Asset Pricing Implications of Disruptive Technological Change

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Abstract

This paper modifies a growth model with standard neoclassical features and isoelastic preferences to include random, low-frequency, large-scale technology shocks. Assuming technology-specific investment, we show the shocks can cause sharp asset-price drops – coupled with relatively mild output fluctuations. We have two agent types, wealthy asset-market participants and middle class. Although the former's consumption plummets when shocks arrive, the latter's behavior smooths aggregate expenditure. We show the model can have a sizeable equity premium, even with a CRRA near 1 and low riskless rate. Furthermore, it is consistent with empirical bull-bear stock-market episodes and accompanying co-movements of financial and real variables.

1 Introduction

This paper studies growth, financial markets, and aggregate risk in an economy subject to randomly timed, large-scale shocks. A number of recent papers examine abrupt, exogenous reductions in output from natural disasters or wars (Rietz [1988], Barro [2006], Barro and Ursua [2008], Gourio [2008], Gabaix [2011, 2012], and others). We, in contrast, focus on shocks from technological revolutions. The literature on general purpose technologies (GPTs), for instance, considers sweeping change that arrives in quantum bundles (e.g. Helpman and Trajtenberg [1998]). Similarly, economic historians long have suggested that progress proceeds in waves, with seminal inventions both causing

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obsolescence and inaugurating new eras of accelerated growth.[1] We analyze the macroeconomic risks that can result from such disruptive episodes, paying particular attention to interactions of financial markets and the real economy. Our goal is to develop a tractable model that can link the periodic arrival and diffusion of major new technologies with key asset pricing facts and low-frequency patterns in time series data.

In the model, shocks from GPT arrivals can give rise to a sizeable equity premium and a low risk-free rate (see Section 4). Importantly, the model's abrupt technological advances tend to have a much more jarring immediate impact on financial markets than on aggregate consumption or GDP. In addition, we show the model is well suited for studying low-frequency co-movements of real and financial variables. For example, it generates, naturally, a sequence of short bear and long bull markets. It also predicts that – as we find in data (e.g., Albuquerque *et al.* [2015]) – during bull (bear) markets (i) price-earnings ratios will rise (plunge) and (ii) output growth, consumption growth, and the equity premium will tend to exceed (fall short of) their unconditional means (see Section 5). The model's mechanisms and interpretations may shed new light on the relationship of bull-bear financial-market cycles to growth and the real economy.

The model has many conventional neoclassical elements, including households that optimize over infinite horizons, isoelastic preferences, and a constant returns to scale aggregate production function with capital and labor inputs. Somewhat less standard assumptions pertain to the special nature and consequences of shocks from GPT arrivals.

First, we assume that GPT shocks that raise the economy's frontier TFP are unpredictable, abrupt, and large — though infrequent. We are thinking of transformative changes in materials and energy sources; the goods available for consumption and investment; and, the nature, organization, and location of work.[2]

Second, capital is technology specific, as in Solow [1960], Laitner and Stolyarov [2003], and others. Although to implement a new GPT the economy needs to accumulate new-vintage capital, we assume that knowledge about a new GPT's potential spreads rapidly. Agents quickly recognize that prior vintages of capital are less productive than new, hence destined for less intensive use in the future. Their market value, accordingly, plummets, bringing a financial calamity to stockholders. In contrast, however, to the disaster papers cited above, where a large shock sharply reduces output, the shock of a GPT arrival in the model leaves current GDP almost unchanged. In fact, the growth surge that follows a GPT's arrival is much longer, and more gentle, because it depends on the accumulation of new-vintage capital.

[1] Examples over the last century would include revolutionary changes in sources of power and transportation, as well as developments in micro-electronics, information processing and communication, and the Internet — e.g., Abramovitz and David [2014], David [1990], Greenwood and Jovanovic [1999], Hobijn and Jovanovic [2001], Alexopoulos and Cohen [2009], Brynjofsson and McAfee [2011], and many others.

[2] The large-scale switch to remote work, if permanent, might provide a recent example of a GPT adoption.

The model's equity premium arises because the arrival of a GPT, which lowers the value of existing capital, also brings impressive new investment opportunities. Agents reduce their consumption to purchase high-yielding assets that finance new-vintage capital. Devastatingly poor returns on existing equities, in consequence, coincide with periods of high marginal utility of consumption. The same mechanism, we show, makes the risk-free rate low: the ever-present possibility of a technological revolution, with its accompanying high returns on new investment, makes bonds, which preserve an investor's principal during the lead-in to a new GPT, an attractive portfolio asset.

Although stockholders' consumption responses to GPT shocks can be large, the model produces realistic aggregate consumption volatility because we assume two agent types and "limited participation." There are high income, high wealth "type-I agents," who own the business-sector capital stock. And, there are middle-class "type-II agents," who consume their labor earnings, and have negligible financial assets. Roughly speaking, we are thinking of type-I households as the upper 3-5% of wealth holders in the *Survey of Consumer Finances.*

Limited participation has, in fact, 3 important roles. (i) It reconciles the model's outcomes with data on aggregate consumption volatility.[3] In particular, type-II households have neither portfolio wealth nor the desire to acquire it. Their sole income source is labor earnings, which are little affected in the short run by a new GPT. Even if type-I households sharply reduce their expenditures as a new GPT arrives, the behavior of type-II households can stabilize aggregate consumption. (ii) Limited participation divides labor earnings between type-I and type-II households, while concentrating asset ownership on the former. The division of labor income determines the extent of type-I households' sensitivity to (portfolio) risk, which, in turn, affects the magnitude of the equity premium.[4] (iii) The income and wealth distribution affects the consumption response of portfolio owners to the model's shocks: when facing unusually good investment opportunities, type-I households' high wealth gives them latitude to pare their consumption substantially without nearing subsistence. (See Section 1.1 for evidence)

Evidence suggests that after a technological advance, a decline in GDP tends to occur prior to enhanced growth (Basu *et al.* [2006], Gali [1999], and others).[5] A GPT's arrival preserves the economy's pre-shock production possibilities. The model, therefore, adds another non-standard element, namely, frictional costs that temporarily reduce output during the general disruption that a new GPT causes. While the present paper's parameter calibrations limit the corresponding output decline to 3%, we show that frictional costs can help to connect our analysis to data.

[3] Mehra and Prescott [1985, 1988] point out that macroeconomic theory addressing asset market facts needs to be disciplined with evidence on aggregate consumption.

[4] Compared to a disaster model where a large shock reduces both labor income and capital income (e.g. Gabaix, 2011), type-I agents' income is partially insured through labor earnings that change little in response to the shock.

[5] Alternatively, we can see that all six major stock market collapses in Albuquerque *et al.* [2015, tab. 2] overlap NBER recessions.

Given the model's features, we need a specialized solution technique. Our procedure has two parts. First, we show that we can modify a familiar detrending technique to fit the model's stochastic environment. Second, although having 2 household types precludes a routine representative-agent approach, we develop an alternative: we devise a sequence of auxiliary maximization problems with $n = 0, 1, ...$ (randomly arriving) new GPTs; show how to solve these problems recursively, yielding a sequence of consumption functions consistent with rational expectations; and, establish that if the latter sequence converges, its limit determines an equilibrium for our model.

This paper's organization is as follows. Section 1.1 examines evidence supporting the model's key assumptions and mechanisms, and Section 1.2 relates our approach to the literature. In Sections 2-3, we formally present our model, define equilibrium, provide conditions for its existence, and present an algorithm for computations. In Section 4, we derive analytical expressions for the equity premium and the risk-free rate. We then present simulations showing the model can generate an equity premium of 3.8-5.5 %/yr, with an intertemporal elasticity of substitution in the range 0.8-1.3 (i.e., a CRRA of 0.75-1.25). In Section 5, we derive analytical expressions for the equity premium, output growth, consumption growth and price-earnings ratios during bull and bear episodes and compare low-frequency patterns in financial-market data to outcomes from the model. Section 6 concludes.

1.1 Assumptions and evidence

There is direct evidence that can offer support for the model's key assumptions and mechanisms.

Several studies highlight the recent impact of new general purpose technologies — e.g., the microprocessor (Hobijn and Jovanovic [2001], Laitner and Stolyarov [2003]) and Internet (Pastor and Veronesi [2009]). Hobijn and Jovanovic, in particular, provide detailed evidence of large-scale changes and technology-specific capital. They link the 1972-1974 bear market to investor recognition of the significance of the microprocessor. Separating firms into those with old versus new-vintage capital, they document divergences after 1972, as follows: the value of firms with old-vintage capital fell by more than 50 percent, never fully recovering. Firms with new-vintage capital, on the other hand, accounted for a subsequent tripling of the market capitalization-to-GDP ratio.[6]

Turning to households' behavior, several recent empirical analyses support a limited-participation specification. Dividing households into those that own stocks and bonds versus those that do not, Attanasio *et al.* [2002] and Vissing-Jorgensen [2002] find behavior much more in accordance with Euler equations for the first group. Estimates of the coefficient of relative risk aversion (CRRA) tend to be exceedingly high for the second group, or for the sample as a whole. For the subsample of

[6]See also Greenwood *et al.* [1997], who argue who argue that investment-specific technological change accounts for over two-thirds of post-WWII productivity growth in the U.S., and Cummins and Violante [2002], who develop updated measurements of the rate of such change.

financial-market participants, however, the estimated CRRA is in the standard range of 1-3 (recall the discussions in Mankiw and Zeldes [1991] and Kocherlakota [1996, sect. II])

In related work, Ait-Sahalia *et al.* [2004] construct a measure of luxury expenditures that, they argue, provides a proxy for stockholder consumption. They find that the growth rate of luxury expenditures has a standard deviation roughly an order of magnitude larger than that for aggregate consumption. They also show that stockholders' consumption growth is strongly positively correlated with stock returns. Their evidence is consistent with our mechanism for the equity premium, which relies on large declines in type-I agent consumption during the periods when stock returns are low. The timing of consumption changes in Ait-Sahalia *et al.*'s figure 1B also seems consistent with the model's mechanism: in data covering 1961-2001, the largest changes in luxury consumption are dips, and the sharpest declines occur in the early to mid 1970s, exactly the period Hobijn and Jovanovic and Laitner and Stolyarov examine, and during 2000-2001, a period Pastor and Veronesi study.

1.2 Relation to the Literature

This paper bridges three strands of the macro-finance literature. First, we extend the body of work that seeks to understand the effects of a technological revolution on financial markets. In this vein, Greenwood and Jovanovic [1999], Hobijn and Jovanovic [2001] and Manuelli [2000] focus on macroeconomic dynamics arising from an anticipated major technological change. Pastor and Veronesi [2009] develop a model in which gradual learning about a new technology's potential drives a boom-bust pattern in the stock market. These papers consider one-time technological events, whereas we analyze recurring, random episodes of disruptive technological change.[7] Our model is thus suitable for studying asset-pricing phenomena related to the periodic diffusions of new technologies and low-frequency movements in the stock market.

Secondly, our work complements the literature that studies aggregate risks stemming from large-scale, infrequent events — so-called "rare disasters." Rietz [1988] hypothesizes that rare, adverse events might significantly contribute to the observed equity premium. Work by Barro [2006] and Barro and Ursua [2008] calibrates the probability and size of such disasters from a large body of cross-country time series data. Gourio [2008] adds recoveries to the Rietz-Barro model, to account for return predictability.[8] Gabaix [2012] extends the framework to rare disasters of variable size. Our work differs with respect to the source of the shocks and the mechanism driving the risk premium. Empirical assessments of Rietz's hypothesis focus on adverse events such as wars and natural catastrophes, and take the processes for dividends to be exogenous. Our paper, in contrast,

[7] Laitner and Stolyarov [2003] does allow recurring changes. Nevertheless, it focuses on a single episode, the advent of the microprocessor chip; it has a single agent type; and, agents have a fixed marginal propensity to consume.

[8] See also Nakamura *et al.* [2013] for updated calibrations.

emphasizes large shocks associated with disruptive technological change. Risk premia in our model are not tied to catastrophic output declines. Rather, they emerge from type-I agents' choices to cut consumption in order to finance new investment precisely when a large shock has caused severe capital losses on existing assets.

Third, a large literature seeks to formulate macroeconomic models capable of replicating key asset pricing facts. The inability of standard real business cycle frameworks to match both the equity premium and the risk-free rate is well-documented, dating back to Mehra and Prescott [1985].[9] More recently, macroeconomists have explored the possible roles of generalized preferences (e.g., Weil [1989], Campbell and Cochrane [1999], Tallarini [2000]), inflexible factor markets (e.g., Boldrin *et al.*, [2001]), multi-input production (Jermann [2010]), transaction costs (e.g., Mehra and Prescott, [2008]), and limited stock market participation (e.g., Guvenen [2009]).[10] Within the literature, our work complements analyses of real business cycle models with investment-specific technological change (ISTC) — e.g., Christiano and Fisher [2003], Papanikolaou [2011], and Kogan and Papanikolaou [2013]. Our paper differs, however, in both modeling and focus. Papanikolaou [2011], for example, develops a two-sector real business cycle model in which the rate of ISTC changes in small increments, and his attention centers on the mapping between ISTC and the cross-section of stock returns. Our framework, by comparison, assumes low-frequency investment-specific technology shocks and studies the link between technology diffusion and time-series outcomes.

Most recently, the macro-finance literature has studied asymmetric benefits of technological innovation — so-called "redistributive growth" —- and the impact of disruptive technological change on income and wealth inequality. Garleanu and Panageas [2018], for instance, focus on the effect of creative destruction on risk-sharing to rationalize portfolio choices and the dispersion of returns across asset classes. Kogan *et al.*. [2019] combine asymmetric benefits from innovation with incomplete markets for intellectual property to explain cross-sectional empirical patterns for stock returns. Again, our paper differs in both modeling and focus. In our model, the benefits of technological change are unrelated to which firm adopts it, and consequently there is no cross-sectional dispersion in stock returns. Our focus is instead on studying time series phenomena and low-frequency comovements in aggregative data.

2 Model

Our framework of analysis is a discrete-time, vintage capital model with recurring, large technology shocks and a production side similar to Laitner and Stolyarov (2003) and Solow (1960). The arrival of a new general purpose technology (GPT) introduces a higher-TFP production function that

[9] See also Kocherlakota [1990, 1996] and Jermann [1998].

[10] For comprehensive surveys, see Kocherlakota [1996] and Mehra [2008].

requires a new input, that is, a new vintage of capital. As new investment begins, the frontier technology diffuses, and aggregate productivity grows. The next GPT arrival interrupts the growth path, at which point aggregate investment switches to the newest vintage, and another diffusion phase starts. The household side of the model consists of two types of agents and features limited participation in asset markets.

In Section 2.1, we derive the aggregate production function, factor prices, market value of capital stock, and detrending formulas. Section 2.2 describes the behavior of the two types of households.

2.1 Output, factor prices and the value of capital

TECHNOLOGICAL PROGRESS GPTs are indexed by j, an integer, with newer technologies having a bigger j. At time t, the economy has a total of J_t distinct technologies, where J_t is a Poisson random variable. Each time a Poisson event occurs, a new technology appears and J increases by 1. The (exogenous) hazard rate is λ, so that the average interval between new technologies is $1/\lambda$. GPT j has a distinct TFP level $\theta_0 \cdot [\theta]^j$, $\theta > 1$. Without loss of generality, set $J_0 = 0$ and $\theta_0 = 1$.

