Problem 1

Imagine that you are being held captive in a dungeon by an evil mathematician with a number of other prisoners, and suppose that every prisoner is given a red or green hat (chosen at random). Your captor will allow you and all the other prisoners to go free if *every* prisoner successfully guesses his or her own hat color, but if even one person guesses incorrectly then the lot of you will be fed to a hungry dragon. You can see everyone's hat but your own, everyone must guess at the same time, and you are allowed no communication of any kind once you are wearing your hats.

(a) Suppose first that every prisoner guesses his or her hat color randomly. If there are 2 prisoners, what is the probability that you will be freed? If there are 3 prisoners, what is the probability that you will be freed? In general, if there are n prisoners, what is the probability that everyone guesses correctly and is set free?

(b) Now suppose that you are allowed to collaborate with your fellow prisoners and formulate a strategy ahead of time to determine how you will guess your hat color. Is there a group strategy you can agree upon that will improve your chances of being set free? As in part (a), begin with the case where there are just 2 prisoners, then consider the case of 3 prisoners, and finally see if you can generalize to the case of n prisoners. In each case, what is the best strategy you can find, and what probability does it give you for escaping?

(c) Based on your answers in (a) and (b), how many other prisoners would you prefer to have with you in your cell?

In thinking about these questions, feel free to test your strategies using the hats provided for you!

Problem 2

Even an evil mathematician would prefer to give you better than a coin flip's chance of surviving, so he resolves to make it easier for you to guess correctly. His first attempt at doing this is to modify the rules slightly in the following way. Instead of having everyone guess their hat color simultaneously, your captor will allow you to guess in sequence one at a time, in any order you choose. Once you have made your guess, you will immediately be freed or fed to the dragon, depending on whether your guess was correct. Everyone else hears your guess and sees your fate. But all the other rules are exactly as before; in particular, when you make your guess you are not allowed to communicate any additional information beyond your guess.

(a) Can you think of a group strategy you can agree upon ahead of time that will maximize your chances of being set free? As before, you may want to begin by considering small groups of two or three prisoners. Test your strategy by acting it out!

(b) Is it possible to devise a strategy which guarantees that everyone will be set free? If not, how many people out of a group of n can you guarantee will be saved?

*(c) Consider how (or whether) your answers to (a) and (b) change if the following wrinkle is added: the prisoners are arranged in a line, front to back, so that each prisoner can only see the hats of the prisoners in front of him or her and none behind. The prisoners still guess sequentially one at a time, and every prisoner can hear everyone else's guesses and whether or not they guessed correctly. Can you devise a good strategy in this case?

Problem 3

After thinking about it some more, our evil mathematician has hit upon a different and fairer way of making it easier for the prisoners to escape. This time, every prisoner must guess simultaneouly as in the original problem, and either all the prisoners will be freed or all will be executed, as in the original problem. In fact, the only difference from Problem 1 is this: the prisoners will be released if *at least one* prisoner guesses correctly.

(a) Will the "strategy" of guessing randomly work better for this scenario than it did in Problem 1? Can you figure out how likely it is for n prisoners employing this strategy to be set free?

(b) For this scenario, it is important to consider first the case where there are just 2 prisoners. For a pair of prisoners, what strategy will maximize the probability that you will be set free and what is that probability? Take as much time as you need to get a good handle on this puzzle, since it is one of the most interesting and important ones of today's session.

(c) In general, for a group of n prisoners, what strategy will maximize the probability that you will be set free and what is that probability?

(d) Why do you think the evil mathematician regards this scenario as a "fairer" test than the first two?

*Problem 4

Having realized that the conditions for Problem 3 made it way too easy for you to escape, the evil mathematician decides to tweak those conditions in the following way: for a group of n prisoners, he will use n different colors for the hats instead of just 2 colors. For instance, if there are 3 prisoners then he will give each a red, green, or blue hat chosen at random. All other conditions are the same as in Problem 3, so in order to be set free at least one prisoner must guess correctly.

*(a) For n prisoners, what is the probability that everyone will be set free if each prisoner guesses randomly?

(b) Notice that for 2 prisoners, this problem is exactly the same as Problem 2. But for 3 prisoners it is different. Is there a strategy for 3 prisoners that will maximize your probability of being set free? If so, what is it and what is this probability?

*(b) Now consider the general case. For n prisoners (and n different colors of hats), what strategy maximizes your probability of being set free, and what is this probability?

Problem 5

Our evil mathematician is still hard at work, devising hat puzzles to torment his prisoners. For each of the variations given in this problem, 3 prisoners are given hats (red or green) and must guess their own hat color without communicating. However, *the prisoners are allowed to guess their hat color at any time, and do not have to guess simultaneously*. The prisoners all go free if the first one to guess guesses correctly, and are all executed if the first one to guess guesses incorrectly. (If two or more prisoners guess at the same time, they must all be correct to be set free).

(a) The three prisoners are arranged in a line so that each can see the prisoners in front of him or her, and the captor tells the prisoners that their hats were chosen randomly from a set of 2 red hats and 2 green hats.

(b) The three prisoners are arranged in a circle so they can see every hat but their own, and the captor tells them that their hats were chosen randomly from a set of 3 red hats and 1 green hat.

(c) This time the three prisoners are again arranged in a line so that each can see only the prisoners in front of him or her, and the captor tells the prisoners that their hats are selected from 3 red hats and 2 green hats.

*Problem 6

The evil mathematician has devised one final puzzle for all those poor exhausted captives who have avoided becoming dragon fodder thus far. For this final problem, three prisoners are randomly given a red or green hat and must simultaneously guess their hat colors without communicating, just as in Problem 1. The difference is that this time each prisoner is allowed to say "pass" instead of guessing a color. For the prisoners to survive, every prisoner who does not pass must guess correctly, and at least one prisoner must guess correctly. What strategy will maximize the probability that the prisoners will go free, and what is this probability?

More Difficult Problems

(a) Generalize Problem 6 to n prisoners (still with 2 colors).

(b) There are infinitely many prisoners, each tatooed with an identifying natural number. Each prisoner can see all the hats but his or her own. The prisoners must guess simultaneously their own hat colors, and are all set free if the number of incorrect guesses is finite. Do the prisoners have a strategy that will guarantee freedom?