Four Number Game

Wayne County Math Teachers' Circle

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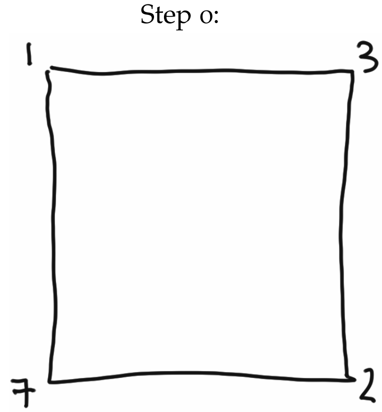
The "Four Number Game" is a simple mathematical game that provides opportunities for mathematical exploration, questioning, conjecture-making, problem-solving, and justification using both geometry and algebra.

How to play

You start by drawing a square and labeling its corners with four whole numbers.

Pay attention to the order and placement of the numbers. Starting from the upper left corner, we'll say we labeled this square using "1, 3, 2, 7."

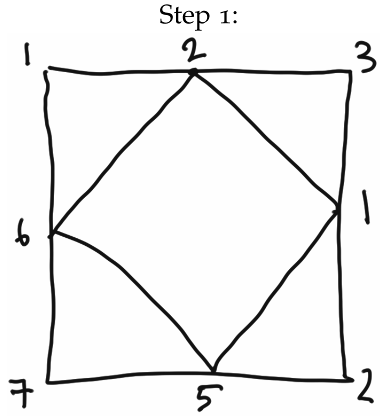
Step 0:



Find the midpoint of each side of the square and connect them to form a smaller square. For each vertex in the new square, label it with the difference between the numbers on the ends of the original edge.

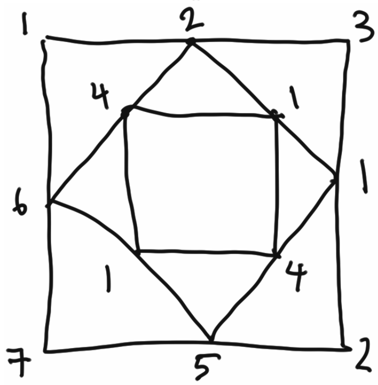
To be totally precise, we are labeling it with the *absolute value* of the difference.

Step 1:

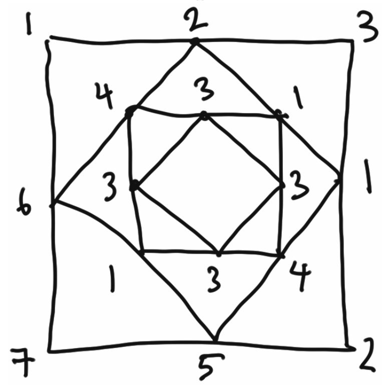


Now find the midpoints of the smallest square and connect them to form an even smaller square. For each vertex in the new square, label it with the difference between the numbers on the ends of the previous edge. Keep repeating that for smaller and smaller squares. Make sure to count your steps.

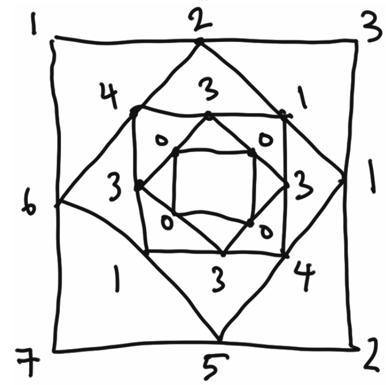
Step 2:



Step 3:



Step 4:



In this game, it took 4 steps until we ended up with all zeros. Notice that playing any further isn't very much fun because we'll just have zeros forever and ever.

Questions and Explorations

This game begs all sorts of questions. Here are just a few:

First Questions

- Will all games end in all zeros? How do we know?

★ Does the initial order of the numbers matter? How will changing their arrangement affect (or not affect) the number of steps the game takes to get to all zeros?

- Are there games that last 10 steps? 11 steps? 100 steps? Can we find games of an arbitrarily long number of steps? Can we find games of any specific number of steps?

Extensions

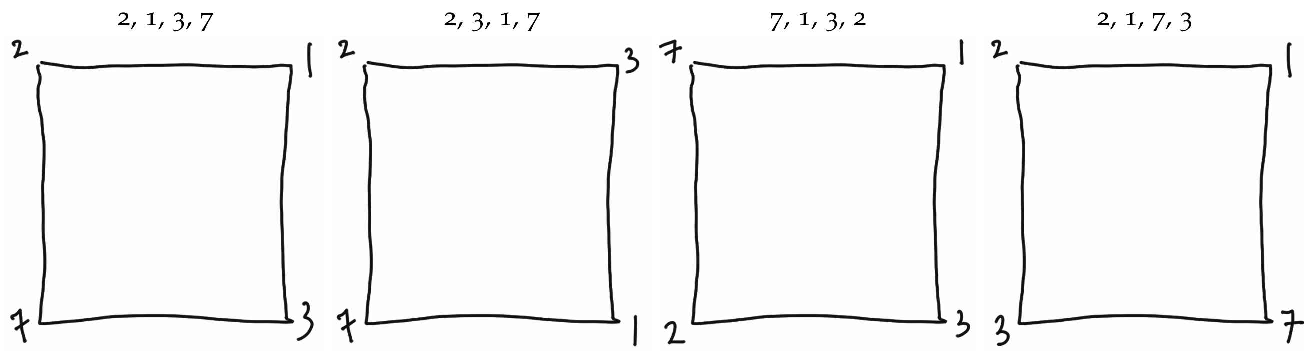
- What if we played a Three Numbers Game, or Five Numbers Game, or Eight Numbers Game. Will these games always end in all zeros?
- What if we could use fractions to begin with? Could we find four numbers to start with so that the game would never end?