VINTAGE CAPITAL Each technology j has a separate production function that requires a capital input, $\mathcal{K}_j$, of distinctive type, which we call vintage j. We have a one sector economy where time-t output can be produced with multiple types of capital, $\{\mathcal{K}_{j,t}\}_{j \le J}$, specific to separate general purpose technologies. GPTs themselves are not privately owned, but technology-specific investments that implement and embody GPTs are assumed to be private, rival goods. Output (the numeraire) is homogeneously divisible into consumption and investment. Investment expenditure $\mathcal{I}_t$ can be transformed one-to-one into capital of any vintage. Let $\mathcal{I}_{jt}$ be time-t investment in capital of vintage $j \le J_t$. At time t, only vintage J_t capital is built because, per dollar's worth, it produces the most output, for any given amount of labor. So, $\mathcal{I}_t = \mathcal{I}_{J_t,t}$. We assume that investment is irreversible, $\mathcal{I}_{jt} \ge 0$. Thus, old vintages cannot be transformed into new.

LABOR SUPPLY The economy is populated by two types of households. Both inelastically supply labor. Let ℓ denote the labor supply of type-I households and μ their measure, and let $\bar{\ell}$ and $\bar{\mu}$ denote the labor supply and measure of type-II households. Then the total supplies from the two household types are $L = \mu \cdot \ell$ and $\bar{L} = \bar{\mu} \cdot \bar{\ell}$, respectively. Labor input is not vintage-specific, and it is a Cobb-Douglas composite of the two types of labor:

$$\mathcal{L} = [L]^\beta \cdot [\bar{L}]^{1-\beta}, \quad \beta \in (0, 1).$$

Without loss of generality, we can normalize $\ell = \mu = \bar{\ell} \cdot \bar{\mu} = 1$.[11] In what follows, type-I agents will own all the equities, and type II agents will not participate in the stock market.

[11]With this normalization, $\mathcal{Y}$ in (1) can be interpreted as output per type-I household.

PRODUCTION FUNCTION The aggregate production function is the sum of outputs across different vintages, with the output-maximizing allocation of labor:

$$\mathcal{Y}_t = \max_{L_{jt}, \bar{L}_{jt}} \sum_{j \le J_t} [\theta]^j [\mathcal{K}_{jt}]^\alpha [\mathcal{L}_{jt}]^{1-\alpha}, \; \alpha \in (0,1), \tag{1}$$

$$\text{s. t.} \sum_{j \le J_t} L_{jt} \le 1, \; \sum_{j \le J_t} \bar{L}_{jt} \le 1, \; L_{jt} \ge 0, \; \bar{L}_{jt} \ge 0,$$

$$\mathcal{L}_{jt} = [L_{jt}]^\beta \cdot [\bar{L}_{jt}]^{1-\beta}.$$

DEPRECIATION In the model, capital depreciates for three reasons. First, there is wear and tear depreciation at rate δ. Second, frictional costs accompanying the arrival of a new GPT destroy some capital, leaving a usable stock that is a fraction Υ of its previous size. Third, a GPT arrival also causes obsolescence, which reduces the value of each dollar's worth of surviving old-vintage capital to $\$\,b$. In the second and third instances depreciation is abrupt, occurring randomly — only at a new GPT's arrival. Combining the latter two types of depreciation, the arrival of a new GPT causes the existing capital stock's value to fall to a fraction

$$\hat{b} \equiv \Upsilon \cdot b \tag{2}$$

of its previous size.[12] The first two types of depreciation imply

$$\mathcal{K}_{jt} = \begin{cases} (1-\delta)\,\mathcal{K}_{j,t-1} + \mathcal{I}_{t-1}, & \text{if } j = J_{t-1} = J_t, \\ \Upsilon\left[(1-\delta)\,\mathcal{K}_{j,t-1} + \mathcal{I}_{t-1}\right], & \text{if } j = J_{t-1} < J_t, \\ (1-\delta)\,\mathcal{K}_{j,t-1}, & \text{if } j < J_{t-1} = J_t, \\ \Upsilon\,(1-\delta)\,\mathcal{K}_{j,t-1}, & \text{if } j < J_{t-1} < J_t. \end{cases} \tag{3}$$

VALUE OF CAPITAL We normalize the price of output to 1 for all times. Let P_{jt} denote the resale value of one unit of $\mathcal{K}_{jt}$. The newest vintage of capital is $J = J_t$, which has price $P_{J,t} = 1$, equaling the marginal cost of transforming output into capital. The resale prices P_{jt} for vintages $j \le J$ obtain from the no-arbitrage condition, namely, that each dollar's worth of capital should produce the same return, regardless of vintage. The relative price of vintages then equals the relative amount of capital services (efficiency units) they provide. We have

Proposition 1: For $j \le J_t$,

$$P_{jt} = [b]^{J_t - j}, \quad b \equiv [\theta]^{-1/\alpha} < 1.$$

[12]For our simulations below, we calibrate $\hat{b}$ from the literature and Υ to cause a modest (i.e., 3 %) drop in GDP. Then we use (2) to determine b.

Proof: See Appendix 1.

The value of the aggregate capital stock at date t is

$$\mathcal{K}_t = \sum_{j \le J_t} P_{jt} \cdot \mathcal{K}_{jt} \,. \tag{4}$$

Proposition 1 shows that $\mathcal{K}_t$ measures the aggregate quantity of vintage-J efficiency units.

We now solve (1) for the aggregate production function. The following proposition parallels analysis in Solow [1960] and Laitner and Stolyarov [2003].

Proposition 2: *Let $\mathcal{Y}_t$ be as in (1) and $\mathcal{K}_t$ be as in (4). Then*

$$\mathcal{Y}_t = [\theta]^{J_t} [\mathcal{K}_t]^{\alpha}. \tag{5}$$

The net marginal revenue product per dollar of physical capital, of any vintage, is

$$\mathcal{M}(\mathcal{K}_t, J_t) = \alpha \frac{\mathcal{Y}_t}{\mathcal{K}_t} - \delta, \tag{6}$$

and, the wage rate, which is the same as the labor income per type I household, is

$$\mathcal{W}(\mathcal{K}_t, J_t) = (1-\alpha)\beta \mathcal{Y}_t. \tag{7}$$

Proof: See Appendix 1.

Because of Proposition 2, we do not need to keep track of the quantities $\mathcal{K}_{jt}$ separately. Given inelastic labor supplies, there are just two aggregate state variables in the production sector, $\mathcal{K}_t$ and J_t. In fact, the next section shows we can use detrended capital, K_t, in place of $\mathcal{K}_t$.

Detrending Before formulating the household decision problem and defining equilibrium, it is convenient to make a change of variables that removes the growth trend. Let $Z = [\theta]^{\frac{1}{1-\alpha}}$ denote the steady-state growth factor for capital, output and consumption associated with a TFP step θ. Define aggregate detrended variables as follows:

$$K_t = \frac{\mathcal{K}_t}{Z^{J_t}}, \quad Y_t = \frac{\mathcal{Y}_t}{Z^{J_t}}, \quad I_t = \frac{\mathcal{I}_t}{Z^{J_t}}, \quad w_t = \frac{\mathcal{W}(\mathcal{K}_t, J_t)}{Z^{J_t}}. \tag{8}$$

Then from Proposition 2, we have

$$Y_t = Y\left(K_t\right) = K_t^{\alpha} \quad \text{and} \quad w_t = w\left(K_t\right) = (1-\alpha)\beta K_t^{\alpha}.$$

In addition, letting m_t denote the net marginal revenue product per dollar of physical capital expressed through the detrended capital stock, (6)-(8) imply

$$m_t = m(K_t) = {K_t}^{\alpha-1} - \delta = \mathcal{M}(\mathcal{K}_t, J_t).$$

The detrending procedure in (8) is strictly analogous to the standard deterministic growth literature. The trend for our model is stochastic because J_t is a random variable.

Summary It is convenient to track the time path of the detrended capital stock: combining (2)-(4) and (8), we have

$$K_t = \begin{cases} (1-\delta) K_{t-1} + I_{t-1}, & \text{if } J_{t-1} = J_t \\ \frac{\hat{b}}{Z} \left[(1-\delta) K_{t-1} + I_{t-1}\right], & \text{if } J_{t-1} < J_t \end{cases} \tag{9}$$

The preceding section shows we can determine $\mathcal{K}_t$ and factor prices if we know K_t and J_t

2.2 Households

The economy has two types of households, type-I households are wealthy, and type-II are middle class.[13] A type-I household is infinitely lived, it receives labor and capital income, and it makes consumption-saving choices. Our analysis excludes residential fixed assets from capital and housing services from output. We assume that only type-I households participate in financial markets and that they finance the economy's non-residential capital stock. Type-I households receive all of the economy's capital income and fraction β of aggregate earnings, with their overall share of aggregate income equal to $\alpha + \beta \cdot (1-\alpha) = \eta$. Type-II households receive just labor income, in the amount of $(1-\eta) \cdot \mathcal{Y}_t$. In effect, type-II households just consume their labor earnings.[14] As stated in the Introduction, the limited participation assumption can discipline the model's quantitative predictions of fluctuations in aggregate consumption, although the paper's qualitative results and analytical characterizations apply for any $\beta \in (0, 1]$.

The remainder of this section focuses on the behavior of type-I households. It is convenient to define a new variable $z_t = J_t - J_{t-1} \in \{0\,,\,1\}$ that registers whether a new technology arrives at date t, or not. As above, set $J_0 = 0$. Then $J_t = \sum_{s=1}^{t} z_s$ all $t \geq 1$. The history of technology shocks from $s = 1$ to t is denoted $z^t = \{z_s\}_{s=1}^{t}$, the set of all possible histories from time $s = 1$ to t is $\mathcal{Z}^t$, and the probability of history z^t is

$$p\left(z^t\right) = \lambda^{J_t} \left(1-\lambda\right)^{t-J_t}. \tag{10}$$

[13] As indicated in the Introduction, our calibrations associate the type-I household with those in the top 3-5 percent of the US wealth distribution.

[14] We could allow type-II households to finance their own residences. For simplicity, however, we exclude housing from the model.

Our timing convention defines most date-t variables at the moment the current technology level Z^t is known.

We use the following notation for the consumption and wealth of type-I households, with capital letters denoting aggregate variables, and lower case letters denoting household-level variables: C_t (c_t) is the consumption level; $X_t = C_t/[Z]^{J_t}$ (x_t) is detrended consumption; A_t (a_t) is detrended beginning of period-t wealth *before* z_t is realized; and, K_t (k_t) is the detrended market value of the capital stock *after* z_t is realized. Below, we refer to a representative type-I household simply as a "household."

Consider the household budget constraint and the law of motion for wealth. The current aggregate state K_t determines factor prices $m(K_t)$ and $w(K_t)$. Let

$$\mathcal{A}\left(k_t, K_t\right) \equiv k_t + m\left(K_t\right) k_t + w\left(K_t\right)$$

denote the period-t resources of a household with current wealth k_t, when aggregate capital is K_t. Household wealth carried into next period, equals

$$a_{t+1} = \mathcal{A}\left(k_t, K_t\right) - x_t.$$

With our timing convention, a_{t+1} is determined before z_{t+1} is known. Before period $t+1$ gets underway, z_{t+1} is realized. If there is no technology shock, J and the price of capital are unchanged from the previous period, so $k_{t+1} = a_{t+1}$. If, however, a new technology arrives in period $t+1$, the aggregate capital stock is re-valued by a factor $\hat{b} < 1$, and, in addition, the trend variable, Z^J, rises by a factor $Z > 1$. Accordingly, when $z_{t+1} = 1$, the realized household (detrended) wealth for period $t+1$ is $k_{t+1} = \hat{b}a_{t+1}/Z$. Thus, the law of motion for k is

$$k_{t+1} = a_{t+1}\omega\left(z_{t+1}\right) = \left(\mathcal{A}\left(k_t, K_t\right) - x_t\right)\omega\left(z_{t+1}\right), \tag{11}$$

where

$$\omega\left(z\right) \equiv \frac{\hat{b}}{Z}z + \left(1 - z\right),\ z \in \{0, 1\}.$$

To formulate the household decision problem, we need to specify the way households form their expectations about $m(K)$ and $w(K)$. In other words, we need a law of motion for the aggregate state. Suppose that households have a common belief about the aggregate consumption decision, $X = \Phi(K)$. The law of motion for the aggregate state is then the aggregate version of (11):

$$K_{t+1} = \Gamma\left(K_t, z_{t+1}; \Phi\right) \equiv \left(\mathcal{A}\left(K_t, K_t\right) - \Phi\left(K_t\right)\right)\omega\left(z_{t+1}\right). \tag{12}$$

The detrended capital stock trajectory, after any history z^t, can be constructed recursively from this law of motion.

Household preferences over consumption are

$$u(c) = \frac{c^\gamma}{\gamma},\ \gamma < 1,\ \gamma \neq 0,$$

and the subjective discount factor is ρ. We restrict the latter to ensure that the household expected lifetime utility is finite:

$$\rho \in (0,1) \quad \text{and} \quad \rho(1-\lambda) + \rho\lambda Z^\gamma < 1. \tag{13}$$

The household problem involves choosing desired wealth for next period $a_{t+1} = a[z^t]$ after any history z^t, prior to knowing z_{t+1}. The household maximizes its expected lifetime utility, with the expectation taken over histories in Z^t, and with the probability of history z^t as in (10). The feasible set for the control variable $a[z^t]$ is the set $\mathbb{K}$ of aggregate states visited by rational expectation equilibrium trajectories. This set is an equilibrium object that Proposition 3 below characterizes.

The period-t (type-I) household payoff function is

$$U\left(a_t, a_{t+1}, z^t\right) = u\left(Z^{J_t} \cdot \left[\mathcal{A}\left(a_t \omega(z_t), K[z^t]\right) - a_{t+1}\right]\right), \tag{14}$$

with Z^{J_t} multiplying detrended variables to recover the household consumption level c_t. Treating z^{-1} and z^0 as known singletons, z_1 and z_0, respectively, and setting $p(z^0) \equiv 1$, $a[z^{-1}] \cdot \omega(z_0) \equiv k_0$, and $K[z^0] \equiv K_0$, the household's overall maximization problem is

$$\begin{aligned} V(K_0, k_0; \Phi) = & \sup_{\{a[z^t]\}_{t=0}^\infty} \sum_{t=0}^\infty \sum_{z^t \in \mathcal{Z}^t} p\left(z^t\right) \rho^t U\left(a\left[z^{t-1}\right], a\left[z^t\right], z^t\right) \\ & \text{s.t. } a\left[z^t\right] \in \mathbb{K},\ k_0 = K_0 \in \mathbb{K}, \end{aligned} \tag{15}$$

where the household faces factor prices determined by K_0, Φ, and (12).

We next proceed to the definition and characterization of equilibrium.

3 Equilibrium

An equilibrium in our model has three requirements: *(1)* households solve utility maximization problem (15) , given common beliefs $\Phi(.)$ and the aggregate state forecast $K[z^{t+1}]$ constructed from (12); *(2)* households' beliefs are consistent with the consumption function solving (15), so that the aggregate state trajectory coincides with its forecast; and, *(3)* the aggregate state trajectory has non-negative gross investment, so that an investment irreversibility constraint is never binding. The formal definition of equilibrium is as follows.

