Math Teacher's Circle

What Color Is My Hat?

Problem 1: Imagine that you are being held captive in a dungeon by an evil mathematician with a number of other prisoners, and suppose that every prisoner is given a red or green hat (chosen at random). Your captor will allow you and all the other prisoners to go free if *every* prisoner successfully guesses his or her own hat color, but if even one person guesses incorrectly then the lot of you will be fed to a hungry dragon. You can see everyone's hat but your own, everyone must guess at the same time, and you are allowed no communication of any kind once you are wearing your hats.

(a) Suppose first that every prisoner guesses his or her hat color randomly. If there are 2 prisoners, what is the probability that you will be freed? If there are 3 prisoners, what is the probability that you will be freed? In general, if there are n prisoners, what is the probability that everyone guesses correctly and is set free?

(b) Now suppose that you are allowed to collaborate with your fellow prisoners and formulate a strategy ahead of time to determine how you will guess your hat color. Is there a group strategy you can agree upon that will improve your chances of being set free? As in part (a), begin with the case where there are just 2 prisoners, then consider the case of 3 prisoners, and finally see if you can generalize to the case of n prisoners. In each case, what is the best strategy you can find, and what probability does it give you for escaping?

(c) Based on your answers in (a) and (b), how many other prisoners would you prefer to have with you in your cell?

In thinking about these questions, feel free to test your strategies using the hats provided for you!

Solution: *All the prisoners should make their guess assuming that the number of red hats is even (or odd). This gives a 50% chance of everyone being correct, no matter how many prisoners there are. To see this, prove by induction that for any $n \geq 1$, exactly half of all the ways of distributing hats to n people will result in an even number of red hats. To see that no strategy can improve on this, notice that given any strategy (i.e., function from $n-1$ to 2), we can associate to any configuration for which this strategy succeeds one for which it fails by switching the hat color of the first person; this association is injective, so the given strategy can succeed at most half the time.*

Problem 2: Even an evil mathematician would prefer to give you better than a coin flip's chance of surviving, so he resolves to make it easier for you to guess correctly. His first attempt at doing this is to modify the rules slightly in the following way. Instead of having everyone guess their hat color simultaneously, your captor will allow you to guess in sequence one at a time, in any order you choose. Once you have made your guess, you will immediately be freed or fed to the dragon, depending on whether your guess was correct. Everyone else hears your guess and sees your fate. But all the other rules are exactly as before; in particular, when you make your guess you are not allowed to communicate any additional information beyond your guess.

(a) Can you think of a group strategy you can agree upon ahead of time that will maximize your chances of being set free? As before, you may want to begin by considering small groups of two or three prisoners. Test your strategy by acting it out!

(b) Is it possible to devise a strategy which guarantees that everyone will be set free? If not, how many people out of a group of n can you guarantee will be saved?

*(c) Consider how (or whether) your answers to (a) and (b) change if the following wrinkle is added: the prisoners are arranged in a line, front to back, so that each prisoner can only see the hats of the prisoners in front of him or her and none behind. The prisoners still guess sequentially one at a time, and every prisoner can hear everyone else's guesses and whether or not they guessed correctly. Can you devise a good strategy in this case?

Solution: *The first prisoner to guess should guess red if and only if he or she sees an even number of red hats. This first prisoner is then freed if and only if there were in fact an odd number of red hats to begin with. Every other prisoner can now successfully guess his or her own hat color using this information. So the strategy is guaranteed to free* $n-1$ *people and has a 50% chance of freeing everyone.*

For part (c), the prisoners should guess back to front. Everybody should start out guessing red if the number of red hats they see in front of them is even, and green otherwise. Then each prisoner should flip this guess every time someone behind him says "red." As before, the first prisoner to guess (i.e., the one in the back of the line) has a 50% chance of being correct, and everyone else is guaranteed to be correct.

Problem 3: After thinking about it some more, our evil mathematician has hit upon a different and fairer way of making it easier for the prisoners to escape. This time, every prisoner must guess simultaneouly as in the original problem, and either all the prisoners will be freed or all will be executed, as in the original problem. In fact, the only difference from Problem 1 is this: the prisoners will be released if *at least one* prisoner guesses correctly.

(a) Will the "strategy" of guessing randomly work better for this scenario than it did in Problem 1? Can you figure out how likely it is for n prisoners employing this strategy to be set free?

(b) For this scenario, it is important to consider first the case where there are just 2 prisoners. For a pair of prisoners, what strategy will maximize the probability that you will be set free and what is that probability? Take as much time as you need to get a good handle on this puzzle, since it is one of the most interesting and important ones of today's session.

