whenever

$$p_x < \frac{1}{r}, \tag{2}$$

purchasing x yields positive profit, so $x_0 \geq Z(0) = 1$. It is clearly wasteful to set $x_0 > \max_{[0,T]} Z(t) = e^{gT}$, so $x_0 \leq e^{gT}$. Therefore, since $1 \leq x_0 \leq e^{gT}$ during the first T periods, $x(t)$ spends $\tau = (1/g) \ln x_0$ periods above the frontier and $T - \tau$ periods below the frontier. Likewise, every time new x is purchased, the size of the purchase will be chosen for x to spend τ periods above the frontier and $T - \tau$ periods below the frontier. Then, the jth purchase of x must be of size $x_j = x_0 e^{jgT} = e^{g\tau} e^{jgT}$. The values of τ and T will maximize the discounted present value of the lifetime profit

$$(\tau, T) = \arg\max \left(\int_0^\infty e^{-rt} y(t)dt - \sum_{j=0}^\infty e^{-jrT} x_j \right)$$

$$= \arg\max \left(\frac{\int_0^\tau e^{-(r-g)t}dt + \int_\tau^T e^{-rt} e^{g\tau}dt - e^{g\tau}p_x}{1 - e^{-(r-g)T}} \right),$$

which is equal to the discounted present value of output minus the discounted present value of the cost of x. This periodic solution is depicted in Figure 1, and we will further prove that it is, in fact, optimal.

Asynchronous Adjustment.—As long as $T > 0$, the qualities of the two inputs grow asynchronously. Every T periods, the expensive input, x, leaps ahead, and then the cheap input, z, catches up gradually. When z has caught up after τ periods, both inputs are kept constant until the next purchase of x.

Excess Capacity.—Whenever x is above the frontier, a fraction of the investment in x sits idle. Let "spare capacity," S, be the average percentage of x that sits idle over the interval $[0, T]$:

$$S = \frac{1}{T} \int_0^\tau \frac{e^{g\tau} - e^{gt}}{e^{g\tau}} dt = \frac{\tau}{T} - \frac{1 - e^{-g\tau}}{gT}.$$

By construction, S is between 0 and 1. As Figure 1 shows, S is increasing and concave in g, and decreasing and convex in r. This makes sense since spare capacity is, in our model, a form of investment. The value of that investment is higher when g is higher because the

[6] In the next subsection, we show that evenly spaced upgrades of x are optimal.

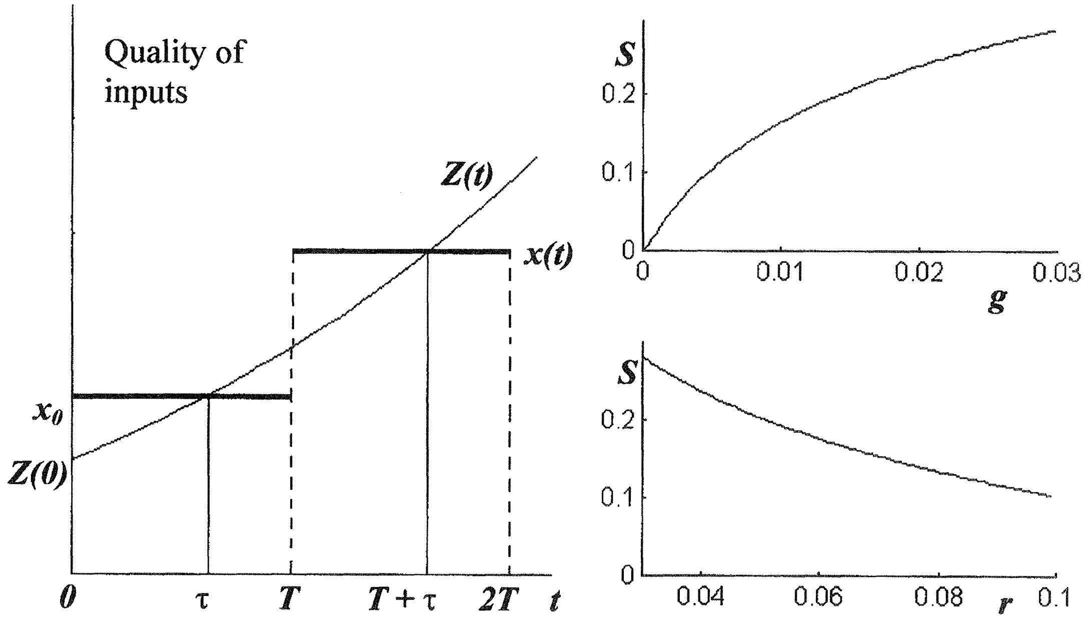

FIGURE 1. THE OPTIMAL SOLUTION AND THE OPTIMAL SPARE CAPACITY, $p_z = 0$

complementary input catches up sooner. The forgone earnings cost, on the other hand, is higher when r is higher. Not portrayed in the figure is a negative relation between S and p_x. As x becomes more expensive, T rises and τ falls, so spare capacity declines monotonically as p_x goes up, until it finally reaches zero when (2) is violated and x is never purchased.

C. *The General Case*

When $p_z = 0$, the adoption policy is asynchronous. But when $p_z > 0$, the firm no longer upgrades z continuously, but, rather, at discrete points in time. At some point, when p_z is sufficiently large, upgrading becomes *synchronous,* and we shall now derive and interpret the conditions that determine when asynchronous upgrading is optimal.

Let T_j ($j = 0, 1, \cdots$) be the dates when the firm upgrades x, and let x_j be the quality of the machine bought at date T_j. Likewise, the firm upgrades z at dates τ_i to quality levels z_i. Then, $x(t)$ and $z(t)$ are increasing step functions, determined fully by their initial conditions x_0 and z_0 and by the sequences $\{x_j, T_j\}$ and $\{z_i, \tau_i\}$. Let $Z(t) = Z(0)e^{gt}$, so that the constraint on the purchases of z reads $z_i \le Z(\tau_i)$, all i.