Definition *An equilibrium is a feasible set* $\mathbb{K}$*, a list of sequences* $\{a[z^{t-1}], k[z^t], K[z^t]\}_{t=0}^{\infty}$ *with* $k[z^t] = a[z^{t-1}]\omega(z_t)$*, and a forecast function* $\Phi(K)$ *satisfying*

1. The policy function $a[z^t]$ *solves household problem (15) given the forecast function* $\Phi(.)$ *and the law of motion for* $K[z^t]$ *defined in (12),*

2. The wealth trajectory chosen by the household coincides with the aggregate state trajectory

$$k[z^t] = K[z^t] \in \mathbb{K} \quad all \quad t \quad and \quad z^t \in \mathcal{Z}^t, \tag{16}$$

3. Investment is non-negative along the equilibrium path,

$$a[z^t] \geq (1-\delta) \cdot K[z^t] \quad all \quad t \quad and \quad z^t \in \mathcal{Z}^t. \tag{17}$$

Our equilibrium requires rational expectations. When (16) holds, the time path of wealth emerging from (type-I) household utility maximization coincides with the time path that agents anticipate for the economy's physical capital stock. The latter, in turn, determines the sequence of future factor prices that agents will face. The solution to fixed-point problem (16) cannot, unfortunately, be straightforwardly derived from the description of a social planner's actions on behalf of the private sector. The model has two agent types, and, although type-I households do all of the wealth accumulation, they claim only a fraction of aggregate income. The next subsection describes our approach for developing a solution.

3.1 Solution methodology

To derive an equilibrium, we solve a sequence of auxiliary problems. That involves constructing growth trajectories compatible with rational expectations for a succession of TFP processes, each limiting the number of future GPT arrivals to a finite number n. For the case $n = 0$, there are no future GPT arrivals; the type-I household utility-maximization problem is non-stochastic; and, solving for an aggregate consumption function $\Phi^0(.)$ consistent with rational expectations is relatively straightforward. For $n = 1$, there is a single future GPT arrival (with hazard λ). We show that we can construct an aggregate consumption function $\Phi^1(.)$ that is consistent with rational expectations prior to the new GPT if we use $\Phi^0(.)$ from the previous step to determine the continuation problem after the GPT arrival. Similarly, at step n, we determine $\Phi^n(.)$ recursively using $\Phi^{n-1}(.)$. As the number of future GPT arrivals, n, becomes larger, we anticipate that the difference between $\Phi^n(.)$ and $\Phi^{n-1}(.)$ will become ever smaller. Proposition 3, Section 3.2, shows that if, in fact, the sequence $\Phi^n(.)$ converges uniformly, we can use its limit to construct a rational expectations equilibrium for our model.

Start with the case $n = 0$. The initial time is $t = 0$. Then $z_s = 0$ all $s \geq 1$. Necessary conditions for type-I agent utility maximization lead to Euler and budget equations

$$u'(X_t) = \rho \cdot [1 + m(K_{t+1})] \cdot u'(X_{t+1}), \tag{18}$$

$$K_{t+1} = K_t + m(K_t) \cdot K_t + w(K_t) - X_t. \tag{19}$$

The Euler equation comes from differentiating (15) with respect to a_{t+1}. For an outcome consistent with rational expectations, we substitute $x_t = X_t$ and $x_{t+1} = X_{t+1}$. Budget equation (19) comes from law of motion (12). To obtain an outcome consistent with rational expectations, we set $k_t = K_t$ and $x_t = X_t$.

[Figure 1 here]

Figure 1 displays the phase diagram. The K-isocline follows from (19): we have $K_{t+1} = K_t$ if and only if

$$X_t = \eta Y(K_t) - \delta K_t,$$

where $\eta = \alpha + \beta(1 - \alpha)$ is the share of gross income for type-I households. After substituting for K_{t+1} from (19), (18) yields the X-isocline: we have $X_t = X_{t+1}$ if and only if

$$1 = \rho \left[1 + m\left(\eta Y(K_t) + (1 - \delta)K_t - X_t\right)\right].$$

As shown in Figure 1, we have a saddle point portrait. For any initial $K_0 \in (0, \breve{K}]$, pick X_0 on the saddle path and define $\Phi^0(K_0) = X_0$. By construction, aggregate consumption function $\Phi^0(.)$ is then consistent with rational expectations and type-I agent utility maximization all $t \geq 0$.[15]

Next, set $n = 1$. That allows one future TFP jump, arriving with hazard λ. The Euler and budget equations for type-I agent utility maximization with the existing technology yield, after setting $X_t = x_t$, $X_{t+1} = x_{t+1}$ and $K_t = k_t$,

$$\begin{aligned} u'\left(X_t\right) &= \rho\left(1-\lambda\right)\left(1+m\left(A_{t+1}\right)\right) \cdot u'\left(X_{t+1}\right) \\ &+ \rho\lambda\hat{b}\left(1+m\left(\frac{\hat{b}A_{t+1}}{Z}\right)\right) \cdot u'\left(Z\Phi^0\left(\frac{\hat{b}A_{t+1}}{Z}\right)\right), \end{aligned} \tag{20}$$

[15]We can see that each auxiliary problem — starting with $n = 0$ — is concave and that the transversality condition sufficient for type-I agent utility maximization holds for the saddle path outcome. We do not belabor this point because, in fact, our derivation of equilibrium in Proposition 3 below depends upon $\Phi^n(.)$, each n, coming from the saddle path solution to (18)-(19) — or their equivalent below — but not upon sufficient conditions for utility maximization in any auxiliary problem.

$$A_{t+1} = K_t + m(K_t) \cdot K_t + w(K_t) - X_t. \tag{21}$$

Fix any $t \geq 0$, and let $z^t = (0, ..., 0)$. As above, we substitute $x_t = X_t$ and $x_{t+1} = X_{t+1}$ into (20) and $k_t = K_t$ and $x_t = X_t$ into (21). Substituting the right-hand side of (21) for A_{t+1} in (20), slight rearrangements yield a mapping

$$(K_t, X_t) \mapsto X_{t+1}. \tag{22}$$

Suppose $z_{t+1} = 0$. Then setting $K_{t+1} = A_{t+1}$ in (21), we have a mapping

$$(K_t, X_t) \mapsto K_{t+1}. \tag{23}$$

Suppose the phase diagram for (22)-(23) has a saddle path (qualitatively) resembling Figure 1. For each each initial value $K_0 \in (0\,,\, \breve{K}]$, use the X_0 on the saddle path to define $\Phi^1(K_0) = X_0$. If we put $X_t = \Phi^1(K_t)$ and $X_{t+1} = \Phi^1(A_{t+1})$ into (20), and if we revert to the problem $n = 0$ at $t+1$ whenever $z_{t+1} = 1$, then Euler equation (22) is consistent with rational expectations as well as type-I household utility maximization.

We iterate in this manner for higher and higher n. The phase diagram for each n maintains exactly the K-isocline of Figure 1. The X-isocline shifts but remains upward sloping. Suppose for each n, the phase diagram exhibits a saddle-path solution that is (i) continuous, (ii) non-decreasing, and (iii) manifests direct motion toward its stationary point, at $\bar{K}^n$. Suppose also that the sequence of functions $\Phi^n(.)$, each defined on $(0\,,\, \breve{K}]$, converges uniformly, with the limit being $\Phi^*(.)$, and that $\bar{K}^n$ converges as well, with the limit being $\bar{K}^*$. Then properties (i)-(iii) carry over to the limit. As we show in the next subsection, we can then determine a rational expectations equilibrium from $\Phi^*(.)$ and $\bar{K}^*$.

3.2 Existence of an Equilibrium

We have

Proposition 3: *Let condition* (13) *hold. Suppose (i) the sequence* $\Phi^n(.)$ *defined in Section 3.1 converges uniformly on* $(0, \breve{K}]$*, (ii) each saddle path* $\Phi^n : (0, \breve{K}] \mapsto (0\,,\, \infty)$ *is non-decreasing and continuous, (iii) each saddle path manifests direct motion towards a unique stationary point* $(\bar{K}^n\,,\, \bar{X}^n)$ *from every initial* $K \in (0\,,\, \breve{K}]$*, and (iv) the sequence* $\bar{K}^n$ *converges to* $\bar{K}$ *with* $\bar{K} \in (0\,,\, \breve{K})$*. Then there exists a limit function* $\Phi^*(.)$ *and invariant set* $\mathbb{K} = [\underline{K}, \overline{K}] \subset (0\,,\, \breve{K}]$ *such that any trajectory* $K[z^t]$ *originating in* $\mathbb{K}$ *and obeying the aggregate law of motion (12) — with aggregate consumption function* $\Phi^*(K)$ *— stays in* $\mathbb{K}$ *for all* $t \geq 0$*; gross investment is non-negative for all* $t \geq 0$*; and, the household wealth trajectory* $k[z^t]$ *with* $k[z^t] = K[z^t]$ *is consistent with utility maximization in (15) and rational expectations.*

Proof: See Appendix 1.

3.3 Computations

For simulations, we implement Proposition 3 using continuous-time auxiliary problems. Continuous-time avoids arbitrariness of the time unit and allows the use of well-known IMSL software for determining saddle paths. This section describes our steps.

The steps are strictly analogous to Section 3.1. We begin with the $n = 0$ case, that is to say, an auxiliary problem with no GPT arrivals. The initial time is $t = 0$. Let the initial TFP index be j. As in Section 2, $Z = [\theta]^{1/(1-\alpha)}$, $K_t = \mathcal{K}_t / Z^j$, and $x_t = c_t / Z^j$. We use $\rho_0 > 0$ with $e^{-\rho_0}$ the continuous-time analog of ρ from Section 2. Assume an individual household takes K_t , $t \geq 0$ as given. Then a type-I household solves

$$v^0(k_0, \{K_s\}_{s\geq 0}, j) \equiv [Z]^{j\cdot\gamma} \max_{x_t} \int_0^\infty e^{-\rho_0 t} u(x_t) dt \tag{24}$$

$$\text{s.t.} \quad \dot{a}_t = (\alpha K_t^{\alpha-1} - \delta) a_t + \beta(1-\alpha) K_t^\alpha - x_t, \qquad a_0 = k_0.$$

Setting $j = 0$ and using a present value Hamiltonian, first-order conditions yield

$$\dot{X}_t = \frac{X_t}{1-\gamma} \cdot \left[\alpha K_t^{\alpha-1} - (\delta + \rho_0)\right], \tag{25}$$

$$\dot{K}_t = \eta K_t^\alpha - \delta K_t - X_t, \tag{26}$$

where, to obtain a rational expectations outcome, we have set $a_t = k_t = K_t$ and $x_t = X_t$.

[Figure 2 here]

Figure 2 illustrates the phase diagram for (25)-(26). The saddle path outcome solves (24) because we have a concave problem and the transversality condition is satisfied. If for each $K_0 \in (0, \breve{K}]$ the corresponding consumption level on the saddle path is X_0, define $\phi^0(K_0) = X_0$. Then $\phi^0(.)$ is the aggregate consumption function consistent with rational expectations for $n = 0$.

Neither $\phi^0(.)$ nor the solution time path $\{K_s, X_s\}_{s\geq 0}$ depends on j. So, setting $n = 0$, for any rational expectations equilibrium,

$$v^n(k_0, \{K_s\}_{s\geq 0}, j) = [Z]^\gamma \cdot v^n(k_0, \{K_s\}_{s\geq 0}, j-1) = ... = [Z]^{j\cdot\gamma} \cdot v^n(k_0, \{K_s\}_{s\geq 0}, 0). \tag{27}$$

Hence, we need only solve (24) for $j = 0$. Continuing with $n = 0$, the envelope theorem yields

$$\frac{\partial v^n(k_0, \{K_s\}_{s\geq 0}, j)}{\partial k_0} = [Z]^{j\cdot\gamma} \cdot u'\left(\phi^n(K_0)\right). \tag{28}$$

Next, consider $n > 0$. Assuming that we have solved for $\phi^0(.), ..., \phi^{n-1}(.)$, which are continuously differentiable by construction, and $v^0(.), ..., v^{n-1}(.)$, which are continuously differentiable in k_0 and concave in k_0 by construction, and that the latter satisfy the analogs of (27)-(28), we recursively derive $\phi^n(.)$ as follows. If the $t = 0$ technology is j, and if we use (27) on the right-hand side, the detrended auxiliary problem is

$$\begin{aligned} v^n(k_0, \{K_s\}_{s\geq 0}, j) \quad &\equiv \quad [Z]^{j\cdot\gamma} \max_{x_t} \int_0^\infty \lambda e^{-\lambda T} \left[\int_0^T e^{-\rho_0 t} u(x_t)\, dt + \right. \\ &\left. + e^{-\rho_0 T} \cdot Z^\gamma \cdot v^{n-1}\left(\frac{\hat{b} a_T}{Z}, \{K_s\}_{s\geq 0}, 0\right)\right] dT \end{aligned}$$

$$\text{s.t. } \dot{a}_t = (\alpha K_t^{\alpha-1} - \delta) a_t + \beta(1-\alpha) K_t^\alpha - x_t, \qquad a_0 = k_0.$$

In other words, we use the rational expectations solution for step $n-1$ as the continuation, after the first GPT arrival, for problem n. Changing the order of integration, we have

$$v^n(k_0, \{K_s\}_{s\geq 0}, j) \equiv [Z]^{j\cdot\gamma} \max_{x_t} \int_0^\infty \lambda e^{-(\lambda+\rho_0)t} \left[u(x_t) + \lambda \cdot Z^\gamma \cdot v^{n-1}\left(\frac{\hat{b} a_T}{Z}, \{K_s\}_{s\geq 0}, 0\right)\right] dt \tag{29}$$

$$\text{s.t. } \dot{a}_t = (\alpha K_t^{\alpha-1} - \delta) a_t + \beta(1-\alpha) K_t^\alpha - x_t, \qquad a_0 = k_0.$$

Using (28) and equating $a_t = k_t = K_t$ and $x_t = X_t$, the analogs of (25)-(26) are

$$\dot{X}_t = \frac{X_t}{1-\gamma} \cdot \left(\lambda \cdot [Z]^{\gamma-1} \cdot \hat{b} \cdot \left[\frac{\phi^{n-1}(\hat{b} K_t / Z)}{X_t} \right]^{\gamma-1} + \alpha K_t^{\alpha-1} - (\delta + \rho_0 + \lambda) \right), \tag{30}$$

$$\dot{K}_t = \eta K_t^\alpha - \delta K_t - X_t. \tag{31}$$

In the phase diagram, the K-isocline remains unchanged as we increase n. The X-isocline shifts but is upward sloping for all n. Provided there is always a saddle path, use its height, X_0, at each $K_0 \in (0\,, \breve{K}]$ to define $\phi^n(K_0) = X_0$. We have a solution for (29) because the criterion is concave and the transversality condition holds (for any finite n). We can verify that analogs of (27)-(28) hold, that v^n is concave and continuously differentiable in k_0, and that ϕ^n is continuously differentiable.

Assuming that each $\phi^n(.)$ is increasing, that the phase diagram has a unique stationary $\bar{K}^n$ and exhibits (direct) convergence to it from all $K_0 \in (0, \breve{K}]$, and that $\phi^n(.) \to \phi(.)$ uniformly on $(0, \breve{K}]$, we base our simulations on $\Phi(.) = \phi^n(.)$ for high enough n. For future reference, note that $\phi^n(.)$ is continuously differentiable by construction.

3.4 Equilibrium growth

Figure 3 illustrates the equilibrium time path of the physical capital stock, $\mathcal{K}_t$, if GPT arrivals occur at times t_j , $j = 1, 2, ...$ Each GPT arrival causes a drop $\mathcal{K}_{t_j} = \hat{b} \cdot \mathcal{K}_{t_j - 0}$, with the decline stemming from both technological obsolescence and frictional costs. Thereafter, the superiority of the new technology enables the economy to support a larger capital-to-labor ratio. So, $\mathcal{K}_t$ grows, asymptotically approaching $[Z]^j \cdot \bar{K}$. The next GPT's arrival cuts the growth short, causing another discontinuous drop. (The corresponding trajectory for the detrended capital stock, $K_t = \mathcal{K}_t / Z^{J_t}$ lies within $(0, \bar{K}]$. At the arrival of a GPT at $t = t_j$, K_t falls to $(\hat{b}/Z) \cdot K_{t-0}$.)