(c) In general, for a group of n prisoners, what strategy will maximize the probability that you will be set free and what is that probability?

(d) Why do you think the evil mathematician regards this scenario as a "fairer" test than the first two?

Solution: *For 2 prisoners, one should chose his partner's hat color and the other should chose the opposite of his partner's hat color. This guarantees that at least one will be correct. For* n *prisoners, the first two can implement this strategy and everyone else can guess whatever they want!*

***Problem 4:** Having realized that the conditions for Problem 3 made it way too easy for you to escape, the evil mathematician decides to tweak those conditions in the following way: for a group of n prisoners, he will use n different colors for the hats instead of just 2 colors. For instance, if there are 3 prisoners then he will give each a red, green, or blue hat chosen at random. All other conditions are the same as in Problem 3, so in order to be set free at least one prisoner must guess correctly.

- *(a) For n prisoners, what is the probability that everyone will be set free if each prisoner guesses randomly?
- (b) Notice that for 2 prisoners, this problem is exactly the same as Problem 2. But for 3 prisoners it is different. Is there a strategy for 3 prisoners that will maximize your probability of being set free? If so, what is it and what is this probability?
- *(b) Now consider the general case. For n prisoners (and n different colors of hats), what strategy maximizes your probability of being set free, and what is this probability?

Solution: *Suppose there are n prisoners, and model colors as natural numbers less than n. Every prisoner should select a different congruence class mod n and guess according to the assumption that the sum of all the hats (mod n) belongs to his chosen congruence class; at least (in fact exactly) one prisoner is guaranteed to guess correctly.*

The probability that n prisoners guessing randomly will succeed is $1 - \left(\frac{n-1}{n}\right)^n$, *which has limit* $1 - e^{-1} \approx .632$ *as* $n \to \infty$.

Problem 5: Our evil mathematician is still hard at work, devising hat puzzles to torment his prisoners. For each of the variations given in this problem, 3 prisoners are given hats (red or green) and must guess their own hat color without communicating. However, *the prisoners are allowed to guess their hat color at any time, and do not have to guess simultaneously.* The prisoners all go free if the first one to guess guesses correctly, and are all executed if the first one to guess guesses incorrectly. (If two or more prisoners guess at the same time, they must all be correct to be set free).

(a) The three prisoners are arranged in a line so that each can see the prisoners in front of him or her, and the captor tells the prisoners that their hats were chosen randomly from a set of 2 red hats and 2 green hats.

(b) The three prisoners are arranged in a circle so they can see every hat but their own, and the captor tells them that their hats were chosen randomly from a set of 3 red hats and 1 green hat.

(c) This time the three prisoners are again arranged in a line so that each can see only the prisoners in front of him or her, and the captor tells the prisoners that their hats are selected from 3 red hats and 2 green hats.

Solutions: (a) *If the prisoner in back sees two hats of the same color, he can successfully guess the opposite color. The middle prisoner can deduce this, and therefore if the prisoner in back does not make a guess after some length of time, the middle prisoner can deduce that his hat color is the opposite of the one he sees in front of him. The prisoner in front should keep silent!*

(b) *In the trivial case where one prisoner wears green, the two who see this can confidently guess red. In the other case, no one speaks for awhile until eventually all three realize they are wearing red hats.*

(c) *If the prisoner in back sees two green hats, he can guess red. So if the front prisoner has a green hat and the back prisoner doesn't guess after awhile, the middle prisoner can guess red. If neither of the back two prisoners makes a guess, the front prisoner can therefore deduce that his hat is red.*

***Problem 6:** The evil mathematician has devised one final puzzle for all those poor exhausted captives who have avoided becoming dragon fodder thus far. For this final problem, three prisoners are randomly given a red or green hat and must simultaneously guess their hat colors without communicating, just as in Problem 1. The difference is that this time each prisoner is allowed to say "pass" instead of guessing a color. For the prisoners to survive, every prisoner who does not pass must guess correctly, and at least one prisoner must guess correctly. What strategy will maximize the probability that the prisoners will go free, and what is this probability?

Solution: *A prisoner who sees the same two colors should guess the opposite, and a prisoner who sees two different colors should pass. The strategy will succeed 75% of the time, which is optimal. To see that this is optimal, let s be an arbitrary strategy and consider how s responds to seeing all red hats. If s tells you to pass or guess green, it will fail when all hats are red; if s tells you to guess red, it will fail whenenver there is exactly one green hat. The same can be said for the case where one sees all green hats. Hence every strategy fails on at least two configurations.*