The Firm's Decision Problem.—Given $x(t)$ and $z(t)$, the firm's discounted lifetime profit is

$$(3)\quad V(x(t), z(t)) = \int_0^\infty e^{-rt} \min\{x(t), z(t)\} dt$$

$$- \sum_{i=0}^{\infty} e^{-r\tau_i} p_z z_i - \sum_{j=0}^{\infty} e^{-rT_j} p_x x_j .$$

The first term is the present value of revenues, which accrue continuously. The remaining two terms are the present values of the costs of the two inputs. The firm maximizes V with respect to $x(t)$ and $z(t)$, subject to given initial conditions x_0, and z_0, and subject to $z_i \le Z(\tau_i)$ for all i.

The Origin and Role of Nonconvex Adjustment Costs.—The crucial assumption determining the optimal solution is lumpy (nonconvex) costs of raising the quality of x and z. In our model they arise because the resale value of the existing input is less than its purchase price. The difference between the two acts as a fixed cost of replacement. Without this cost, both inputs would be upgraded continuously and would follow the frontier: $x(t) = z(t) \equiv Z(t)$. To simplify the analysis we assume that resale value of old equipment is zero—it is scrapped.

Asynchronous Replacement.—The fixed costs of raising x and z will now prevent the firm from setting $x(t) \equiv Z(t)$ and $z(t) \equiv Z(t)$ all the time. The results of the previous subsection should lead one to expect that, by continuity, for low values of p_z, the firm will upgrade z more often than x, so that the firm *still* will not set $x(t) = z(t)$ all the time and that, instead, the firm will sometimes buy these inputs at different dates. The firm will trade off the following two considerations:

(a) buying x together with z all the time to avoid letting x sit idle while z catches up, and
(b) buying $x > Z$ up front and then replacing only z until it catches up with x and thereby saving on the fixed cost of repeat purchases of x.

If (a) is more important than (b), the optimal policy will be synchronous. If (b) dominates, however, the policy will be asynchronous. The above intuition is confirmed by the following result which characterizes the optimal solution to (3) from any initial conditions. The result presumes that

$$p_x + p_z < \frac{1}{r}, \tag{4}$$

a condition that guarantees that upgrading is more profitable than maintaining the current input levels forever.

THEOREM 1: *From some date $\tau_0 < \infty$, the optimal solution to (3) is fully characterized by a vector $(n, T, t_1, \ldots, t_{n-1})$ such that:*

(*i*) *Input x is upgraded periodically with a constant period T. That is, $T_j = jT$ for every j.*
(*ii*) *Input x is purchased at most as often as z, and the relative frequency of upgrading (number of upgrades of z per one upgrade of x) is constant and equal to n. This says that z may be upgraded while x is constant, which implies that same x is used with several different z's.*
(*iii*) *Input z is always upgraded to the frontier, i.e., $z_i = Z(\tau_i)$, and $\tau_i = \tau_{jn+k} = jT + t_k$, $t_0 = 0$, $k = 0, \ldots, n - 1$. This says that x is always purchased together with z, but sometimes only z is purchased, but x is not. Thus, when $n > 1$, upgrades are asynchronous.*
(*iv*) *$x_j = max_{[jT,(j+1)T]} z(t) = Z(jT + t_{n-1})$. This says that x is purchased up front ($x_j \geq Z(T_j)$ for every j) and the quality of x matches the quality of the best z that is used together with this x.*

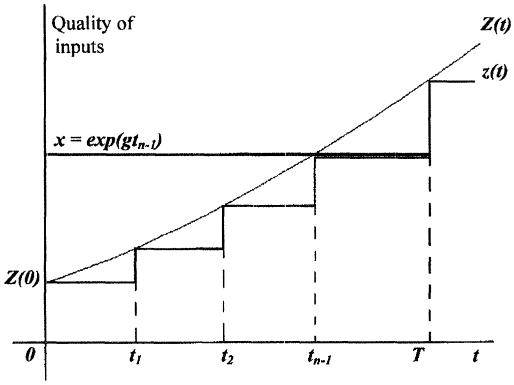

FIGURE 2. THE OPTIMAL SOLUTION

PROOF:
In the Appendix.

The theorem states that the periodic solution—synchronous or asynchronous—is globally stable. Moreover, the approach to the periodic upgrading pattern occurs not asymptotically, but in finite time, τ_0.

Figure 2 is the general case analog of Figure 1. It illustrates the statement of the theorem and depicts the optimal solution $(x(t), z(t))$ on the interval $[0, T]$. Note that for the policy that the figure illustrates:

(1) Every nth purchase of inputs is synchronized because of the extreme complementarity. Because of this, $x(t)$ and $z(t)$ both spend some time below $Z(t)$. In the case of $z(t)$, this is because the positive cost of upgrades forces them to be spaced out. And

$x(t)$ must be below $Z(t)$ precisely because just prior to the nth (synchronized) purchase, x and z must be equal.

(2) Purchases of inputs are generally asynchronous: $n - 1$ purchases of z occur with no accompanying adjustment in x.

(3) $x(t) \geq z(t)$ always, and only x ever has spare capacity. Spare capacity is the highest right after a purchase of x, and it then declines as z gradually catches up with x. This implies that productivity, z, rises as input x ages.

(4) Investment has spikes when inputs are purchased. Spikes are relatively bigger for x than they are for z, because x is replaced less often.[7]

(5) Since x is replaced less often than z, input x is, on average, older.

