Activities
Wayne County Math Teachers' Circle
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Problems

1. Lattice squares

Say that a point in the coordinate plane is a *lattice point* if both of its coordinates are integers. Say that a square is a *lattice square* if all of its vertices are lattice points. What are the possible areas of a lattice square?

2. Sum of two squares

Which numbers can be written as a sum of two (perfect) squares? For example:

$$\begin{aligned} 5 &= 2^2 + 1^2, \\ 8 &= 2^2 + 2^2, \\ 13 &= 3^2 + 2^2, \\ 16 &= 4^2 + 0^2, \end{aligned}$$

but 7 is not a sum of two squares. (Can you think of an efficient way to verify this last fact?)

3. Age problem

A census taker[1] approaches a woman leaning on her front gate and asks about her children. She says, "I have three children and the product of their ages is 72. The sum of their ages is the number on this gate." The census taker does some calculation and claims not to have enough information. The woman enters her house, but before slamming the door tells the census taker, "I have to see to my eldest child who is in bed with measles." The census taker departs, satisfied.

What are the ages of the three children?

4. Age problem: More challenging version

I say to my friend Bob, "The product of the ages of my music students is 2450, and the sum of their ages is twice your age."

Bob thinks for a while, but still doesn't know the ages of my students.

I add, "I'm older than any of them".

"Now I know!" says Bob.

Find the ages of everyone in this story: me, my students, and Bob.

[1]Versions of this problem have been around for decades. This version is taken from Wikipedia.

Solutions

1. Lattice squares

The area of a square is completely determined by the length of one of its sides. A side of a lattice square must start and end at lattice points. The horizontal distance between these two points is some integer. Call it C. The vertical distance between these two points is an integer. Call it D. By the Pythagorean Theorem, the length of the side is $\sqrt{C^2 + D^2}$, and so the area of the square is $C^2 + D^2$. That is, the area must be a sum of squares.

Conversely, suppose that a number A can be written in the form $C^2 + D^2$, where C and D are nonnegative integers. Then draw a line segment from the origin to the point (C, D), and this will form the first side of a lattice square whose area is A.

Conclusion: A number is the area of a lattice square if and only if it is the sum of two squares.

2. Sum of two squares

The complete solution was first announced in a letter written by Fermat and dated December 25, 1640. Thus, the result is sometimes called...

Fermat's Christmas Theorem. *Let n be a natural number, and write n as a product of powers of distinct primes:*

$$n = p_1^{r_1} p_2^{r_2} \cdots .$$

If the factorization includes any prime that is congruent to 3 modulo 4, taken to the power of an odd exponent, then n cannot be written as a sum of two squares. Otherwise, it can.

Many proofs have been published, but the first didn't appear until over 100 years after Fermat's letter, so we do not expect to prove the theorem during our activity. But here are some observations that one might make along the way, and proofs of them.

Observation 1. *If a number n can be written in the form $4m + 3$ for an integer m, then n cannot be a sum of two squares.*

Proof. Let a be even. Then $a = 2r$ for some integer r, so $a^2 = 4r^2$.

Let b be odd. Then $b = 2s + 1$ for some integer s, so $b^2 = (2s + 1)^2 = 4s^2 + 4s + 1 = 4(s^2 + s) + 1$.

In other words, the square of an even number has the form $4m$ for some m, and the square of an odd number has the form $4m+1$ for some m. Here are all of the possible results of adding two squares:

$+$	$4m$	$4m+1$
$4m'$	$4(m+m')$	$4(m+m')+1$
$4m'+1$	$4(m'+m)+1$	$4(m'+m)+2$

Note that the form 4(some integer) $+\,3$ never appears above. □

Observation 2. *If a is a sum of two squares, and b is an integer, then ab^2 is a sum of two squares.*

Proof. We're given that $a = c^2 + d^2$ for some integers c and d, then $ab^2 = (cb)^2 + (db)^2$. □

Observation 3. *If M and N can each be written as a sum of two squares, then so can their MN.*

Proof. Let's write $M = a^2 + b^2$ and $N = c^2 + d^2$. If you're willing to use the imaginary number i, a square root of -1, then you can view a sum of two squares as a difference of two squares, and then you can factor it:

$$M = a^2 + b^2 = a^2 - (bi)^2 = (a + bi)(a - bi)$$

Similarly, $N = (c + di)(c - di)$. Therefore,

$$\begin{aligned} MN &= \big[(a+bi)(c+di)\big]\big[(a-bi)(c-di)\big] \\ &= \big[(ac-bd)+(ad+bc)i\big]\big[(ac-bd)-(ad+bc)i\big] \\ &= (ac-bd)^2 + (ad+bc)^2 \end{aligned}$$

□

3. Age problem

We don't know the number on the woman's front gate, but the census taker does, and therefore he or she knows the sum of the ages of the children. We also know that this is not enough information to determine the ages.

Therefore, it must be the case that of all triples of numbers whose product is 72, there must be two with the same sum. This problem is discussed in detail on Wikipedia[2], where you can see all of the triples whose product is 72. Each triple has a distinct sum, with the exception of the fact that $(2, 6, 6)$ and $(3, 3, 8)$ both add up to 14. Therefore, 14 must be the number on the gate. Since the woman refers to her "eldest" child,[3] we conclude that the ages of her chilren are 3, 3, and 8.

If this problem looks too complicated for your students, show them a simpler version, where the product of the ages is 36. You'll find it described on the same Wikipedia page.

4. Age problem: More challenging version

As in the previous problem, it must be the case that there are two different sequences of numbers that have the same sum, and that both have the product 2450. This time, we don't know how many people we're dealing with. But since they are music students, let's assume that their ages are all greater than 1 and less than 140. Moreover, since the sum is twice Bob's age, it must be an even number. This cuts out a lot of possibilities.

From among the remaining possibilities, you'll see that there are only two sequences whose sums are equal and whose products are both 2450:

$$49 \times 10 \times 5 = 50 \times 7 \times 7 = 245049 + 10 + 5 = 50 + 7 + 7 = 64$$

Now we know that Bob's age is half of 64 or 32. But initial he can't tell if the ages are 49, 10, and 5, or 50, 7, and 7. Once he hears that I am older than all of the students, then he can tell. The only way that this extra information could make a difference is if I am 50, implying that the ages are 49, 7, and 7.

[2] `https://en.wikipedia.org/wiki/Ages_of_Three_Children_puzzle`

[3] There is one flaw in this problem. It's possible for two siblings to be born within a year of each other. Thus, measured in a whole number of years, their ages could be the same, and yet they are not twins.