[Figure 3 here]

Figure 3 also illustrates the equilibrium path of the GDP. The diffusion of a new technology depends upon the economy's acquisition of new-vintage capital, which proceeds at the flow rate of investment. If a GPT arrives at $t = t_j$, the economy possesses no capital at that instant suitable for taking advantage of the new technology. Absent frictional costs, the path would be continuous. Our frictional costs, however, cause the GDP to decline modestly (being multiplied by a factor Υ^{α}) — allowing the time path to be more consistent with evidence, as noted in Section 1 (see also Section 5 below).

4 Asset Returns

This section studies the model's equilibrium asset returns. We derive analytical expressions for riskless and risky rates, examine the economic mechanisms that determine them, and present simulations illustrating the quantitative outcomes that the model can generate. All of the analysis in Sections 4-5 utilizes the continuous-time formulation of Section 3.3.

4.1 Risk-free and Risky Rates of Return

Consider the risk-free interest rate. For expositional convenience, we first set the period length to h and then present expressions with $h \to 0$. Let ρ_0 be the instantaneous discount rate.

The riskless rate between dates t and $t + h$, $r_{t,t+h}$, obeys the Euler equation

$$u'(C_t) = (1 + (r_{t,t+h} - \rho_0) h) \cdot \mathbb{E}[u'(C_{t+h})].$$

Use the notation $\dot{C}_t$ for the rate of change in consumption when there is no GPT arrival at t, and let $\tilde{C}_{t+h}$ be per capita type-I household consumption at $t+h$ if there is an arrival at t. We have

$$\begin{aligned}\mathbb{E}\left[u'\left(C_{t+h}\right)\right] &\approx (1-\lambda h)\left(u'\left(C_t\right)+u''\left(C_t\right)\cdot \dot{C}_t\cdot h\right)+\lambda h\cdot u'\left(\tilde{C}_{t+h}\right)\\ &\approx u'\left(C_t\right)+u''\left(C_t\right)\cdot \dot{C}_t\cdot h+\lambda h\cdot\left(u'\left(\tilde{C}_{t+h}\right)-u'\left(C_t\right)\right).\end{aligned}$$

So, a first-order approximation for the Euler equation yields

$$u'(C_t)\approx u'(C_t)+u''(C_t)\cdot\dot{C}_t\cdot h+\lambda\cdot h\cdot[u'(\tilde{C}_{t+h})-u'(C_t)]+[r_{t,t+h}-\rho_0]\cdot h\cdot u'(C_t).$$

Collecting terms, taking the limit $h\to 0$, and switching to detrended consumption X_t, the risk-free rate at time t is

$$r_t=r(K_t)=\rho_0+(1-\gamma)\cdot\frac{\dot{X}_t}{X_t}-\lambda\cdot\frac{\Delta u'_t}{u'(X_t)}. \tag{32}$$

where, given equilibrium aggregate consumption function $\Phi(.)$,

$$\frac{\Delta u'_t}{u'\left(C_t\right)}=\lim_{h\to 0}\frac{u'(\tilde{C}_{t+h})-u'(C_t)}{u'(C_t)}=\frac{u'\left(Z\cdot\Phi^*\left(\hat{b}K_t/Z\right)\right)-u'(\Phi^*\left(K_t\right))}{u'(\Phi^*\left(K_t\right))}. \tag{33}$$

The risky return per dollar's investment in physical capital over time interval $[t\,,\,t+h)$, with $h>0$ small, is $R_{t,t+h}$ with

$$1+R_{t,t+h}h=\begin{cases}1+m_t h, & \text{with prob. } 1-\lambda h,\\ \hat{b}+m_t h & \text{with prob. } \lambda h.\end{cases} \tag{34}$$

The expected value of the risky rate of return at time t is then

$$\bar{R}_t=\bar{R}\left(K_t\right)=m\left(K_t\right)-\lambda\left(1-\hat{b}\right), \tag{35}$$

with the component $\lambda\cdot(1-\hat{b})$ reflecting agents' time-t expectation for the part of the return that cannot be known *ex ante.*

In the model, type-I households own the entire physical capital stock, and we can characterize aggregate saving and wealth accumulation in terms of supply and demand for that one asset. Nevertheless, we can also imagine, for example, that a constant fraction ν of the capital stock is financed through riskless short-term bonds, and the rest through shares of common stock. Because our type-I households are all alike, we then assume each has the same portfolio composition. Appendix 2 suggests a calibration for ν. Importantly, the model's equilibrium $\Phi(.)$ and time path $\{K_t, r_t, \bar{R}_t\}_{t\geq 0}$ are independent of the value of ν. A specific ν does, however, enable us to determine the expected return per dollar's worth of common stock, $\bar{R}^S_t$, and the expected equity premium, $\bar{E}_t$.

Let

$$\bar{R}_t = (1-\nu)\cdot \bar{R}_t^S + \nu \cdot r_t\,.$$

Then

$$\bar{R}_t^S = \bar{R}^S(K_t) = \frac{\bar{R}_t - \nu \cdot r_t}{1-\nu} \tag{36}$$

$$\bar{E}_t = \bar{E}(K_t) = \bar{R}_t^S - r_t = \frac{\bar{R}_t - r_t}{1-\nu} \tag{37}$$

4.2 Simulation Methodology

We use simulations to study the size of the equity premium that the model can generate. For each simulation, we compute an equilibrium aggregate consumption function $\Phi(.)$. A Poisson process determines GPT arrivals.

The arrival times for a Poisson process are realizations from an exponential distribution. Setting initial time $t_0 = 0$, we use a random number generator and λ to determine exponential-distribution realizations Δ_j , $j = 1, ..., N$, where $N = 1000$ in practice. Set $t_j - t_{j-1} = \Delta_j$ each j. Then from a given initial condition K_0, we determine K_t from the continuous-time analog of (9): for $t \in [t_{j-1}, t_j)$, we solve the differential equation

$$\dot{K}_s = \eta K_s^\alpha - \delta K_s - \Phi(K_s), \quad K_{t_{j-1}} \text{ given}, \tag{38}$$

where $\eta = \alpha + \beta \cdot (1-\alpha)$ as above. And, for $t = t_j$, we set

$$K_t = (\hat{b}/Z)\cdot K_{t_j - 0}. \tag{39}$$

Proceeding for $j = 1, ..., N$, we simulate the time path of K_t for $0 \le t \le T$ where $T \equiv t_N$. We set a starting value for K as follows. First, set $K_0 = \bar{K}$, where $\bar{K}$ is the stationary value in the limiting phase diagram. After running through (38)-(39) from $K_0 = \bar{K}$ to K_T, reset $K_0 = K_T$ and repeat (38)-(39) for $j = 1, ..., N$. The results below utilize K_t , $0 \le t \le T$, from the second step.

Our long-run average riskless and risky rates and equity premium are, respectively,

$$\bar{r} = \frac{\sum_{j=1}^N \int_{t_{j-1}}^{t_j} r(K_s)\, ds}{T}, \tag{40}$$

$$\bar{R} = \frac{\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \bar{R}(K_s)\, ds}{T}, \tag{41}$$

$$\bar{E} = \frac{\bar{R} - \bar{r}}{1-\nu}. \tag{42}$$

4.3 Parameter constraints

Appendix 2 presents baseline parameter values. However, we require 3 constraints to hold, as described in this section.

Long-Run Growth Constraint. The first constraint connects 4 production-side parameters. Baseline parameter values from Appendix 2, Table A1, are

$$(\lambda, \alpha, \hat{b}, g) = (0.04, 0.30, 0.40, 0.0210) \tag{43}$$

Although the Appendix derives the values from independent sources, the model's structure requires that the product of the frequency of GPT arrivals and the magnitude of the corresponding technological advances be consistent with the long-run average rate of growth of per capita GDP. As (A21) in Appendix 3 shows, that yields a condition

$$g = \frac{\lambda}{1-\alpha} \cdot [\ln(0.97) + \alpha \cdot \ln(1/\hat{b})] \tag{44}$$

Because (44) must hold, Table 1 below presents 4 sets of simulations, labeled Trials 1-4, each preserving 3 values from (43) but deriving a substitute for the fourth from (44).

Riskless Rate Constraint. In each simulation, we choose ρ_0 so that the model's average riskless rate, $\bar{r}$, is consistent with data from Table 2, column 4, below, namely, $\hat{r} \equiv 0.0232$. In each trial, and for each assumed γ (see below), we iterate on ρ_0 until the simulated $\bar{r}$ equals $\hat{r}$. Table 1 displays the resulting values of ρ_0.

Consumption Volatility Constraint. A longtime concern for models with large shocks has been the potential for greater drops in aggregate consumption in the analysis than those observed in practice (Mehra and Prescott [1987]). Matching the observed frequency of large aggregate consumption declines forms the basis of our third constraint, which we use to calibrate γ. The value of γ, in turn, determines the coefficient of relative risk aversion (CRRA), $1-\gamma$, and the intertemporal elasticity of substitution (IES), $1/(1-\gamma)$.

In each trial, we perform simulations for 5 values

$$\gamma \in \{-0.75, -0.50, -0.25, 0.00, 0.25\} \tag{45}$$

In each simulation, after establishing ρ_0 we calculate a time path of $\mathcal{K}_t$, $0 \leq t \leq T$, year by year, simultaneously determining annual aggregate consumption. We assess the fraction of times the fall in annual aggregate consumption from one year to the next exceeds 5% (10%). Call the fraction $F05$ ($F10$). For each trial, we choose the γ — designated with an asterisk in Table 1 — for which the tuple ($F05, F10$) best fits our data according to a sum of squares criterion.[16]

[16]Note that a larger γ implies a larger IES, which, in turn, leads to a higher $F05$ and $F10$; and, both $F05$ and

In the end, the designated values of γ are relatively large. The arrival of a new GPT leads to severe capital losses in type-I households' portfolios. That, and the favorable new investment opportunities arising immediately thereafter, cause type-I households temporarily to save more, and consume less. On the other hand, labor earnings, which technology shocks influence little in the short run, determine the consumption of type-II households. The smooth consumption of Type-II households attenuates the impact on aggregate consumption of the sharp responses of type-I households to GPT arrivals, allowing a high IES to satisfy the third constraint. A high IES helps make a low risk-free interest rate possible below.

4.4 Quantitative assessment

Table 1 illustrates that our model can match a low value for $\hat{r}$ yet produce a sizable equity premium.[17] This section examines the table's outcomes in detail.

[Table 1 here]

LOW RISKLESS RATE. As stated, each simulation begins by determining the $\rho_0 > 0$ that allows $\bar{r} = \hat{r}$. Table 1 presents the outcomes. We first investigate the main determinants of ρ_0.

Figure 4 presents a key graph. Consider formula (32) for the riskless rate. We can decompose $\Delta u'_t$ from the formula into the sum of its linear approximation and higher order terms:

$$\Delta u'_t = \Delta X_t \cdot u''(X_t) + \tilde{u}'_t = -(1-\gamma) \cdot \frac{\Delta X_t}{X_t} \cdot u'(X_t) + \tilde{u}'_t \tag{46}$$

where

$$\Delta X_t \equiv Z \cdot \Phi\left(\hat{b} K_t / Z\right) - \Phi(K_t)\,.$$

Provided $\Delta X_t < 0$ (which is the case in all our simulations), Figure 4 shows that both components in (46) are positive and that $\tilde{u}'_t$, in particular, can be quite large.

Substituting (46) into (32) and taking expectations,

$$\bar{r} = \mathbb{E}_t[r_t] = \rho_0 + (1-\gamma) \cdot \mathbb{E}_t\left[\frac{\dot{X}_t + \lambda \cdot \Delta X_t}{X_t}\right] - \lambda \cdot \mathbb{E}_t\left[\frac{\tilde{u}'_t}{u'(X_t)}\right].$$

The second term is the product of the inverse of the IES and the sum of the average consumption growth rate between GPT arrivals and the average consumption decline at arrivals, the latter

$F10$ tend to be less than (greater than) their empirical counterparts for $\gamma = -0.75$ ($\gamma = 0.25$) in each trial.

[17] As stated above, the simulations are done in fortran using IMSL routines. We use a multiple shooting algorithm for the saddle paths, saving the path as a cubic Hermite spline. The calculations for each Table 1 entry (including determining ρ_0) take 2-4 seconds.

weighted by the frequency of arrivals.[18] The second term, therefore, equals $(1-\gamma)\cdot g$, with g the measured long-run average percentage growth rate of aggregate output and, hence, of type-I households' consumption. Substituting the growth rate,

$$\bar{r} = \rho_0 + (1-\gamma)\cdot g - \lambda \cdot \mathbb{E}_t \left[\frac{\tilde{u}'_t}{u'(X_t)} \right] \tag{47}$$

Each simulation chooses the ρ_0 making $\bar{r} = \hat{r}$. Expression (47) shows that all else the same, a larger IES leads to a larger ρ_0 — hence a better chance of satisfying (13). The logic is familiar from non-stochastic growth models — a faster growth rate necessitates a higher $r_t - \rho_0$ for the Euler equation to hold, but a higher IES moderates g's impact. The final term of (47), reflecting the model's nonlinear response to shocks, raises ρ_0 further. And, Figure 4 shows that the larger the model's shocks, the larger the term will be.

[Figure 4 here]

Table 1 designates the simulations most consistent with our constraints. The model's large shocks together with the high IES that its treatment of aggregate consumption makes possible enable us to find a satisfactory ρ_0 in each trial. Notice that the magnitude, as well as the non-negativity, of ρ_0 is important: the larger the ρ_0 compatible with $\bar{r} = \hat{r}$, the lower the corresponding equilibrium capital stock — implying a higher $\bar{R}$ and, in turn, a higher $\bar{E}$ (recall (37)). The next subsection concentrates on the equity premium.

Large-Scale Shocks and the Equity Premium. The size of the equity premium that the model can generate is a focus of this paper. A major determinant of the premium is our large-scale shocks.

We can write the Euler equation for ownership of physical capital from t to $t+h$:

$$\begin{aligned} u'(C_t) \;&\approx\; [1 + (m_t - \rho_0)\, h] \cdot (1-\lambda h) \cdot \left(u'(C_t) + (\gamma - 1)\, u'(C_t)\, \frac{\dot{C}_t}{C_t} h \right) \\ &\quad + \left[\hat{b} + (m_t - \rho_0)\, h \right] \cdot \lambda h \cdot u'\left(\tilde{C}_{t+h} \right) \end{aligned}$$

Collecting terms, taking the limit $h \to 0$, switching to detrended consumption X_t, and using (41), we have

$$\bar{R}_t = \rho_0 + (1-\gamma)\cdot \frac{\dot{X}_t}{X_t} - \lambda \cdot \hat{b} \cdot \frac{\Delta u'_t}{u'(X_t)}\,.$$

Then (32) and (37) imply

[18] Notice that $\dot{X}_t/X_t = \dot{C}_t/C_t$ and, using the notation of (33), that $\Delta X_t/X_t = \lim_{h\to 0} (\tilde{C}_{t+h} - C_t)/C_t$.

$$\bar{E}_t = \frac{1}{1-\nu} \cdot \lambda \cdot (1-\hat{b}) \cdot \frac{\Delta u_t'}{u'(X_t)}. \tag{48}$$

Table 1 and (48) can help us to understand the importance of large-scale shocks. Compare, first, Trials 1 and 4 of Table 1. Trial 4 has less frequent shocks, the same $\hat{b}$, but a lower g. Fix $\gamma = 0.00$, for instance. Then we see that λ is two-thirds as large in Trial 4 as in Trial 1 and that the same is true for the resulting $\bar{E}$. Since only λ is changing in (48), the proportionate decline in $\bar{E}$ is straightforward to interpret.