Properties (2)–(5) hold if and only if $n > 1$ is optimal.[8] It may instead be optimal that $n = 1$, in which case x and z behave as if they were one input. This means that upgrades will be perfectly synchronized and neither input will have spare capacity at any time.[9]

The nature of the solution depends critically on whether or not $n > 1$. Generally, when z is replaced frequently enough—and this will happen when z is cheap enough—it never pays the producer to buy x together with z every time and repeatedly incur the adjustment cost of x. Then $n > 1$ is optimal. We will see frequent upgrades of z if:

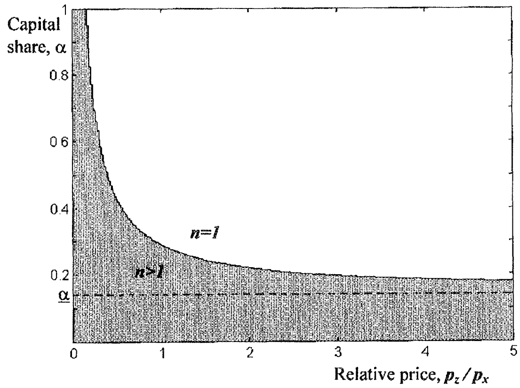

FIGURE 3. CAPITAL SHARES AND RELATIVE PRICES OF INPUTS FOR WHICH $n > 1$ IS OPTIMAL

(a) The *relative* price of z, p_z/p_x, is low. Although the optimal time between replacements of z depends on both prices, p_z has a stronger effect on it because every purchase of inputs involves z, but only every nth purchase involves x. The analysis in Subsection B showed that when $p_z = 0$, n is infinite, because z is upgraded every instant. By continuity, for small positive values of p_z, $n > 1$ is optimal for all p_x that satisfy (4).

(b) If both inputs are sufficiently cheap, it is optimal to replace z often enough to make $n > 1$ optimal, no matter what the relative price is. We choose $\alpha = r(p_x + p_z)$ as a measure of the capital share.[10] When $\alpha \leq \underline{\alpha}$, $n > 1$ is always optimal.[11]

Figure 3 portrays these results graphically. It shows the area where $n > 1$ is optimal as a function of relative price, p_z/p_x, and the share of capital, α. If p_z/p_x is sufficiently low, z becomes so much cheaper than x that it does not pay to replace x as frequently as z no matter

[7] The next section describes precisely how the spikes are measured.

[8] An example where n is apparently greater than one is modems and computers, which are complements in the Internet connection technology. Most desk PCs come equipped with a modem. But there is also a retail market for modems which means that some users prefer to upgrade the modem only, but not the computer. Thus investment is asynchronous: some upgrades involve buying one input, but not the other. The purchase of each type of machine is indivisible, but modems are a lot cheaper than computers, and the nonconvexity is more important in the case of a computer purchase, and so computers are upgraded less often.

[9] Beginning from time τ_0, this synchronous policy has both inputs upgraded periodically, with a constant period T. The level of both inputs at the time of upgrading is equal to the frontier at that time. That is, for every, $T_i = \tau_i = \tau_0 + T_i$, $z_i = x_i = Z(\tau_0 + Ti)$.

[10] This measure equals to the total cost of the two inputs divided by the present value of output that can be produced from them.

[11] The value of $\underline{\alpha}$ depends on r and g and can be computed only numerically. The analytical lower bound for $\underline{\alpha}$ equals

$$\underline{\alpha} \geq 1 - \frac{\left(\frac{1}{2}\right)^{\gamma} - \frac{\gamma}{2}}{(1 - \gamma)}, \qquad \text{where } \gamma = \frac{g}{r}.$$

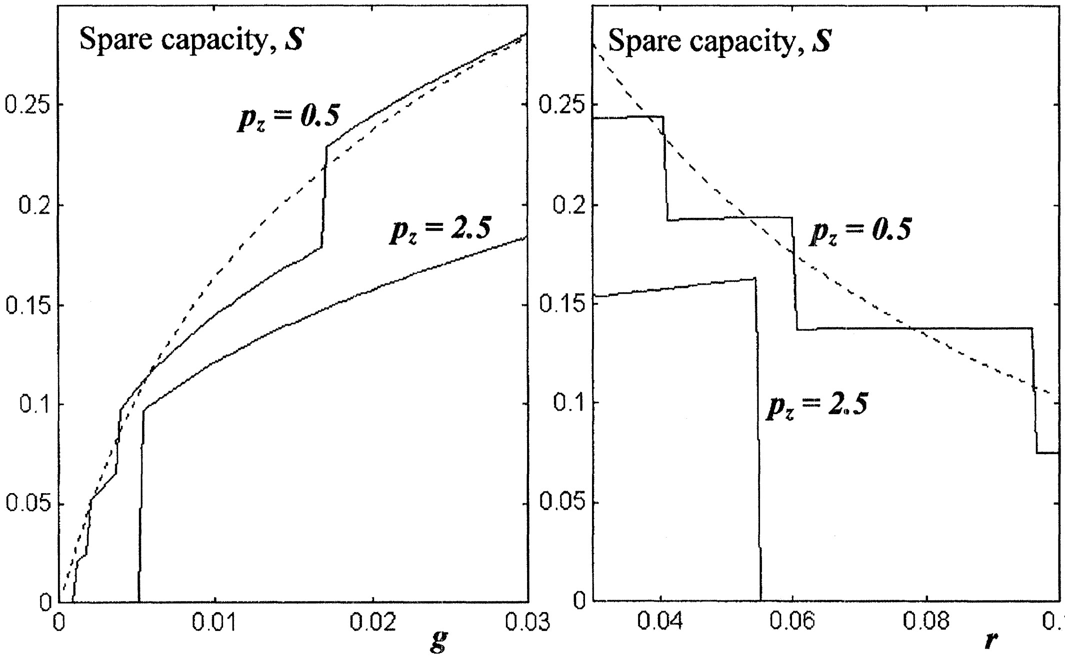

FIGURE 4. OPTIMAL SPARE CAPACITY AS A FUNCTION OF g AND r FOR DIFFERENT VALUES OF p_z

what the value of α is. The convex borderline on the figure corresponds to the maximum value of α for which $n > 1$ is still optimal, given the relative price p_z/p_x. As the relative price p_z/p_x increases, the maximum α for which $n > 1$ is optimal decreases, until α reaches $\underline{\alpha}$ in the limit.

