Wayne County Math Teachers' Circle Immersion

Problem Session #1: Lockers and more Lockers

11 August 2015

At a large high school, there are 10000 lockers all on one wall of a long corridor. The lockers are numbered, in order, 1, 2, 3, . . . , 10000, and to start, each locker is closed. There are also 10000 students, also numbered 1, 2, 3, . . . , 10000. The students walk the length of the corridor, opening and closing lockers according to the following rules.

Many of you may have seen this problem before. If so, try to think through the solution, rather than remember it. Also try to support your group members without giving them the answer. Finally, if you are already really comfortable with the solution: reflect on problem solving strategies helpful in solving the problem, which math practices are used in solving the problem, how you might facilitate this problem (or a simpler version) with your class, and what other mathematically interesting questions you might ask about the context.

1. Student 1 opens every locker.

2. Student 2 closes every second locker (starting with the 2nd locker).

3. Student 3 changes the state of every third locker (starting with the 3rd locker), closing it if it is open, and opening it if it is closed.

. . .

k. Student k changes the state of every kth locker (starting with the kth locker).

. . .

10000. Student 10000 changes the state of the 10000th locker.

When all 10000 students have walked the corridor, which lockers end up open?

What other questions might be interesting to ask about this problem context?

Problem 1. If the students go down in a different order, is the result changed?

Problem 2. What if student 3 is ill and had to skip her turn? What if she took a second turn when the teacher was not looking? What if students 3 and 9 are ill? 3 and 10?

Problem 3. Suppose that we can send any students we like down the corridor. If, when we are done, we want only locker 1 open and all others closed, then which students should go? What if we want only locker 3 open? What if we want only locker 9 open? Both lockers 3 and 9 and no other lockers?

Problem 4. Suppose that we want only the lockers with prime numbers open. Which students should be sent down the corridor?

Problem 5. Let L be any subset of $\{1, 2, ..., 10000\}$, the set of the first 10000 positive integers. Is there a set of students that you can send down the corridor so that when all of these students have gone, the set of open lockers is exactly those with a number in L?

Problem 6. Let S_1 and S_2 be two different groups of students. Each is sent down a row of lockers. Is it possible that the students from the two groups leave exactly the same lockers open?

Problem 7. A number is called square-free if it is not divisible by the square of any prime number (so the number 1 is square free.)

(a) List the first 15 square-free numbers.

(b) If we send all of the students with square free numbers down the corridor, which lockers will be open when the activity is done?

Problem 8. Suppose that we send down the corridor exactly those students with perfect square numbers (e.g., 1, 4, 9, 16, ...) Which lockers are left open when this activity is concluded? What if we send only the students with numbers that are perfect cubes?