Next, compare Trials 1 and 3. When we move from Trial 1 to 3, we are consolidating smaller technology shocks to larger (a lower $\hat{b}$), less frequent (a smaller λ) events of the same cumulative magnitude. The effect is dramatically different from the preceding paragraph. Fix $\gamma = 0.00$, for example. Then $\bar{E}$ is 8.04%/yr in Trial 3 but 5.77%/yr in Trial 1. In other words, instead of a drop of 192 basis points in moving from Trial 1 to 4, we have a 227 basis point increase. The effect is a direct consequence of the convexity of marginal utility in Figure 4 — when we lower $\hat{b}$, $\Delta X_t < 0$ has larger amplitude, and the graph shows the impact on $\bar{E}$ can be substantial.

Figure 4 and the comparisons above show the importance of large-scale shocks for the relatively high equity premiums of the starred columns in Table 1. Parenthetically, Figure 4 also indicates why a high IES alone does not ensure a large $\bar{E}$: though an increase in the IES tends to increase the absolute size of ΔX_t, it simultaneously attenuates the curvature of the marginal utility function. The overall effect is ambiguous. In fact, within every row of Table 1, increases in γ at first raise $\bar{E}$ but then lower it.

Earnings and the Equity Premium. Shifting from Trial 1 to 2 lowers the frequency of GPT arrivals without changing the shock size; hence, we might expect the equity premium to decline proportionately to λ. That does not happen, however, revealing a second significant determinant of the equity premium, as follows.

Our financial-market participants, type-I households, have a mixture of labor and portfolio income. When a new GPT arrives, the marginal product of labor changes little in the short run. The stability of its labor compensation increases a type-I household's tolerance for portfolio risk — tending to make the equilibrium equity premium smaller.

Importantly, when we shift from Trial 1 to 2, type-I households' income shares have to change to meet the long-run growth constraint (44). The share of GDP flowing to type-I households is $\alpha + \beta \cdot (1-\alpha)$, with α giving the capital-income component, and $\beta \cdot (1-\alpha)$ labor's contribution. Given a value for α, Appendix 2 calibrates β to match income in the SCF. Trial 2 raises α to satisfy (44). To preserve consistency with the SCF, we must then lower β. Both changes lower $\beta \cdot (1-\alpha)$. So, α rises and $\beta \cdot (1-\alpha)$ falls. That increases type-I agents' sensitivity to risk. In the end, despite a noticeably lower λ, Trial 2's equity premium is only 32 basis points below Trial 1's.[19,20]

FRICTIONAL COSTS AND THE EQUITY PREMIUM. The arrival of a new GPT causes temporary frictional losses in GDP. Such costs, as Section 1 notes, can aid in the model's interpretation of time series data. However, they are not a major determinant of $\bar{E}$ in our calibrations.

To illustrate their quantitative significance, eliminate the frictional costs in Trial 2. In other words, set $\Upsilon = 1$. Keep $\gamma = 0.25$, and readjust α to maintain constraint (44). Then in equilibrium, $\alpha = 0.3643$ and $\bar{E} = 0.0500$, the latter having fallen 40 basis points.

4.5 Financial-Variable Moments

Table 2 presents detailed results from the starred cases for Trials 2 and 4. We exclude Trial 1, as its frequency of GPT arrivals is higher than our baseline calibration, and Trial 3, because its $\hat{b}$ is lower than current evidence seems to support.

[Table 2 here]

To derive entries for Table 2, we simulate (38)-(39) one year at a time, producing annual figures to compare with the data of the table's columns 4-5. The size of the model's equity premium in Trial 2 (4) is similar to the data of column 5 (4). To illustrate that we can match the equity premium from the longer time series without adopting a lower rate of technological progress, we also present Trial 2A. Trial 2A changes only a single parameter from Trial 2, lowering the debt fraction, ν, from 40% to 25%.

The next section compares our simulation outcomes with low-frequency financial-market data.

5 Low Frequency Financial Market Movements

Albuquerque *et al.* [2015], for example, use statistical filtering to study U.S. stock market-data for 1869-2013. Their analysis divides the time series into a sequence of bear-then-bull episodes. This section compares patterns that they and others find in practice with our model's outcomes, and it examines the model's interpretations.

[19] To preserve a credible level of total labor compensation for type-I households, we can assume that they, the wealthiest agents in the economy, tend to have jobs requiring technology-specific human capital. Trial 2's $\mathcal{K}_t$ can then be a combination of human and physical capital (both measured in dollars). Appendix 4 provides a detailed analysis of an income share calibration consistent with the SCF data.

[20] Many equity-premium studies use a Lucas [1978] model (or an A-K model) in which shocks affect all income (see also Gabaix [2011]). This subsection illustrates that the present paper's neoclassical production function approach, with modest effects of GPT shocks on labor income, makes achieving a large equity premium more difficult.

5.1 Bear and Bull Episodes

Albuquerque *et al.* find that bear-then-bull episodes last about 18 years on average, with bears being much shorter (averaging about 3 years) than bulls (averaging about 15 years). Outcomes from our model are quite similar.

As in Section 4, we can specify 2 financial markets for our model, one for common stock and another for riskless short-term bonds. Assume the ratio of bond financing to the total value of $\mathcal{K}_t$ is a constant $\nu \in [0, 1)$. Let new GPTs arrive at times t_j , $j = 1, 2, ...$ Figure 3 shows that at t_j the value of the physical capital stock, which corresponds to the combined value of the 2 financial markets, abruptly drops from $\mathcal{K}_{t_j-0}$ to $\mathcal{K}_{t_j} = \hat{b} \cdot \mathcal{K}_{t_j-0}$. With ν constant, the stock market falls from $(1-\nu) \cdot \mathcal{K}_{t_j-0}$ to $\hat{b} \cdot (1-\nu) \cdot \mathcal{K}_{t_j-0}$.[21] In other words, the arrival of a new GPT causes a bear market — in fact, a stock-market crash. During $[t_{j-1}, t_j)$, on the other hand, both $\mathcal{K}_t$ and $\mathcal{Y}_t$ rise (see Figure 3) and there are no capital losses. We can think of the intervals between bears, therefore, as bull markets.

On the one hand, there turns out to be quite a close match between the qualitative descriptions that Albuquerque *et al.* select for financial-market events and our model's proposed mechanism: Albuquerque *et al.*'s Table 2 associates most U.S. bull markets 1869-2007 – 4 out of 6 to be exact – with exceptional technological progress or the spread of new technologies, just as our framework attributes all of its bulls to the diffusion of new GPTs.

On the other hand, the length of our bear-then-bull episodes is roughly consistent with evidence as well. Our baseline calibration has 4 seminal inventions per century, implying a hazard for GPT arrivals $\lambda = 0.04$. Albuquerque *et al.*'s second table assigns about one-third of severe stock-market downturns to wars and financial panics. If we were to enhance our analysis to incorporate non-technological shocks of that frequency, the hazard for bear markets would rise to $\lambda = 0.06$, with the corresponding average episode length being 16-17 years.

Likewise, our model's bears are short and the bulls long — as is true in the data. The model's explanation for the brevity of bears is that knowledge can spread rapidly. In contrast, the halcyon environment after a GPT's arrival can last through the interval between bears, no matter how long, because it depends upon the accumulation of new-vintage capital, an asymptotic process.

The model's episodes resemble Hobijn and Jovanovic [2001] (recall Section 1.1) as well. As in Hobijn and Jovanovic, the value of old capital never recovers after a crash. Rather, in both our model and Hobijn and Jovanovic, post-crash growth in $\mathcal{K}_t$ depends exclusively upon investments in new-vintage capital.

[21] Suppose, for example, $\nu = 0.40$ and $\hat{b} = 0.40$. Then the funding of a \$100 firm at time $t_j - 0$ is \$60 stock and \$40 bonds. At $t = t_j$, the firm's value drops to \$40. For ν to be constant, we might suppose the firm is immediately sold to a buyer with \$24 of equity financing and \$16 of bond financing.

5.2 The Equity Premium during Bulls and Bears

Albuquerque et al. find the equity premium is above average during bull markets and below average — in fact, negative — in bears. Our model exhibits an analogous pattern.

The model's return per dollar's worth of physical capital over the interval $[t, t+h), h > 0$, is a random variable $R_{t,t+h}$. Recall (34). Letting $h \to 0$, we have $R_t = m(K_t)$ if we restrict ourselves to known bull-market times $t \in [t_{i-1}, t_i)$. Using the notation of (38)-(42), and letting the superscript "B" denote a value restricted to bull markets, we have

$$\bar{R}^B = \frac{\sum_{j=1}^N \int_{[t_{j-1},t_j)} m(K_s) ds}{T}.$$

Using (34) again, the unconditional average return is

$$\bar{R} = \bar{R}^B + \frac{\sum_{j=1}^N \int_{[t_{j-1},t_j)} \lambda \cdot (\hat{b} - 1) ds}{T} = \bar{R}^B - \lambda \cdot (1 - \hat{b}).$$

The riskless rate, r_t, is non-stochastic and bounded; thus, $\bar{r}$ is not affected by our instantaneous bears. So, $\bar{r} = \bar{r}^B$. Continuing with the same notation,

$$100 \cdot [\bar{E}^B - \bar{E}] = 100 \cdot \left[\frac{\bar{R}^B - \bar{r}^B}{1 - \nu} - \frac{\bar{R} - \bar{r}}{1 - \nu} \right] = 100 \cdot \lambda \cdot \frac{1 - \hat{b}}{1 - \nu} = 4.0\%/yr \tag{49}$$

for either Trial 2 or 4. By comparison, $\bar{E}^B$ exceeds $\bar{E}$ by 5.5%/yr in Albuquerque *et al.*

The model's return on physical capital and equity premium are negative for bear markets. Let $t = t_j$ be a GPT arrival date. Then as $h \to 0$, (34) shows

$$R_t = m_t + \lim_{h \to 0} \frac{\hat{b} - 1}{h} = -\infty \,,$$

$$E_t = \frac{1}{1 - \nu} \cdot \left(m_t + \lim_{h \to 0} \frac{\hat{b} - 1}{h} - r_t \right) = -\infty \,.$$

5.3 Output and Consumption Growth

Let $\bar{G}_{\mathcal{Y}}$ be the average annual percentage growth rate for $\mathcal{Y}_t$, and let $\bar{G}_{\mathcal{Y}}^B$ be the same if we restrict our attention to bull markets. Continue with the notation of (38)-(42). If $\Delta \mathcal{Y}_t / \mathcal{Y}_t$ is the discontinuous change in GDP at each bear-market date $t = t_j$ (caused, in the model, by frictional costs from disruptive GPT arrivals), our calibrations imply $\Delta \mathcal{Y}_t / \mathcal{Y}_t = -0.03$. Note that $\dot{\mathcal{Y}}_s / \mathcal{Y}_s = \dot{Y}_s / Y_s$ each bull-market date s. The logic used to construct (47) shows

$$
\begin{aligned}
\bar{G}^B_{\mathcal{Y}} - \bar{G}_{\mathcal{Y}} &= 100 \cdot \left[\frac{\sum_{j=1}^{J} \int_{[t_{j-1},t_j)} \dot{Y}_s / Y_s ds}{T} - \frac{\sum_{j=1}^{J} \int_{[t_{j-1},t_j)} \left(\dot{Y}_s / Y_s + \lambda \cdot \Delta \mathcal{Y}_s / \mathcal{Y}_s \right) ds}{T} \right] \\
&= 100 \cdot 0.04 \cdot 0.03 = 0.12\%/\text{yr}.
\end{aligned}
$$

for either Trial 2 or 4. Table 1 in Albuquerque *et al.* finds 0.1%/yr.

Recall that $\mathcal{C}_t$ ($\mathcal{X}_t$) is aggregate (type-I household aggregate) consumption. Then

$$
\mathcal{C}_t = \mathcal{X}_t + (1 - \alpha) \cdot (1 - \beta) \cdot \mathcal{Y}_t. \tag{50}
$$

For Trial 2 (4), the simulated bull-market consumption growth differential, $\bar{G}^B_{\mathcal{C}} - \bar{G}_{\mathcal{C}}$, is 0.33%/yr (0.38%/yr). The analog from Albuquerque *et al.*'s Table 1 is 0.3%/yr.

The unconditional long-run average growth rate for GDP and aggregate consumption must, naturally, be equal. The results above then have consumption growing faster during bulls than GDP. The model's explanation is that type-I households' consumption drops sharply when a GPT arrives and recovers during the ensuing bull, while type-II households' consumption always moves in-step with GDP. The economy's average propensity to consume then tends to be low (high) at the beginning (end) of a bull — with, from accounting, the average propensity to invest having the reverse pattern.

5.4 Price-Earnings Ratios

Commentators often note that the stock-market's price-to-earnings (PE) ratio tends to rise during bull markets. For example,

> "An assessment of a bull market isn't complete without looking at ... PE ratios. Generally, the ratio rises during a bull market ...."[22]

Our model offers an explanation.

Suppose we equate the denominator of the PE ratio with the marginal product per dollar's worth of capital (i.e., per \$ $(1-\nu)$ of common stock) less long-run average depreciation plus interest on debt. Then using (36),

$$
PE_t = \frac{1 - \nu}{m(K_t) - \lambda \cdot (1 - \hat{b}) - \nu \cdot r(K_t)}. \tag{51}
$$

[22] https://marketrealist.com/2016/06/importance-pe-ratio-bull-market/

As Figure 3 shows, K_t rises during a bull market — the consequence of capital deepening after the technology has improved. That will drive $m(K_t)$ down, tending to raise the PE ratio. The PE ratio will reset — as does K_t — during a bear.

In each simulation of Table 2, we can derive PE ratios at the start of each bull, 1 and 2 and 3 quarters since the shock, and at the end. Then for Trials 2, 2A, and 4, respectively, we find PE ratios of 5.02, 8.42, 12.43, 16.40, and 19.93; 5.32, 9.28, 14.38, 19.93, and 25.38; and, 5.59, 9.94, 15.20, 20.50, and 25.29. In practice, we expect a PE ratio between 10 and 20, occasionally higher — roughly what the computations exhibit.

6 Conclusion

This paper studies disruptive technological change as a source of macroeconomic risk. Our model features large, but infrequent, technology shocks that we associate with arrivals of new general purpose technologies. Each sharply improves society's frontier technology and causes sweeping change to the economy. An attribute of GPT shocks that is key to our analysis is that a new type of capital has to be built to realize the productivity advantage of a new technology. Consequently, each shock reduces the value of existing assets (through obsolescence), though, at the same time, raises returns on new investment. Asset holders respond to the latter by cutting their consumption in order to invest, driving a substantial equity premium and a low risk-free rate. Simulations illustrate that the model can support an equity premium in the range of 4-6%/yr. — with isoelastic utility, a CRRA near 1.0, and a risk-free interest rate calibrated to 2.3%/yr.