Spare capacity, S, is another measure of asynchronization. S is zero when $n = 1$, and positive when $n > 1$. Figure 4 shows how g and r affect S for different values of p_z. The dashed lines on the figure show the limit case when $p_z = 0$, displayed earlier in Figure 1. The intuition remains the same as discussed at the end of subsection B.

Spare capacity increases in g for two reasons. First, higher g raises the opportunity cost of staying below the frontier making upgrades of z more frequent, which in turn increases n. Second, for a fixed n, higher g lowers the fraction of time that x works at full capacity.

Spare capacity does not decrease monotonically in r. It decreases only when n falls as r rises. This happens because higher discounting raises the opportunity cost of up front purchase of x, which lowers n. For a fixed n, however, spare capacity increases in r, because with higher r, inputs are replaced less often.[12]

For higher values of p_z, spare capacity is less, because the fraction of time that x works at full capacity increases with p_z.

Nonconvex adjustment costs are *sufficient* to deliver these conclusions and they also are *necessary*. Without them, the firm would upgrade each input continuously, and the (s, S) nature of the optimal policy would disappear.

Mehmet Yorukoglu (1998) has results similar to ours. His calibrated model has a firm making two upgrades in information-technology capital for every one upgrade in physical capital. He assumes a Cobb-Douglas production function so that the elasticity of substitution between the two types of capital is unity. In our case, this elasticity is zero. In this sense, our result is stronger in showing that even perfect complementarity does not guarantee synchronized investment.

[12] This happens because for higher r the present value of output is lower.

II. Application: Equipment and Structures

This section estimates the model using industry-level data, and finds that the model fits relatively well. Some features of asynchronous policies discussed in points (1) through (5) following Theorem 1 are present in plant-level data too. These data indicate that plants have spare capacity that declines with their age. On the one hand, plants are not modified much after they are built: Table 2 of Raford Boddy and Gort (1971) shows that the bulk of firms' spending on plants is on *new* plants, not existing ones. On the other hand, a plant tends to acquire more and more equipment as it ages: Table 3 in Douglas W. Dwyer (1998) shows that as a plant ages, its capital stock grows. One would guess that most of this added capital is equipment. At least, this is what one can infer from James Bessen's (1997) finding that as a plant ages, the capital that it rents grows faster than the capital that it owns, and most new rentals are likely to be equipment. Over 90 percent of spending on new *equipment* is made for existing, as distinct from new, plants (Gort and Boddy, 1967 p. 398). Gort and Boddy also find that reserve capacity in new plants far exceeds that of old plants. Finally, John D. McClelland (1997) finds that investment in structures is more spiked than equipment investment.

In the model, input x is the one that, when $n > 1$, has spare capacity from time to time; input z never does. In light of the plant-level evidence, we label x to be structures, and z to be equipment. When $n > 1$, structures will periodically have spare capacity that declines as equipment is upgraded.[13]

Outline of the Facts and of How the Model Explains Them.—By endogenizing replacement, the ages of the two inputs, and their cost shares, the model has a chance to explain some stylized facts on the relation between an industry's rate of technological progress, the age of its stock of structures and equipment, and the shares of structures and equipment. Here are some facts that emerge from industry-level and plant-level data, and an outline of how our model explains them:

(i) The share of equipment exceeds the share of structures, and the average age of equipment is, in each industry, lower than that of structures. In our model, the share of equipment could exceed the share of structures for two reasons: (a) equipment may be more expensive, and (b) equipment may be replaced more often. If (b) is true, equipment will be younger than structures.

(ii) High-productivity growth industries use younger equipment and structures. In the model this occurs because a high g raises the forgone output costs of lagging behind the frontier technology. This effect is present regardless of whether upgrading is synchronous or not.

(iii) The correlation between spending on equipment and its age is positive and much stronger than the corresponding correlation for structures. In the model this occurs because when structures are expensive, the firm reduces their spare capacity, and hence their average age.

(iv) Investment in structures is more spiked than equipment investment. In the model this occurs because structures are replaced less often than equipment.

(v) As plants age, their productivity rises. In the model this occurs when $n > 1$ because equipment in the same plant is periodically upgraded.

Outline of the Procedure.—We do not observe quality of the two inputs x and z and we do not have industry-level data on the quality-adjusted prices of the two inputs. As a result we shall not be able to distinguish cross-industry variation in p_x and p_z from variation in the coefficients of x and z in the production function. We shall normalize by assuming that all firms in all industries have the production function (1), but that the parameters g, p_x, and p_z are industry specific. We also assume that all firms face the same r.

[13] For example, most new office buildings have many more electrical sockets and telephone jacks than are actually used. This "excess wiring" is done because in the future the building may house more appliances, and adding wiring is too costly. The same goes for air-conditioning and ventilation.

We shall obtain industry-specific estimates of g, p_x, and p_z, and plug them into the decision problem of the firm. We shall then compute the optimal policy of a firm and ask how much of the cross-industry variation in the ages of structures and equipment the model can explain.

An Assumption About the Industry-Specific Age Distributions.—We cannot measure p_x and p_z directly, but we have data on the percentage of value added spent on x and z in each industry. We choose values of p_x and p_z so that when each producer behaves optimally, the predicted expenditure shares of the industry match the observed shares. This requires that we aggregate the spending of each industry's producers. To carry this out, we shall make the following assumption.

ASSUMPTION: *The age of structures is distributed uniformly within an industry, and this distribution is fixed over time.*

This assumption allows us to relate the maximum age of structures, an endogenous variable in our model, to the average age of structures observed in cross-sectional data, and to the expenditure shares, which are then constant over time. The following lemma formally describes the procedure for predicting the age of equipment and the age of structures.