Question ★: "Does the initial order of the numbers matter?"

Spoiler Alert: This question is particularly interesting pedagogically because it leads (perhaps surprisingly) to an investigation of symmetries of the square. The Common Core State Standards (in both 8th grade and HS geometry) have moved towards a transformational approach to geometry. One standard this leads to is investigation of symmetries of polygons. This game gives us an opportunity to explore those symmetries in an authentic, motivated context. How is that? Let's see.

Getting Started

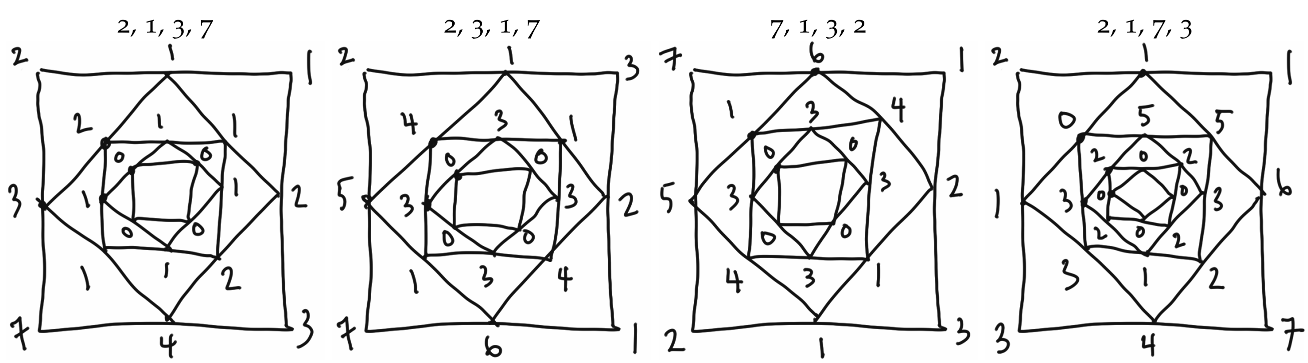
Once we pose the question, we start tackling it experimentally—trying all sorts of arrangements of the numbers. Below are a few that might come up:



What happens when you play out these games? Do any of the games look familiar? How so? Why do they look familiar? How can you tell if two games will end up "the same"?

Moving Ahead

Here are pictures of the games played out.



Let's look at two of them in detail.

First notice that every difference that shows up in 2, 3, 1, 7, also shows up in 7, 1, 3, 2. Why is that? Can you describe a symmetry of the square that would turn one game into another game? Symmetries may include rotating the square or flipping it over a diagonal, vertical, or horizontal line. A little examination allows us to see that we can turn the 2, 3, 1, 7 game into the 7, 1, 3, 2 game by flipping the whole thing over a horizontal axis going through the middle of the square. But we could have done that before playing the game, so only the initial arrangement of numbers was important.

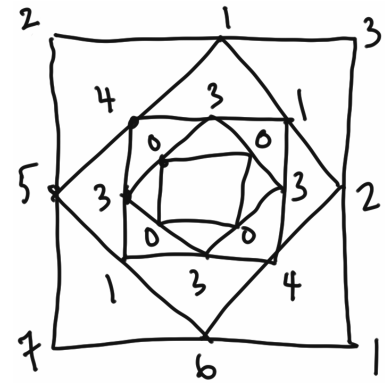


Figure 1: 2, 3, 1, 7

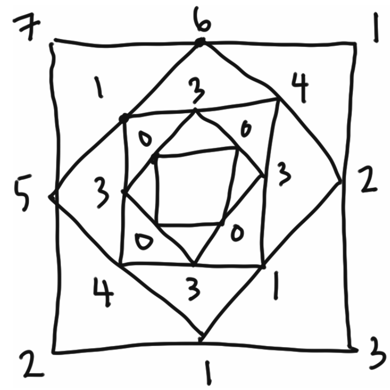


Figure 2: 7, 1, 3, 2

A Complete Description

We are now ready to fully describe how the initial order of the numbers does (or doesn't) matter. That is, we can answer the question: "**How many nonequivalent games can we play given four distinct starting numbers?**"

Let's assume our original numbers were 5, 6, 7, 8 and our original game board started with 5 in the upper left corner and labeled the corners clockwise in increasing order. Here is one possible argument.

Step 1: We can start by computing how many different ways there are to write out the numbers on the corners of the squares. Following a familiar argument for counting permutations, we see that we have four choices for the upper left corner, then three choices remaining for the upper right corner, then two choices remaining for the lower right corner, then only one number for the lower left corner. This gives us $4 \times 3 \times 2 \times 1 = 24$ ways of labeling the corners.

Step 2: Next we can see that two games will be equivalent if we can obtain one from the other by rotating or flipping the game board. There are four rotations of our game board (the original, rotated $90°$ clockwise, rotated $180°$ clockwise, and rotated $270°$ clockwise). In all of those cases the numbers 6, 7, 8 will follow sequentially clockwise from 5. If we flip our game board over and "look" at the back of it the numbers 6, 7, 8, will follow sequentially *counter-clockwise* from 5. Although the numbers are still in the same order, they are ordered in the opposite direction. There are also 4 ways the numbers can be ordered in this counter-clockwise way, so that gives us 8 total equivalent boards.

Convince yourself there are no more than this!

Step 3: If we start with any other labeling using the numbers 5, 6, 7, 8, there are always 8 equivalent boards to it. This means that of our 24 possible labelings, there are at most 3 labelings that are different because there are three sets of 8 that are the same.