Section 5 extends the analysis to examine the relation of our framework's outcomes to episodes in equity markets. We show that asset-price declines accompanying GPT arrivals in the model correspond to bear markets in practice, and the new technology's diffusion to bull markets. Given that mapping, the model can offer interpretations of the duration, periodicity, and nature of low-frequency empirical patterns.

The paper develops a new, tractable model that utilizes a standard neoclassical framework but can provide a unifying perspective on asset pricing facts and co-movements of real and financial variables accompanying bull-bear stock market episodes. The model thus suggests new ways of using financial data to study long-run growth and technological change.

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Appendix 1: Proofs

Proof of Proposition 1: As in the text, assume $\theta_0 = 1$. Let $\mathcal{W}_t$ and $\bar{\mathcal{W}}_t$ be the wage of type-I and type-II households. Let $\mathcal{L}^*_{jt}$, L^*_{jt}, $\bar{L}^*_{jt}$ denote the solution to (1), and let

$$\mathcal{Y}_{jt} = [\theta]^j [\mathcal{K}_{jt}]^\alpha [\mathcal{L}_{jt}]^{1-\alpha}. \tag{A1}$$

Step 1 Claim $L^*_{jt} = \bar{L}^*_{jt}$, $\sum_{j\leq J_t} \mathcal{L}^*_{jt} = 1$ and $\mathcal{Y}_{jt}/L^*_{jt} = y_t$ is independent of j.

Output maximization in (1) requires equating the marginal product of labor across vintages:

$$\beta(1-\alpha)\frac{\mathcal{Y}_{jt}}{\mathcal{L}^*_{jt}}\frac{\mathcal{L}^*_{jt}}{L^*_{jt}} = \mathcal{W}_t, \tag{A2}$$

$$(1-\beta)(1-\alpha)\frac{\mathcal{Y}_{jt}}{\mathcal{L}^*_{jt}}\frac{\mathcal{L}^*_{jt}}{\bar{L}^*_{jt}} = \bar{\mathcal{W}}_t. \tag{A3}$$

Sum (A1) over j. Then using (A2)-(A3), relate $\mathcal{W}_t$ and $\bar{\mathcal{W}}_t$ to aggregate output:

$$\beta(1-\alpha)\mathcal{Y}_t = \beta(1-\alpha)\sum_{j\leq J_t}\mathcal{Y}_{jt} = \mathcal{W}_t\sum_{j\leq J_t}L^*_{jt} = \mathcal{W}_t, \tag{A4}$$

$$(1-\beta)(1-\alpha)\mathcal{Y}_t = (1-\beta)(1-\alpha)\sum_{j\leq J_t}\mathcal{Y}_{jt} = \bar{\mathcal{W}}_t\sum_{j\leq J_t}\bar{L}^*_{jt} = \bar{\mathcal{W}}_t.$$

It follows that

$$\frac{\mathcal{W}_t}{\bar{\mathcal{W}}_t} = \frac{\beta}{1-\beta}.$$

Dividing (A2) by (A3) and then using the preceding line,

$$\frac{\mathcal{W}_t\cdot L^*_{jt}}{\bar{\mathcal{W}}_t\cdot\bar{L}^*_{jt}} = \frac{\beta}{1-\beta} \Leftrightarrow L^*_{jt} = \bar{L}^*_{jt}.$$

It follows that $\mathcal{L}^*_{jt} = \left[L^*_{jt}\right]^\beta\cdot\left[\bar{L}^*_{jt}\right]^{1-\beta} = L^*_{jt}$ and $\sum_{j\leq J_t}\mathcal{L}^*_{jt} = \sum_{j\leq J_t}L^*_{jt} = 1$. Since $\mathcal{L}^*_{jt} = L^*_{jt}$, (A2) implies that

$$\mathcal{Y}_{jt}/\mathcal{L}^*_{jt} = \frac{\mathcal{W}_t}{\beta(1-\alpha)} \equiv y_t. \tag{A5}$$

Step 2 Claim

$$\theta^{\frac{j}{\alpha}}\frac{\mathcal{K}_{jt}}{\mathcal{L}^*_{jt}} = \theta^{\frac{J}{\alpha}}\frac{\mathcal{K}_{Jt}}{\mathcal{L}^*_{Jt}}. \tag{A6}$$

The claim follows from (A5) and (A1):

$$\frac{\mathcal{Y}_{jt}}{\mathcal{L}^*_{jt}} = \frac{\mathcal{Y}_{Jt}}{\mathcal{L}^*_{Jt}} \Leftrightarrow \theta^{\frac{j}{\alpha}}\frac{\mathcal{K}_{jt}}{\mathcal{L}^*_{jt}} = \theta^{\frac{J}{\alpha}}\frac{\mathcal{K}_{Jt}}{\mathcal{L}^*_{Jt}}.$$

Step 3 Let $J = J_t$. We have $P_{Jt} = 1$. With wages as above, the marginal revenue product at time t per \$1's worth of $\mathcal{K}_{Jt}$ is $\alpha \cdot \mathcal{Y}_{Jt}/\mathcal{K}_{Jt}$. For \$1's worth of $\mathcal{K}_{jt}$ with $j < J$, it is $\alpha \cdot \mathcal{Y}_{jt}/[\mathcal{K}_{jt} \cdot P_{jt}]$. Depreciation per \$1's worth is the same for both. Thus, the returns will be the same if

$$\frac{\alpha \cdot \mathcal{Y}_{Jt}}{\mathcal{K}_{Jt}} = \frac{\alpha \cdot \mathcal{Y}_{jt}}{\mathcal{K}_{jt} P_{jt}} \iff \frac{\alpha \cdot \mathcal{Y}_{Jt}}{\mathcal{L}^*_{Jt}} \frac{\mathcal{L}^*_{Jt}}{\mathcal{K}_{Jt}} = \frac{\alpha \cdot \mathcal{Y}_{jt}}{\mathcal{L}^*_{jt}} \frac{\mathcal{L}^*_{jt}}{\mathcal{K}_{jt}} \frac{1}{P_{jt}} \iff \frac{\mathcal{K}_{Jt}}{\mathcal{L}^*_{Jt}} = P_{jt} \frac{\mathcal{K}_{jt}}{\mathcal{L}^*_{jt}},$$

where the last iff uses (A5). So, for equal returns per \$1, we need

$$P_{jt} = \frac{\mathcal{K}_{Jt}/\mathcal{L}^*_{Jt}}{\mathcal{K}_{jt}/\mathcal{L}^*_{jt}} = \frac{1/\left[\theta\right]^{J/\alpha}}{1/\left[\theta\right]^{j/\alpha}} = \theta^{\frac{j-J}{\alpha}},$$

with the second equality following from (A6). Thus, we can have compatibility with profit-maximizing behavior – with markets for all vintages clearing – provided P_{jt} satisfies the last expression. ■

Proof of Proposition 2: Evaluate aggregate output in (1) and the aggregate value of capital in (4) using (A6) and $\sum_{j \le J_t} \mathcal{L}^*_{jt} = 1$ from the proof of Proposition 1:

$$\begin{aligned}
\mathcal{Y}_t &= \sum_{j \le J_t} \left[\theta\right]^j \left[\mathcal{K}_{jt}\right]^\alpha \left[\mathcal{L}^*_{jt}\right]^{1-\alpha} = \sum_{j \le J_t} \left[\theta^{\frac{j}{\alpha}} \mathcal{K}_{jt}\right]^\alpha \left[\mathcal{L}^*_{jt}\right]^{1-\alpha} \\
&= \sum_{j \le J_t} \left[\frac{\theta^{\frac{j}{\alpha}} \mathcal{K}_{jt}}{\mathcal{L}^*_{jt}}\right]^\alpha \mathcal{L}^*_{jt} = \left[\frac{\theta^{\frac{J}{\alpha}} \mathcal{K}_{Jt}}{\mathcal{L}^*_{Jt}}\right]^\alpha \sum_{j \le J_t} \mathcal{L}^*_{jt} = \theta^J \cdot \left[\frac{\mathcal{K}_{Jt}}{\mathcal{L}^*_{Jt}}\right]^\alpha,
\end{aligned}$$

$$\begin{aligned}
\mathcal{K}_t &= \sum_{j \le J_t} P_{jt} \cdot \mathcal{K}_{jt} = \theta^{-\frac{J}{\alpha}} \sum_{j \le J_t} \theta^{\frac{j}{\alpha}} \mathcal{K}_{jt} = \theta^{-\frac{J}{\alpha}} \sum_{j \le J_t} \frac{\theta^{\frac{j}{\alpha}} \mathcal{K}_{jt}}{\mathcal{L}^*_{jt}} \mathcal{L}^*_{jt} \\
&= \frac{\mathcal{K}_{Jt}}{\mathcal{L}^*_{Jt}} \sum_{j \le J_t} \mathcal{L}^*_{jt} = \frac{\mathcal{K}_{Jt}}{\mathcal{L}^*_{Jt}}.
\end{aligned} \tag{A7}$$

Combining the two expressions above establishes (5).

From Step 3 in the proof of Proposition 1, the marginal revenue product of every vintage is the same and equal to

$$\alpha \theta^J \left[\frac{\mathcal{K}_{Jt}}{\mathcal{L}^*_{Jt}}\right]^{\alpha - 1} - \delta = \alpha \theta^J \left[\mathcal{K}_t\right]^{\alpha-1} - \delta = \alpha \frac{\mathcal{Y}_t}{\mathcal{K}_t} - \delta,$$

where the first equality follows from (A7) and the second one from (5). This establishes (6). Lastly, (7) follows directly from (A4). ■

Proof of Proposition 3: Assume at first that $\mathbb{K} = \left[\underline{K}, \bar{K}\right] \subset (0, \bar{K}]$ is well defined.

Since the functions Φ^n are continuous and converge uniformly on $(0, \breve{K}]$, $\Phi^*(K) = \lim_{n \to \infty} \Phi^n(K)$

exists and is continuous on $\mathbb{K}$. Since each $\Phi^n$is non-decreasing on $(0, \breve{K}]$, the same is true for Φ^* on $\mathbb{K}$.

For any $K_t \in \mathbb{K}$, motion on the equilibrium saddle path insures $A_{t+1} \geq K_t$; hence, gross investment is positive.

Next, we prove

Claim: *There is an interval* $\mathbb{K} = \left[\underline{K}, \bar{K}\right]$ *with* $0 < \underline{K} < \bar{K} < \infty$ *such that we can assume all of our rational expectations equilibrium state realizations* K_t are contained in $\mathbb{K}$.

Proof of the claim, Step 1: Letting $m(K)$ be as in Section 2, define

$$\psi_0(A) = \rho(1-\lambda)(1+m(A)),$$
$$\psi_1(A) = \rho\lambda\hat{b}Z^{\gamma-1}\left(1+m\left(\frac{\hat{b}A}{Z}\right)\right).$$

Then we can show there are unique, positive roots $\underline{A}_0$ and $\underline{A}_1$, respectively, to $\psi_0(A) = 1$ and $\psi_1(A) = 1$.

Function $m(A)$, defined in the text, is continuous, strictly decreasing, and has $\lim_{A\to 0} m(A) \longrightarrow \infty$ and $\lim_{A\to\infty} m(A) = -\delta$. Therefore, $\psi_0(A)$ and $\psi_1(A)$ are both continuous, strictly decreasing, and have $\lim_{A\to 0}\psi_0(A) = \infty = \lim_{A\to 0}\psi_1(A)$ and, given (13) and $\hat{b} < 1 < Z$,

$$\lim_{A\to\infty}\psi_0(A) = \rho(1-\lambda)(1-\delta) \in (0,1),$$
$$\lim_{A\to\infty}\psi_1(A) = \rho\lambda\hat{b}Z^{\gamma-1}(1-\delta) \in (0,1).$$

Thus, our unique, positive roots, $\underline{A}_0$ and $\underline{A}_1$, exist.

Step 2 Setting $\underline{K} = \left[\hat{b}/Z\right] \cdot \min\{\underline{A}_0, \underline{A}_1\}$, we show that the interval $\mathbb{K} = \left[\underline{K}, \bar{K}\right]$ is non-empty.

Using the notation $\psi_0(\cdot)$ and $\psi_1(\cdot)$ in Euler equation (20) evaluated on the equilibrium trajectory, we have

$$u'(\Phi^*(K_t)) = \psi_0(A_{t+1}) \cdot u'(\Phi^*(A_{t+1})) + \psi_1(A_{t+1}) \cdot u'\left(\Phi^*\left(\frac{\hat{b}A_{t+1}}{Z}\right)\right). \tag{A8}$$

Both right-hand side terms are positive and $K_t = \bar{K}$ must yield a solution. By construction, $K_t = \bar{K}$ yields $A_{t+1} = \bar{K}$. For equality in (A8), we then must have $\psi_0\left(\bar{K}\right) < 1$. Then $\psi_0(\underline{A}_0) = 1$ implies $\underline{A}_0 < \bar{K}$. Hence, $\underline{K} < \underline{A}_0 < \bar{K}$, establishing Step 2.

Step 3 We show that $\mathbb{K} = \left[\underline{K}, \bar{K}\right]$ is an invariant set, i.e. $K_t \in \mathbb{K} \Rightarrow K_{t+1} \in \mathbb{K}$**.**

First, $K_t \in \mathbb{K} \Rightarrow K_t \leq \bar{K}$. Hence, motion on the equilibrium saddle path implies $A_{t+1} \leq \bar{K}$. Then $\hat{b}A_{t+1}/Z \leq \bar{K}$ as well. So, $K_{t+1} \leq \bar{K}$.

Second, we show $K_t \in \mathbb{K} \Rightarrow K_{t+1} \geq \underline{K}$.

Case 1: $K_t \in \mathbb{K}$ and $A_{t+1} \geq \underline{A}_1$. Then $A_{t+1} \geq \hat{b}A_{t+1}/Z \geq \hat{b}\underline{A}_1/Z \geq \underline{K}$. So, $K_{t+1} \geq \underline{K}$.

Case 2: $K_t \in \mathbb{K}$ and $A_{t+1} < \underline{A}_1$. Then $\psi_1(A_{t+1}) > 1$. From (A8) and the monotonicity of Φ^*, we then have

$$u'(\Phi^*(K_t)) > \psi_1(\underline{A}_1)\, u'\left(\Phi^*\left(\frac{\hat{b}A_{t+1}}{Z}\right)\right) = u'\left(\Phi^*\left(\frac{\hat{b}A_{t+1}}{Z}\right)\right)$$
$$\Rightarrow \Phi^*(K_t) < \Phi^*\left(\frac{\hat{b}A_{t+1}}{Z}\right) \Rightarrow K_t < \frac{\hat{b}A_{t+1}}{Z}.$$

So, $A_{t+1} \geq \hat{b}A_{t+1}/Z > K_t \geq \underline{K}$. Hence, $K_{t+1} \geq \underline{K}$. Cases 1-2 establish Step 3. ■

Proof of Proposition 3 continued We have shown that all of the rational expectations equilibrium aggregate state realizations are contained in a compact interval $\mathbb{K}$ that is bounded away from 0. Since $\mathbb{K}$ is the feasible set for the household problem (15) and the payoff function U in (14) is strictly increasing in a_t, strictly concave in a_t and a_{t+1}, and continuously differentiable, the household problem satisfies the assumptions of Theorem 4.15 in Stokey and Lucas (1989). Applying the theorem establishes that the first-order condition and the transversality condition,

$$\lim_{t\to\infty} T_t \equiv \lim_{t\to\infty} \sum_{z^t \in \mathcal{Z}^t} p\left(z^t\right) \rho^t \frac{\partial U\left(a_t, a_{t+1}, z^t\right)}{\partial a_t} a_t = 0, \tag{A9}$$

are sufficient for optimality in (15).