LEMMA 1: *If the Assumption holds, each producer in the industry follows the optimal policy* $(n, t_0 = 0, t_1, \ldots, t_{n-1}, t_n = T)$ *and* θ_x, θ_z, g, *and* r *are given, the predicted average ages of equipment and structures are*

$$\hat{A}_z = \frac{1}{2T} \sum_{k=0}^{n-1} (t_{k+1} - t_k)^2, \; \hat{A}_x = \frac{T}{2},$$

$$\theta_x = \frac{g p_x e^{g t_{n-1}}}{\sum_{k=0}^{n-1} (1 - e^{-g(t_{k+1}-t_k)})},$$

$$\theta_z = \frac{g n p_z}{\sum_{k=0}^{n-1} (1 - e^{-g(t_{k+1}-t_k)})}.$$

PROOF:
In the Appendix.

Note that the Assumption does not preclude the existence of net capital formation which, in our model, takes place in terms of efficiency units, not physical quantities.

The Data.—To test the model, we use the cross-sectional data for 21 2-digit manufacturing industries. The age data for 1989 are from the U.S. Department of Commerce, Bureau of Economic Analysis (1993).[14] The data on capital expenditures for 1989 are from the Annual Survey of Manufactures and have been provided by the Bureau of the Census, Manufacturing and Construction Division, Information Services Center. For each industry, the growth of labor productivity, g, is measured by the growth of real value added per hour worked over the postwar period, using the National Bureau of Economic Research (NBER) Productivity Database compiled by Eric J. Bartelsman and Wayne B. Gray.[15]

RESULTS:
The model predicts values of n that are much higher than the observed ratios of A_x/A_z.[16] As a result, the model on average underpredicts A_z by 3.3 years, and overpredicts A_x by 2.7 years. Since the relationship between $\hat{A}_z$ and n is negative, higher than observed values of n imply lower values of $\hat{A}_z$, causing A_z to be underpredicted (Table 1).

The Age of Equipment and Structures by Industry.—Table 3 (at the end of this section) reports the correlations between the actual and predicted

[14] Tables A.1 of U.S. Department of Commerce, Bureau of Economic Analysis. The age data are available only for 2-digit manufacturing industries.

[15] Value added (*VADD*) is divided by shipments deflator (*PISHIP*) and hours of production workers (*PRODH*). The parameter g is taken to be the average growth rate of *VADD*/(*PISHIP* × *PRODH*) over the time period 1959–1991. Results using number of production workers rather than hours are similar and hence only the latter are reported.

[16] In part we obtain higher-than-observed values of n, because in our model the technological progress in equipment is the sole cause of growth of labor productivity. In reality, this is only one of the factors, so the growth rate of labor productivity will, in general, be higher than g in our model.

TABLE 1—COMPARISON OF CORRELATION COEFFICIENTS

Correlation	Data	Model
$\rho(g, A_z)$***	−0.16	−0.55*
$\rho(g, A_x)$	−0.32**	−0.89*
$\rho(\theta_z, A_z)$	0.36**	0.72*
$\rho(\theta_x, A_x)$***	−0.13	−0.08
$\rho(A_z, A_x)$***	0.67*	0.45*

* Significant at 5 percent.
** Significant at 10 percent.
*** Hypothesis $\rho = \hat{\rho}$ cannot be rejected.

TABLE 2—PREDICTIONS COMPARED TO DATA

Variable	Data	Model
Mean A_z	5.92	2.64
Mean A_x	9.51	12.18
Mean A_x/A_z	1.67	6.86

ages of equipment and structures.[17] Both correlation coefficients are positive and significant, with $\rho(A_z, \hat{A}_z)$ having the p-value of 0.01 and $\rho(A_x, \hat{A}_x)$ having the p-value of 0.1. The model fits better for A_z than it does for A_x, probably because it implies a positive relationship between g and n, which, however, is negative and insignificant in the data. This raises the cross-industry dispersion of n, and this in turn raises the variance of $\hat{A}_x$, thereby lowering $\rho(A_x, \hat{A}_x)$.

The Relation Between Productivity Growth and the Ages of the Inputs.—Equipment and structures tend to be younger in the fast-growing industries. The model predicts this relationship correctly, but exaggerates its magnitude (Table 2). This is in part because the relationship between the parameters and the endogenous variables is deterministic: all other things being equal, a higher growth rate will definitely result in structures and equipment becoming younger. Besides, the model reacts to an increase in g in two ways, which work in the same direction: (1) equipment is replaced more often, because the opportunity cost of holding it increases, and (2) n increases, which makes the average age of equipment even less. The second effect, absent in the data, is another explanation why the model "overreacts" to changes in g.

The Relation Between the Cost Shares and Age.—The relation between spending and age for equipment differs from the corresponding relation for structures. For equipment, industries with bigger spending (in terms of share) will tend to have older equipment. This is intuitive—expensive equipment must work longer to justify its cost. The model matches this observation, although the effect is exaggerated.

For structures, however, the opposite is true; industries that spend more on structures will have *younger* structures. The effect is fairly weak and the model matches the empirical correlation rather closely. While the same force that makes more expensive equipment older works for structures, a bigger cost of structures induces producers to keep less spare capacity (i.e., a lower n). This second effect is absent for equipment, which makes the ages of the two inputs respond differently to an increase in spending. Presumably, then, the second effect is stronger, so more expensive structures are slightly younger, on average.

Complementarity of x and z in the Production Function.—Table 2 shows that in the data the correlation between A_x and A_z is highly significant and positive. If the inputs were independent in production (and not complementary), this observation would be inconsistent with the model. The model explains about two-thirds of this correlation. This may be due to the positive relationship between g and n that is absent in the data and increases the variance of $\hat{A}_x$.

Equipment Investment in New Versus Old Plants.—How well does the model predict spending on new equipment made by *existing* plants, a number that Boddy and Gort report at over 90 percent? Whenever (in the model) equipment is *not* purchased together with a structure, we will think of it as a purchase done by an existing plant. Under the Assumption the share of spending on equipment made by existing plants is $1 - 1/n$ (see the Corollary to Lemma 1 in the Appendix for the

[17] The correlation coefficient is invariant to affine transformations of the data, which makes the results insensitive to possible multiplicative and aditive measurement errors.

TABLE 3—DATA BY INDUSTRY AND PREDICTIONS FOR THE AGES OF EQUIPMENT AND STRUCTURES

Industry	θ_z, percent	θ_x, percent	g, percent	A_z, years	$\hat{A}_z^*$, years	A_x, years	$\hat{A}_x^{**}$, years
Food	5.20	1.44	2.90	6.32	1.94	9.23	11.66
Tobacco	1.84	0.36	5.21	5.78	0.36	8.46	8.92
Textile	7.64	1.24	3.59	5.67	2.36	10.87	11.78
Apparel	2.03	0.79	2.94	4.63	0.72	9.55	11.45
Lumber and wood	5.95	1.25	2.03	4.43	3.34	10.24	13.38
Furniture	3.33	1.52	1.80	4.08	2.04	8.65	14.25
Paper	15.00	1.91	3.05	6.02	5.30	8.95	10.61
Printing	4.73	1.55	1.85	3.91	2.88	7.64	14.40
Chemicals	7.86	1.51	3.93	6.26	2.15	9.04	10.73
Petroleum and coal	10.51	2.50	4.78	7.58	2.39	11.11	9.56
Rubber and plastic	8.35	1.73	2.34	4.37	4.25	8.85	12.74
Leather	2.05	0.82	1.99	5.90	1.08	11.09	14.02
Stone, clay, glass	7.70	1.14	1.76	6.53	5.21	11.05	15.62
Primary metal	9.53	1.17	1.38	10.69	6.54	12.80	13.07
Fabricated metal	5.24	1.08	1.45	8.15	4.22	10.07	16.86
Industrial machinery	5.25	1.21	2.47	7.38	2.36	9.43	11.79
Electronics	6.95	1.54	3.93	5.78	1.91	7.91	9.56
Motor vehicles	6.53	0.85	3.18	5.60	2.22	8.78	11.10
Other transportation	4.02	2.13	2.66	5.76	1.60	7.97	12.81
Instruments	4.65	1.26	4.44	3.95	1.10	7.85	9.88
Miscellaneous	3.64	0.89	2.66	5.56	1.46	10.14	11.67
Mean	6.09	1.33	2.87	5.92	2.64	9.51	12.48
Standard deviation	3.16	0.49	1.10	1.60	1.62	1.34	2.07

* $\rho(A_z, \hat{A}_z) = 0.51$ (p-value 0.01).
** $\rho(A_x, \hat{A}_x) = 0.29$ (p-value 0.1).

proof). Using the values of n predicted by the model for different industries, we find that the average share across all industries is 0.79, which is close to what Boddy and Gort report.

The Relative Size of Investment Spikes. —Consider Figure 2 once more. The predicted size of spikes can be measured by spending on an input during the spike divided by average spending between the spikes. For input x this measure is simply equal to T, because all spending is done in one out of T periods. By the same logic, the relative size of each spike for input z is at most $\max_{1 \leq i \leq n}(t_i - t_{i-1})$. Since $T \geq \max_{1 \leq i \leq n}(t_i - t_{i-1})$, spikes are bigger for x then they are for z.

McClelland[18] finds that investment in structures is lumpier than it is in equipment. He studies a balanced panel of 330 pulp and paper mills over the period 1974–1993. The breakdown of investment was 7.5 percent in structures, and 92.5 percent in equipment. For each plant, he ranks its annual investment rates. Since there are 20 years in all, had investment been distributed uniformly over time, each year should have witnessed 5 percent of investment—both in structures and in equipment. In fact, both types of a plant's investment were concentrated at certain dates, but more so for structures than for equipment: The top-ranked year accounts for 34.5 percent of all the investment in structures, but for only 23 percent of all the investment in equipment, and the top-ranked two-year period accounts for 49 percent of all the investment in structures but for only 37 percent of all the investment in equipment.

As Table 3 shows, the paper industry is, in fact, an outlier in that the breakdown of investment of 11 percent on structures and 89 percent on equipment (which is close to the breakdown in McClelland's sample) is much more heavily weighted toward equipment than any other in-

[18] In work done for—but not reported in—McClelland (1997).

dustry. This may in fact be the outcome of regulation—the industry has relatively few new plants because it is difficult to get an environmental permit to build a new one.

III. Conclusion

Our paper has combined two assumptions that the literature has treated extensively, but separately: Complementarity and nonconvex costs of adjustment. We have analyzed production processes that require two complementary capital inputs subject to improvements in quality. We showed that if the costs of adjusting the inputs are not convex, a firm may choose to buy the inputs at different dates.

When a firm buys its inputs at different dates, spare capacity appears in one input. Thus, we have explained spare capacity not as arising from unexpected shifts in demand, but, rather, as originating from nonconvex costs of adjustment. Moreover, if the two inputs are plant and equipment, productivity grows as a plant ages, investment spikes for structures exceed those for equipment, and other implications arise that plant-level and cross-industry data confirm.

On a theoretical level, we have carried out a full-dynamic analysis of a two-capital model with lumpy costs of adjustment. We showed that the steady state (whether asynchronous or not) is globally stable which only partially confirms Eytan Sheshinski's (1990) conjecture that a steady state would be locally stable only if investment patterns in it are synchronous. But asynchronous adjustment arises on a large subset of the parameter space. One cannot then analyze a technology in isolation using a standard one-input vintage-capital model because the optimal strategies for upgrading the two technologies are interrelated.