To see that the first-order condition for (15) coincides with (A8), collect the terms in (15) that depend on $a_{t+1} = a\left[z^t\right]$, differentiate with respect to a_{t+1} and evaluate the resulting expression at $x_t = X_t = \Phi^*(K_t)$, $k_t = K_t$ and $a_t = A_t$.

It is left to show that the rational expectations equilibrium trajectory satisfies the transversality condition (A9). This involves two steps.

Step 1 Show that

$$\sigma_t = \sum_{z^t \in \mathcal{Z}^t} p\left(z^t\right) \rho^t Z^{\gamma J_t} < 1 \text{ all } t \geq 1.$$

Since z^t is a sequence of realizations of $t+1$ i.i.d. Bernoulli random variables, we have

$$\sigma_t = \sum_{z^t \in \mathcal{Z}^t} \lambda^{J_t} (1-\lambda)^{t-J_t} \rho^t Z^{\gamma J_t} = \sum_{z^t \in \mathcal{Z}^t} \left(\rho\lambda Z^\gamma\right)^{J_t} \cdot \left(\rho(1-\lambda)\right)^{t-J_t}$$
$$= \sum_{J=0}^{t} \binom{t}{J} \cdot \left(\rho\lambda Z^\gamma\right)^J \cdot \left(\rho(1-\lambda)\right)^{t-J} = \left(\rho\lambda Z^\gamma + \rho(1-\lambda)\right)^t \text{ all } t \geq 1.$$

Thus, (13) implies that

$$\lim_{t\to\infty} \sigma_t = 0. \tag{A10}$$

Step 2 We show that (A9) holds.

Lemma 1 establishes that $a\left[z^{t-1}\right]\omega\left(z_t\right)=K\left[z^t\right]\in\left[\underline{K},\bar{K}\right]$. Following the notation of Section 2:

$$\begin{aligned}\frac{\partial U}{\partial a_t}a_t &= u'\left(c_t\right)\cdot Z^{J_t}\cdot\omega\left(z_t\right)a_t\cdot\left(1+m\left(K\left[z^t\right]\right)\right)\\ &\leq u'\left(x_t\right)\cdot Z^{\gamma J_t}\cdot\bar{K}\cdot\left(1+m\left(K\left[z^t\right]\right)\right)\\ &\leq u'\left(\Phi^*\left(\underline{K}\right)\right)\cdot Z^{\gamma J_t}\cdot\bar{K}\cdot\left(1+m\left(\underline{K}\right)\right).\end{aligned}$$

Then

$$0\leq T_t\leq u'\left(\Phi^*\left(\underline{K}\right)\right)\cdot\bar{K}\cdot\left(1+m\left(\underline{K}\right)\right)\cdot\sigma_t.$$

So, (A10) establishes (A9). ■

Appendix 2: Baseline parameter calibrations

The parameters of the model include

$$(g,\alpha,\lambda,\hat{b},d,\nu,\beta,\Upsilon,b,\bar{\delta},\delta,\theta,Z,\rho,\gamma). \tag{A11}$$

Section 4.3 in the text discusses the calibration of ρ and γ. Here, we present our baseline calibrations for the remaining parameters of (A11) — summarized in Table A1 below.

On the basis of U.S. data for 1870-1987, Maddison [1991, tab. 3.3] estimates an average annual per capita growth rate $g=0.0210$. Gollin [2002] estimates a factor share for capital $\alpha=0.30$. We assume type-I households receive all of capital's share and a fraction β of labor's share, $1-\alpha$. In Kennickell [2009], the 2007 Survey of Consumer Finances seems representative of 1989-2007. The share of net worth for the top 5% (of the distribution of family net worth) is 55-60%. We assume these families own the economy's stock of plant and equipment. They correspond to the model's type-I households — with remaining families financing only their own residences (which are not part of our model). Kennickell determines the income share of the same top wealth holders to be 33%. Let $\widehat{\mathcal{Y}}$ be measured GDP excluding services of residential housing, investment in residential housing, and depreciation on residential housing. Let $\widehat{\mathcal{D}}$ be measured depreciation excluding depreciation of residential housing. Then we derive β from

$$\frac{1}{3}\cdot[\widehat{\mathcal{Y}}-\widehat{\mathcal{D}}]=[\alpha\cdot\widehat{\mathcal{Y}}-\widehat{\mathcal{D}}]+\beta\cdot(1-\alpha)\cdot\widehat{\mathcal{Y}}. \tag{A12}$$

Using 2007 U.S. data, we have $\widehat{\mathcal{Y}}=\$12,308.3\,bil/yr$ and $\widehat{\mathcal{D}}=\$1458.3\,bil/yr$; so, given $\alpha=0.30$, we compute $\beta=0.1563$.

Let λ be the hazard for the arrival of new GPTs. Gordon [2012, p. 1-2] assumes 4-5 arrivals in about 125 years, implying $0.032 \leq \lambda \leq 0.04$; Abramovitz and David [2014] have 4-5 arrivals in a century, implying $0.04 \leq \lambda \leq 0.05$; and, Abuquerque et al. [2015, tab. 2] have 4-5 arrivals in 139 years (see the interpretation in Section 5.1 of the text), implying $0.03 \leq \lambda \leq 0.035$. Our baseline assumption is $\lambda = 0.04$.

Considering the introduction of the microprocessor chip as a seminal invention in the early 1970s, Laitner and Stolyarov [2003] estimate that each dollar's worth of existing capital fell in value to \$0.3866. Here, we set $\hat{b} = 0.40$. We assume frictional costs at the time of a GPT's arrival reduce the GDP. We model this as a destruction of part of the capital stock. Assuming a drop in GDP of 3% (i.e., temporary disappearance of trend growth, plus an additional 1% decline) and letting the fraction of each dollar's worth of capital surviving destruction be Υ, our aggregate production function implies

$$\Upsilon = [0.97]^{1/\alpha}. \tag{A13}$$

Letting \$$b$ be the value per dollar's worth of surviving capital after also taking into account obsolescence, we have

$$b \cdot \Upsilon = \hat{b}. \tag{A14}$$

Using U.S. data 1953-2001, Laitner and Stoyarov [2003] estimate an average (total) depreciation rate $d = 0.0752$. In our model,

$$d = \delta + \bar{\delta}$$

with δ is the rate of wear and tear deprecation of capital and $\bar{\delta}$ the rate of depreciation from technological obsolescence. Think of $\bar{\delta}$ as a long-run average rate. Fix any time t and small $h > 0$. The amount of depreciation from technological obsolescence during the interval $[t, t+h)$ is

$$\bar{\delta} \cdot h = \begin{cases} 0, & \text{if no GPT arrival } [t, t+h) \\ 1 - \hat{b}, & \text{if a new GPT arrives during } [t, t+h). \end{cases}$$

Since the probability of a GPT arrival during the same interval is $\lambda \cdot h$, we have

$$\bar{\delta} = \lambda \cdot (1 - \hat{b}), \tag{A15}$$

$$\delta = d - \lambda \cdot (1 - \hat{b}). \tag{A16}$$

Letting ν be the fraction of bond financing for the physical capital stock, we follow Papanikolaou [2011], Boldrin et al. [1995], and Barro [2006, p.843] in setting $\nu = 0.40$.

Proposition 1 in the text shows

$$[\theta]^{-1/\alpha} = b. \tag{A17}$$

Our detrending procedure uses Z with

$$Z = [\theta]^{\frac{1}{1-\alpha}}. \tag{A18}$$

Parameters set independently		
Parameter	Value	Reference (see text)
g	0.021	Maddison [1991]
α	0.30	Gollin [2002]
λ	0.04	Gordon [2012], Abramovitz & David [2014], Albuquerque *et al.* [2015]
$\hat{b}$	0.40	Laitner & Stolyarov [2003]
d	0.0752	Laitner & Stolyarov [2003]
ν	0.40	Barro [2006], Papanikolaou [2011], Boldrin *et al.* [1995]
Parameters derived from calibration restrictions		
Parameter	Value	Formula (see Appendix 1)
β	0.1563	(A12)
Υ	0.9035	(A13)
b	0.4427	(A14)
$\bar{\delta}$	0.0240	(A15)
δ	0.0512	(A16)
θ	1.2769	(A17)
Z	1.4179	(A18)

Table A1. Baseline parameter values.

Appendix 3: Long-Run Growth Constraint

Our theoretical framework implies a consistency condition that relates the frequency and size of technological improvements to the rate of long-run growth. While the condition is not imposed in Appendix 2, our analysis requires it. This section derives it.

The growth factor from each new GPT is Z. Suppose we have a large number, N, of new GPTs and they arrive at t_1, t_2,...,t_N. Set $\Delta_j \equiv t_{j+1} - t_j$. Then

$$[Z]^N \approx \prod_{j=1}^{N} e^{g \cdot \Delta_j}.$$

So,

$$\ln(Z) \approx \frac{g \cdot \sum_{j=1}^{N} \Delta_j}{N}, \tag{A19}$$

with equality for $N \to \infty$. Since GPT arrivals are Poisson, the interval lengths Δ_j are independent samplings from an exponential distribution with mean $1/\lambda$. Letting $N \to \infty$ in (A19) and using the weak law of large numbers, we set

$$\ln(Z) = \frac{g}{\lambda}. \tag{A20}$$

Combining (A13)-(A14) and (A17)-(A18) with (A20), we have

$$g = \lambda \cdot \frac{\alpha}{1-\alpha} \cdot \ln\left(\frac{1}{b}\right) = \lambda \cdot \frac{\alpha}{1-\alpha} \cdot \ln(\Upsilon/\hat{b}) = \lambda \cdot \frac{\alpha}{1-\alpha} \cdot \ln\left(\frac{[0.97]^{1/\alpha}}{\hat{b}}\right). \tag{A21}$$

Appendix 4: Human Capital

This appendix accompanies the subsection "Earnings and the Equity Premium" in Section 4 of the text.

Production function. Without loss of generality, set the technology index to $j = 0$ and the time to $t = 0$. Let $\bar{\mathcal{K}}$ be dollar's worth of physical capital and $\tilde{\mathcal{K}}$ be dollar's worth of human capital. Replace aggregate production function (1) with

$$\mathcal{Y}_0 = [\bar{\mathcal{K}}_0]^{\bar{\alpha}} \cdot [\tilde{\mathcal{K}}_0]^{\tilde{\alpha}} \cdot [L_0]^{\beta \cdot (1-\alpha)} \cdot [\bar{L}_0]^{(1-\beta)\cdot(1-\alpha)}, \quad \alpha \equiv \bar{\alpha} + \tilde{\alpha}. \tag{A22}$$

Let $\mathcal{K}_0$ be the economy's total dollar's worth of capital. If we assume physical and human capital have the same depreciation schedule, a Pareto efficient private sector will allocate investment such that $\bar{\mathcal{K}}_0$ and $\tilde{\mathcal{K}}_0$ maximize (A22) subject to

$$\bar{\mathcal{K}}_0 + \tilde{\mathcal{K}}_0 \leq \mathcal{K}_0. \tag{A23}$$

Solving the maximization problem, we have

$$\bar{\mathcal{K}}_0 = \frac{\bar{\alpha}}{\alpha} \cdot \mathcal{K}_0 \quad \text{and} \quad \tilde{\mathcal{K}}_0 = \frac{\tilde{\alpha}}{\alpha} \cdot \mathcal{K}_0. \tag{A24}$$

Substituting from (A24) into (A22), we have

$$\mathcal{Y}_0 = [\frac{\bar{\alpha}}{\alpha}]^{\bar{\alpha}} \cdot [\frac{\tilde{\alpha}}{\alpha}]^{\tilde{\alpha}} \cdot [\mathcal{K}_0]^{\alpha} \cdot [L_0]^{\beta \cdot (1-\alpha)} \cdot [\bar{L}_0]^{(1-\beta)\cdot(1-\alpha)}. \tag{A25}$$

Type-I Households' Incomes. Thinking of Trial 2 in our Table 1, let $\bar{\alpha} = 0.3000$, $\tilde{\alpha} = 0.0854$, and $\alpha = 0.3854$. Physical capital's (gross) compensation and total labor compensation in the economy will be

$$\bar{\alpha} \cdot \mathcal{Y}_0 \quad \text{and} \quad [(1-\alpha)+\tilde{\alpha}] \cdot \mathcal{Y}_0 = (1-\bar{\alpha}) \cdot \mathcal{Y}_0, \tag{A26}$$

respectively. Thus their ratio matches the calibration of Trial 1.

We calibrate β from (A12) in all cases. In Trials 1 and 3-4, the ratio of type-I to type-II household labor income is

$$\frac{\beta}{1-\beta} \approx 0.185. \tag{A27}$$

In Trial 2, β is about two-thirds smaller (assuming we continue with the same $\widehat{\mathcal{Y}}$ and $\widehat{\mathcal{D}}$ as before). Using the algebra from (A12), the ratio of comprehensive labor incomes for type-I and type-II households is, however,

$$\frac{(1/3) \cdot [\widehat{\mathcal{Y}} - \widehat{\mathcal{D}}] - (\bar{\alpha}/\alpha) \cdot [\alpha \widehat{\mathcal{Y}} - \widehat{\mathcal{D}}]}{(2/3) \cdot [\widehat{\mathcal{Y}} - \widehat{\mathcal{D}}]} \approx 0.140, \tag{A28}$$

which is only about one-quarter smaller than (A27).[23]

Appendix 5: Data for Table 2

This appendix outlines the construction of columns 4-5 of Table 2 in the text.

We use annual data covering the period 1871-2018. The stock price and earnings data are from the S&P composite index data in Shiller (1989, Ch 26, Series 1, 3), updated to 2018. The series for real per-capita consumption is taken from Shiller (1989, Table 26.2 series 9) for 1889-1929 and spliced with NIPA after 1929. All the rates calculated from the data are geometric rates, $\ln(1+R)$. We define the risk-free rate as the real annualized rate of return on prime 4-6 month commercial paper as reported in Shiller (1989, Ch 26, Series 4).[24]

For the 1947-2018 sub-sample, we use additional data on capital structure to construct moments for the "unlevered" return on equity, the direct analog of our R_t, using the empirical time series for R_t^S, r_t, and ν_t.

[23] A more detailed treatment would assume NIPA measures incorrectly omit on-the-job investments in human capital from investment output, with $\mathcal{D}$ similarly understated. A more thorough analysis would then tend to make (A27) and (A28) even closer.

[24] The Federal Reserve Board discontinued its 6-month commercial paper rate series in August 1997. After this date, Shiller uses the 6-month certificate of deposit rate, secondary market, FRED series CD6NRNJ, with FDIC as a data source.

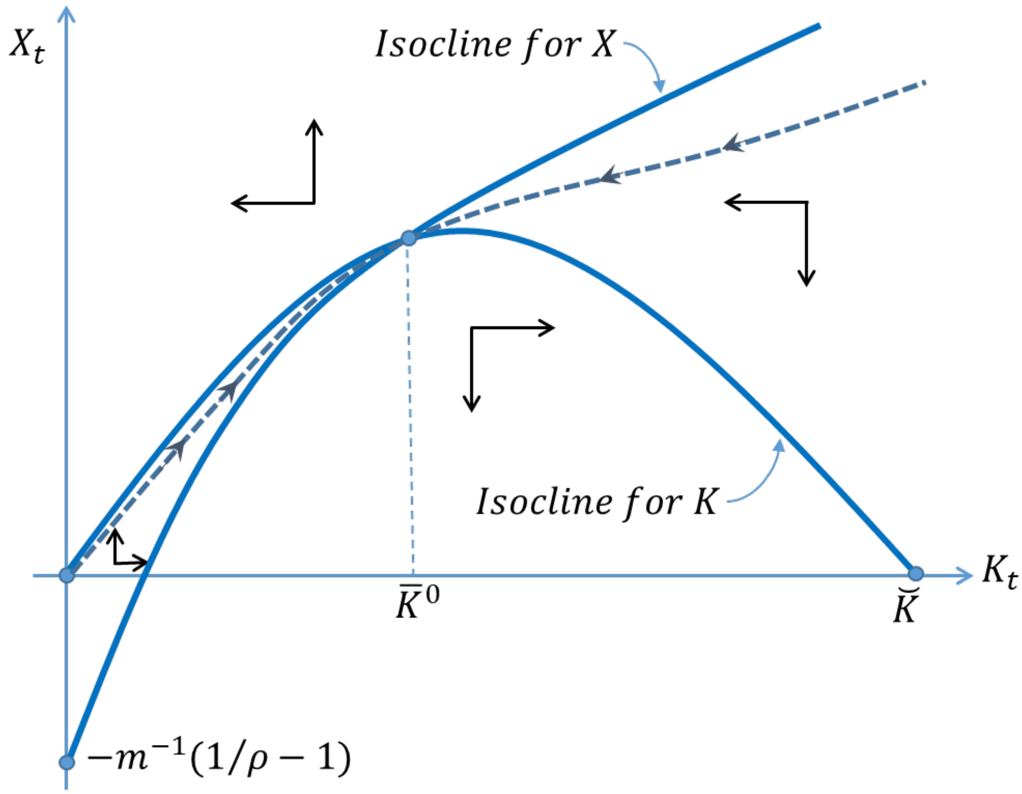

Figure 1. Phase diagram for $n = 0$.

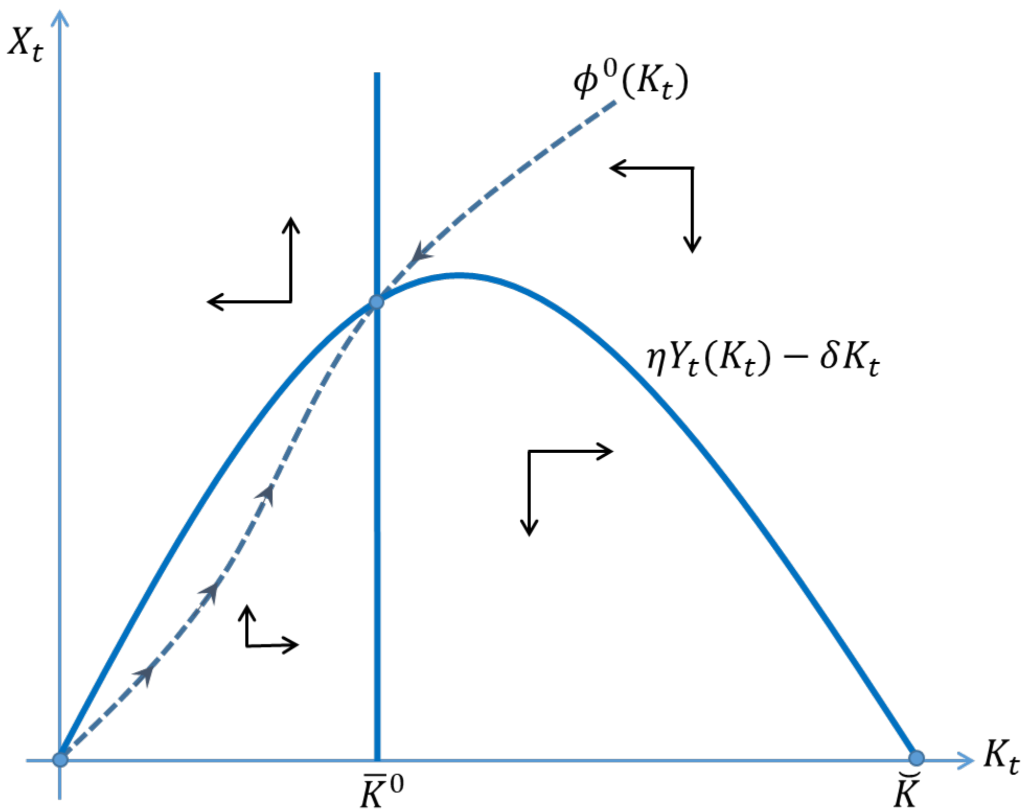

Figure 2. Phase diagram for $n = 0$, continuous time auxiliary problem.

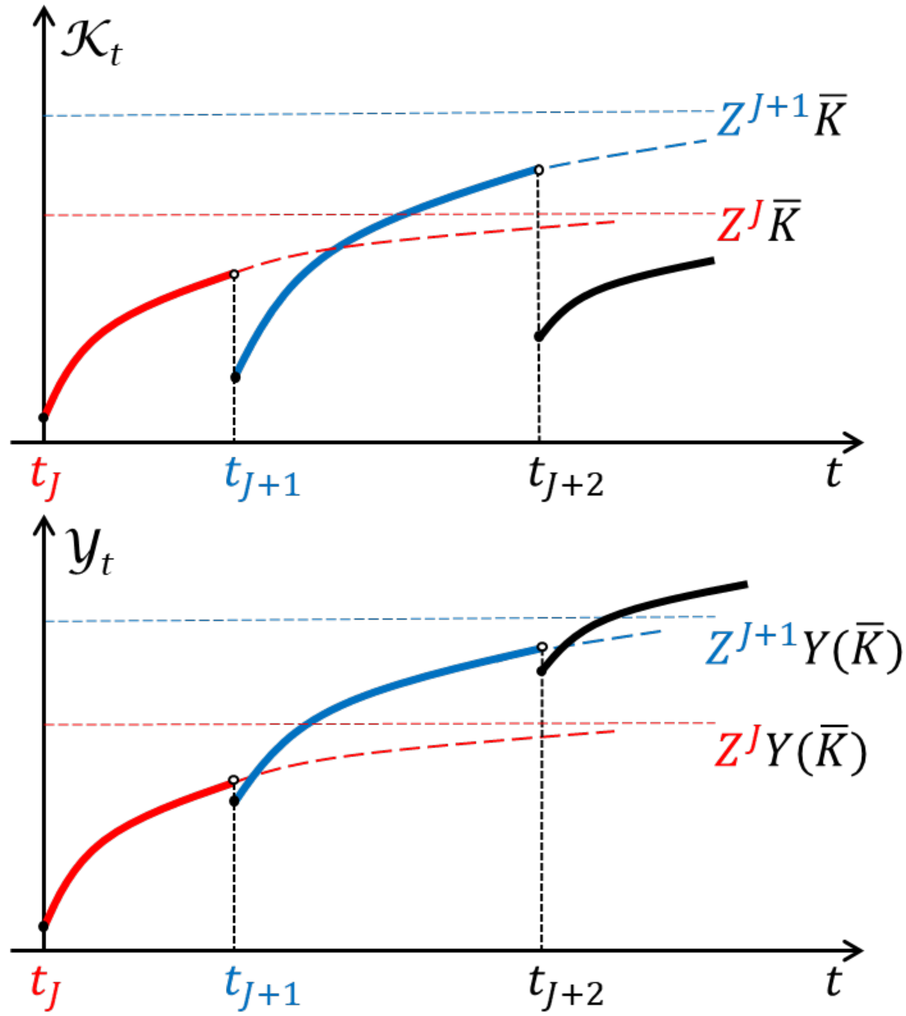

Figure 3. Equilibrium trajectories for capital stock and output.

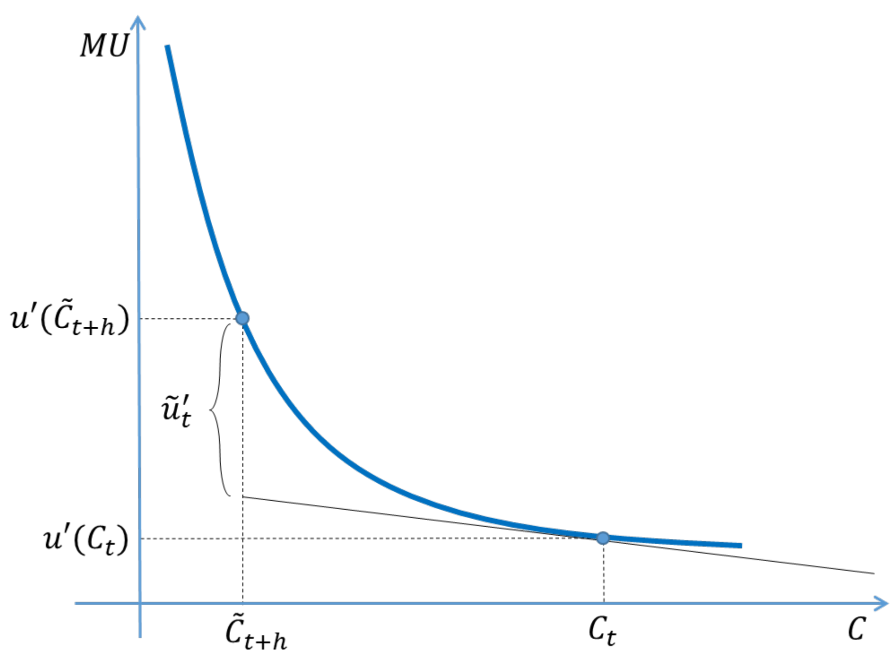

Figure 4. Change in marginal utility after a large drop in consumption.

Simulated variable	$\gamma = -0.75$	$\gamma = -0.50$	$\gamma = -0.25$	$\gamma = -0.00$	$\gamma = 0.25$
Trial 1: $\alpha = 0.30$; $\hat{b} = 0.40$; $g = 0.021$; and, treated as residual, $\lambda = 0.0601$[1/] [2/]					
Subjective disc. rate, ρ_0	na[3/]	0.0063	0.0133	0.0195	0.0247
Equity Premium, $\bar{E}$	na	0.0521	0.0557	0.0577	0.0577
Annual aggregate consumption per capita[4/]					
Prob Decline ≥ 5%	na	0.0173	0.0256	0.0331	0.0405
Prob Decline ≥ 10%	na	0.0002	0.0026	0.0094	0.0177
Best Match (*)				*	
Trial 2: $\hat{b} = 0.40$; $g = 0.021$; $\lambda = 0.04$ and, treated as residual, $\alpha = 0.3854$[1/] [5/]					
Subjective disc. rate, ρ_0	na[5/]	0.0029	0.0119	0.0202	0.0275
Equity Premium, $\bar{E}$	na	0.0378	0.0452	0.0508	0.0545
Annual aggregate consumption per capita[4/]					
Prob Decline ≥ 5%	na	0.0032	0.0110	0.0184	0.0247
Prob Decline ≥ 10%	na	0.0000	0.0000	0.0029	0.0097
Best Match (*)					*
Trial 3: $\alpha = 0.30$; $\lambda = 0.04$; $g = 0.021$; and, treated as residual, $\hat{b} = 0.2654$[1/] [6/]					
Subjective disc. rate, ρ_0	0.0054	0.0145	0.0224	0.0291	0.0344
Equity Premium, $\bar{E}$	0.0640	0.0722	0.0776	0.0804	0.0806
Annual aggregate consumption per capita[4/]					
Prob Decline ≥ 5%	0.0125	0.0182	0.0250	0.0268	0.0307
Prob Decline ≥ 10%	0.0000	0.0025	0.0077	0.0129	0.0183
Best Match (*)			*		
Trial 4: $\alpha = 0.30$; $\lambda = 0.04$; $\hat{b} = 0.40$; and, treated as residual, $g = 0.0140$[1/] [7/]					
Subjective disc. rate, ρ_0	0.0075	0.0123	0.0168	0.0208	0.0242
Equity Premium, $\bar{E}$	0.0325	0.0354	0.0374	0.0385	0.0384
Annual aggregate consumption per capita[4/]					
Prob Decline ≥ 5%	0.0099	0.0157	0.0209	0.0255	0.0306
Prob Decline ≥ 10%	0.0000	0.0002	0.0021	0.0076	0.0139
Best Match (*)				*	

Table 1. Simulated equity premium and consumption volatility. Mean riskless rate $r = 2.32\%/yr$ all cases.

1. See Appendix 1 for sources on (α, b, λ, g). In each trial, one of these four parameters is calculated from the long-run growth constraint (44) of Section 4.3.
2. Trial 1 other parameters: $\beta = 0.1563, \Upsilon = 0.97, Z = 1.4179, \delta = 0.0391, \bar{\delta} = 0.0361, T = 16.628, \nu = 0.40$
3. No ρ_0 satisfying (13) attains $\bar{r} = 2.32\%/yr$.
4. U.S. data 1871-2018 shows the probability for an annual decline in per capita aggregate consumption of ≥ 5% (≥ 10%) of 0.0234 (0.0078). The "best match" has the lowest sum of squared log difference (simulated vs actual) for the two measures of decline. See text.
5. Trial 2 other parameters: $\beta = 0.0605, \Upsilon = 0.97, Z = 1.6905, \delta = 0.0512, \bar{\delta} = 0.0240, T = 25, \nu = 0.40$
6. Trial 3 other parameters: $\beta = 0.1563, \Upsilon = 0.97, Z = 1.6905, \delta = 0.0458, \bar{\delta} = 0.0294, T = 25, \nu = 0.40$
7. Trial 4 other parameters: $\beta = 0.1563, \Upsilon = 0.97, Z = 1.4318, \delta = 0.0512, \bar{\delta} = 0.0240, T = 25, \nu = 0.40$

Variable	Trial:			Data [4/]	
	2 [1/]	2A [2/]	4 [3/]	1871-2018	1947-2018
Means					
Stock return, $\bar{R}^S$	7.76	6.67	6.18	6.46	6.93
Risk-free rate, $\bar{r}$	2.32	2.32	2.32	2.32	1.22
Equity premium, $\bar{E}$	5.45	4.36	3.85	4.14	5.72
Return on capital, $\bar{R}$	5.58	5.58	4.63	na	5.90
Sharpe ratio, $\bar{E}/\sigma(E)$	0.27	0.27	0.19	0.24	0.37
Standard deviations					
Stock return, $\bar{R}^S$	19.13	15.40	19.01	16.97	15.60
Risk-free rate, $\bar{r}$	6.43	6.43	5.58	6.24	2.93
Equity premium, $\bar{E}$	19.97	15.97	20.01	17.28	15.50
Return on capital, $\bar{R}$	11.85	11.85	11.54	na	10.94
Coefficients of variation					
Stock return, $\bar{R}^S$	2.46	2.31	3.08	2.63	2.25
Risk-free rate, $\bar{r}$	2.77	2.77	2.40	2.69	2.41
Equity premium, $\bar{E}$	3.67	3.67	5.20	4.17	2.71
Return on capital, $\bar{R}$	2.12	2.12	2.49	na	1.82

Table 2. Model-Data Comparisons. Units for rates are percent.

Source: see text and Table 1.

1. Parameters as in Table 1, trial 2, with $\gamma = 0.25$ ($CRRA = 0.75$).
2. Parameters as in Table 1, trial 2, with $\gamma = 0.25$ ($CRRA = 0.75$), and $\nu = 0.25$.
3. Parameters as in Table 1, trial 4, with γ = 0.00 (CRRA=1.00).
4. See Appendix 5.