The bigger point of this exercise is one that runs counter to conventional wisdom. The point is that even extreme complementarity does not necessarily produce perfect comovement. Inputs may be strong complements, but they may not behave as one, and one may adjust before the other. We have analyzed only how a single firm should optimally behave, but we expect that our results imply some lead-lag relations for the macroeconomy: Investment in lumpy items should precede investments on other, more smoothly adjustable inputs.

Appendix

The following claim gives a necessary condition for the optimum.

Claim 1: In the optimum, x is not upgraded in the points where $z(t)$ is constant. That is, for every j, $T_j = \tau_i$ or for some i.

PROOF:

Suppose this is not true. Take an arbitrary interval $[t_1, t_2)$, where $z(t) = z =$ constant. Suppose there is $T_j \in (t_1, t_2)$. Then, the present value of profit on $[t_1, t_2)$ will be given by

$$\frac{\min\{x_{j-1}, z\}}{r}(e^{-rt_1} - e^{-rT_j}) + \frac{\min\{x_j, z\}}{r}(e^{-rT_j} - e^{-rt_2}) - p_x x_j e^{-rT_j}$$

$$= e^{-rT_j}\left(\frac{\min\{x_j, z\}}{r} - \frac{\min\{x_{j-1}, z\}}{r} - p_x x_j\right) + \frac{\min\{x_{j-1}, z\}}{r} e^{-rt_1} - \frac{\min\{x_j, z\}}{r} e^{-rt_2}.$$

This is a monotonic function of T_j, which is why setting $T_j = t_1$ or $T_j = t_2$ is always preferred to any $T_j \in (t_1, t_2)$. To finish the proof, observe that when we exclude all the open intervals where $z(t)$ is constant, this leaves us only with the points $\{\tau_i\}_{i=0}^{\infty}$ as possible times for upgrades in x.

PROOF OF THEOREM 1:

If $(x(t), z(t))$ is the optimal solution to (3), it must be the case that

$$x_j = \max_{t \in [T_j, T_{j+1})} z(t). \tag{A1}$$

Indeed, if for some j, x_j is strictly more, then we can cut the cost without changing output by purchasing less x. If x_j is strictly less, then we can cut the cost without changing output by purchasing less z.

Consider the interval $[T_j, T_{j+1})$. Suppose that z is purchased $n(j)$ times on this interval. By Claim 1, z must be purchased at least at time T_j, so $n(j) \geq 1$. Let t_k, $k = 0,..., n(j) - 1$ be the time elapsed from date T_j to the kth upgrade of z on the interval $[T_j, T_{j+1})$. Notice that $t_0 = 0$ (Claim 1). Then the present value of profit at time T_j on $[T_j, T_{j+1})$ [taking (A1) into account] will be given by[19]

$$\text{(A2)}\quad v_j = \sum_{k=0}^{n(j)-2} \frac{z_k}{r}(e^{-rt_k} - e^{-rt_{k+1}} - rp_z e^{-rt_k}) + \frac{z_{n(j)-1}}{r}(e^{-rt_{n(j)-1}} - e^{-r(T_{j+1}-T_j)} - rp_z e^{-rt_{n(j)-1}} - rp_x).$$

In the optimum, every term of this sum must be positive,[20] because otherwise the sequence of upgrading dates cannot be optimal. Suppose, for example, that the last term is negative. Then it pays to decrease $z_{n(j)-1}$ at least down to the level of $z_{n(j)-2}$. But then, z should not be upgraded at all at time $t_{n(j)-1}$, so this time cannot be a part of optimal upgrading sequence.

Since every term in (A2) is positive, it is optimal to upgrade z to the frontier, i.e., to set $z_k = Z(T_j + t_k) = Z(0)e^{g(T_j+t_k)}$. Substituting this into (A2), we get

$$v_j = \sum_{k=0}^{n(j)-2} \frac{1}{r} e^{gt_k}(e^{-rt_k} - e^{-rt_{k+1}} - rp_z e^{-rt_k}) + \frac{1}{r} e^{gt_{n(j)-1}}(e^{-rt_{n(j)-1}} - e^{-r(T_{j+1}-T_j)} - rp_z e^{-rt_{n(j)-1}} - rp_x).$$

Note that v_j depends only on the difference $T_{j+1} - T_j$, which we will denote ΔT_j. Let

$$\text{(A3)}\quad (n(j, \Delta T_j), t_1(\Delta T_j),..., t_{n(j)-1}(\Delta T_j)) = \arg\max v_j(\Delta T_j).$$

[19] If $n(j) = 1$, only the last term of this sum is left.

[20] Condition $r(p_x + p_z) < 1$ ensures that strictly positive profit is feasible.

Then the optimal value function on $[\tau_0, \infty)$ can be written as

$$\max_{\{\Delta T_j\}} \hat{V} = Z(\tau_0) \max_{\{\Delta T_j\}} \sum_{j=0}^{\infty} \exp\{-(r-g) \times \sum_{k<j} \Delta T_k\} \max v_j(\Delta T_j)$$
$$= Z(\tau_0)\,(\max v_0(\Delta T_0) + e^{-(r-g)\Delta T_0} \max_{\{\Delta T_j\}} \hat{V}).$$

Thus the optimal value function depends only on ΔT_0. This proves that x is upgraded with a constant period $T = \Delta T_0$. It follows from (A3) that the optimal relative frequency of upgrades is also constant: $n = n(\Delta T_0)$.

Finally, it is left to show that τ_0 is finite. Let τ_0 be the time when both inputs are purchased together for the first time. Let $V_0(z_0, x_0, \tau_0)$ be the optimal value function on $[0, \tau_0)$. Then, the value function on $[0, \infty)$ is equal to

$$V = Z(0)(V_0(z_0, x_0, \tau_0) + e^{-(r-g)\tau_0}\hat{V}).$$

The derivative of this expression with respect to τ_0 is equal to

$$\frac{\partial V}{\partial \tau_0} = e^{-r\tau_0} r \min\{x_0, z_0\} - Z(0)(r-g)e^{-r\tau_0}e^{g\tau_0}\hat{V}.$$

The first term is the marginal benefit from keeping x and z unchanged at time τ_0, and the second term is the opportunity cost of delaying the purchase of both inputs. The first term is equal to $e^{-r\tau_0} r \min\{x_0, z_0\}$ because on $[0, \tau_0)$ only one input is purchased and the other one is equal to its initial value. Therefore, either $x(\tau_0) = z(\tau_0) = x_0$ or $x(\tau_0) = z(\tau_0) = z_0$. For τ_0 to be finite it is sufficient that $\hat{V} > 0$ (as it is in the optimum).

PROOF OF LEMMA 1:

Under the Assumption, the age of structures is uniform on $[0, T]$. Since this distribution does not change over time and quality of structures grows at rate g, it must be the case that structures are replaced continuously, and the new structure has e^{gT} times better quality than the one it replaces. Structure of age t was the best in the industry t periods ago, and will be replaced when its quality becomes the lowest in the industry. This implies that the quality of structure age t relative to the quality of the oldest structure changes according to

$$\frac{x_t}{x_T} = e^{gT}e^{-gt},\ t \in [0, T].$$

During the lifetime of a particular structure, equipment is replaced n times (when structures reach age $t_1, \ldots, t_n = T$). Every replacement of equipment puts the producer back on the frontier, so when equipment is replaced at time t_k, the new equipment has $\exp(t_k - t_{k-1})$ times better quality. Structures have spare capacity, which is the largest initially and declines with every replacement of equipment. After the kth replacement of equipment spare capacity becomes $e^{g(t_k - t_{k-1})}$ times less than before the replacement. This implies that the quality of equipment used together with the structure age t changes according to

$$\frac{z_t}{x_t} = e^{-g(t_{n-1} - t_k)},\ t \in (t_k, t_{k+1}],\ k = 0 \cdots n - 1.$$

Substituting here the expression for x_t, we get

$$z_t = x_T e^{g(T - t_{n-1})} e^{-g(t - t_k)},\ t \in (t_k, t_{k+1}],$$

$$k = 0 \cdots n - 1.$$

Since $z_t \leq x_t$ at all times, the instantaneous output of the producer who has the structure age t is equal to z_t: $y_t = z_t$. Now we can compute the instantaneous output of the industry by integrating over the age of the structures:

$$Y = \int_0^T \frac{1}{T} y_t dt$$

$$= \sum_{k=0}^{n-1} \frac{1}{T} \int_{t_k}^{t_{k+1}} x_T e^{g(T - t_{n-1})} e^{-g(t - t_k)} dt$$

$$= \frac{1}{gT} x_T e^{g(T - t_{n-1})} \sum_{k=0}^{n-1} (1 - e^{-g(t_{k+1} - t_k)}).$$

For the kth replacement of equipment, quality z is replaced by quality $ze^{g(t_k - t_{k-1})}$. These replacements take place when structures reach the age $t_1, \cdots t_n$. Then the spending on equipment of the producer who makes the kth replacement is equal to p_z times quality of equipment immediately prior to replacement times $e^{g(t_k - t_{k-1})}$, and the total instantaneous spending on equipment is given by

$$E_z = \sum_{k=1}^{n} \frac{1}{T} p_z z_{t_{k-1}} e^{g(t_k - t_{k-1})}$$

$$= \sum_{k=1}^{n} \frac{1}{T} x_T p_z x_T e^{g(T - t_{n-1})} e^{-g(t_k - t_{k-1})} e^{g(t_k - t_{k-1})}$$

$$= \sum_{k=1}^{n} \frac{1}{T} p_z x_T e^{g(T - t_{n-1})} = \frac{n}{T} p_z x_T e^{g(T - t_{n-1})}.$$

Since structures are replaced at age T only, and the new structure is e^{gT} times better than the old one, the instantaneous spending on structures is

$$E_x = \frac{1}{T} p_x x_T e^{gT}.$$

The expenditure share is the ratio of industry spending to industry output:

$$\theta_x = \frac{E_x}{Y} = \frac{g p_x e^{g t_{n-1}}}{\sum_{k=0}^{n-1} (1 - e^{-g(t_{k+1} - t_k)})},$$

$$\theta_z = \frac{E_z}{Y} = \frac{g n p_z}{\sum_{k=0}^{n-1} (1 - e^{-g(t_{k+1} - t_k)})}.$$

Finally, the age distribution for equipment can be obtained from the age distribution of structures by integrating over all the producers who use equipment of the same age. This distribution is the sum of n uniform distributions on $[0, t_{k+1} - t_k]$. In other words, a measure $(t_{k+1} - t_k)/T$ producers uses equipment between ages 0 and $t_{k+1} - t_k$, with the average age of $(t_{k+1} - t_k)/2$. Industry average is the sum of averages for the kth group of producers weighted by the size of the group:

$$\hat{A}_z = \frac{1}{2T} \sum_{k=0}^{n-1} (t_{k+1} - t_k)^2.$$

For structures, obviously, $\hat{A}_x = T/2$.

COROLLARY:

The share of spending on equipment made by existing plants is equal to $1 - 1/n$.

PROOF:

The spending on equipment made by existing plants is equal to total spending on equipment minus spending on equipment that is done together with the purchase of a new structure. At the time the structure is replaced, the spending on equipment equals $(n/T) p_z x_T e^{g(T - t_{n-1})}$. Then the share of equipment purchased by existing plants equals

$$\eta = \frac{E_z - \frac{n}{T} p_z x_T e^{g(T - t_{n-1})}}{E_z} = 1 - \frac{1}{n}.$$

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