Understanding the concept

Motivation

In general, the Perfect Bayesian Equilibrium (PBE) is the concept we are using when solving dynamic games with incomplete information (such as signaling game and reputation game). Usually, there will be two counterparts in the game, one in informed and the other not (informed workers and uninformed firms, informed normal incumbent and uninformed entrant). The uninformed player tries to infer the informed player’s private information from his behavior to make a decision. The informed player takes this inference into account to make her choice. To make an inference properly, the uninformed player must take the above effect into account. Unlike our usual Nash equilibrium definition, we cannot say anything about players’ best responses without the knowledge of the uninformed players’ inference rule. It’s obvious that for different inference rule, the optimal decision of the players can be different. In this situation, one natural way to predict the result of the game is to build a concept that explicitly incorporate the inference rule (or belief system, in our usual terminology) and the best responses are defined according to this specific rule. Another thing we need to keep in mind is that, out inference rule must follow appropriate rules.

Definition

Following the motivation part, we are now ready to specify the concrete definition of PBE. The major departure from Nash Equilibrium here is that we explicitly incorporate
belief system (inference rule for the uninformed player) into the equilibrium concept. I will briefly summarize the key points in the definition of PBE below. To make the interpretation simple, I didn’t follow the strict definition here. You should check the formal definition given in the lecture and try to come up with your own intuitive interpretation.

To be specific, a **PBE has two parts**, one is a strategy profile $s^*$ (as usual) and the other is a belief system $\mu^*$ (inference rule) (such as the firms’ common probability assessment that the work is of high ability if she chooses effort level $e$, or the entrant’s belief about the incumbent’s probability of being normal if he chooses to fight). Like usual, we need to put restrictions on the strategy profile and belief system to define the equilibrium.

i **Consistency:** the belief system should be consistent (Bayes Rule) with the strategy profile.

ii **Best responding:** strategies are best responses (dynamically) given the belief system.

To understand the definition, it’s important to keep in mind that, in any PBE you find, the strategy profile and belief system must come together. We’ve already explained this in the motivation part. The second thing is understand the two restrictions given in the definition. The best responding (or sequential rationality) condition is usual, justifiable by our usual Nash thinking. The consistency requirement actually has two part: the technical part is also natural, simply saying that, when the uninformed players make inference from the informed players’ strategy, they base the inference on statistical intuition, i.e. they update their belief by Bayes Rule (if you have trouble in understanding this, see lecture note for concrete examples); the other part is more conceptual, that is why we require the actions should match with the belief. To answer this question, it’s important to remember yourself that PBE is a definition we constructed. Like lots of other economic definitions, one has to make her own judge (by certain rational) about certain aspects of the concept. If one thinks further, the type of consistency in PBE is pretty natural, at least from an economist’s point of view, since that’s the minimum condition to keep the whole system look “stable”. Whether this matches with the reality is not what we can really handle in our course. Actually, there are other related concepts developed by putting different restriction on the belief system (such as sequential equilibrium), you are welcome to talk with me if interested in further material.
Solving for PBE

I will briefly talk about the solution concept to PBE in this section, and accompanying with two examples from homework (the drinking beer signaling game and the simple bargaining game with a fair type player).

The key here is to understand the definition and follow the definition. That way you won’t be confused about all those seemingly different questions.

What we are looking for?

Whenever facing a PBE question, the fist thing to keep in mind is that you are looking for a pair of things (from definition): a strategy profile and a consistent belief system. By pair we mean the belief system and strategy profile are mostly one-to-one on the equilibrium path\(^1\). This is because of consistency, once you know the equilibrium strategy profile, you know exactly what is the belief on equilibrium path, and vice versa.

Moreover, we are looking for a particular pair which also satisfies the best response condition (given the belief). This will help us pin down or the possible PBEs.

Steps

There is no universal steps in solving a particular question, but I came up with a possible one for those who like to have some procedures to follow. You can and should summarize your own solving strategy, just keep in mind the definition.

1. Understand the incomplete information situation (informed and uninformed players) and how the belief system work (how the uninformed player makes inference and what’s the optimal strategy given the belief).

2. Find out possible pairs (strategy profile and consistent belief system).

3. For all those possible pairs, check the best response conditions for all players, and given the above consistent belief system (on equilibrium path), survivors are potential PBEs.

\(^1\)We know from the definition of PBE that the consistency doesn’t take care of the off-equilibrium path, which means we have freedom to choose the off-equilibrium belief to support the equilibrium (think about the signaling game we talked in the lecture and session).
4. Depend on the situation, sometimes you need to specify the off-equilibrium belief to make sure the no profitable deviation condition also holds on off-equilibrium path (this can be combined with step 3).

In practice (homework or exam questions), the professor usually will walk you through those steps actually.

**Beer drinking signaling game**

1. Situation: player 2 has to infer player 1’s type from the number of beers he drinks. The belief of strong type will be a function of bottle of beers drunk (denote $\mu(e_i)$, $i \in \{W, S\}$ as the belief about the probability of player 1 being strong). And the optimal strategy depends on the belief in such a way that she will choose to Fight when $\mu < 1/2$, to Escape when $\mu > 1/2$ (indifferent when $\mu = 1/2$, but we usually ignore this situation, since its degenerate and won’t add intuition but just complicate the analysis).

2. Two types of possible pairs, one is separating, $(e_W \neq e_S, \mu(e_S) = 1, \mu(e_W) = 0)$, the other is pooling $(e_W = e_S = e^*, \mu(e^*) = \Pi)$.

3. (I only consider the case where $\Pi < 1/2$ in this part) For separating case, best responses for weak players requires $e_W = 0$ and

$$0 \geq 1 - \frac{2}{3}e_S$$

And for strong players requires

$$1 - \frac{1}{4}e_S \geq 0$$

This gives us $e_S \in \{2, 3\}$.

For pooling case, best responses both players requires

$$0 - \theta e^* \geq 0$$

This gives us $e^* = 0$.

4. Moreover, to make the no profitable condition also hold on the off-equilibrium path, for separating case, we need to require $\mu(e) < 1/2$, for all $e < e_S$; for the pooling case, we need to require $\mu(e) < 1/2$ for all $e > 0$. 

4
Bargaining game with a fair type player (reputation)

1. Situation: player 2 has to infer player 1’s type from the offer he made. The belief of being Fair type will depend on the belief about Normal player’s strategy. Moreover, given a belief of Normal player’s strategy $q$, the belief should be

$$p = \frac{z}{z + (1 - z)q}$$

And the optimal strategy depends on the belief in such a way that

- Accept if $p > 1/3$
- Reject if $p < 1/3$
- Indifferent if $p = 1/3$

2. Two types of possible pairs, one is pure strategies, ($q = 0$, player 1 never offers (0.5,0.5), player 2 accepts an offer of (0.5,0.5)) or ($q = 1$, player 1 always offers (0.5,0.5), player 2 rejects an offer of (0.5,0.5) if $z < 1/3$, accepts an offer of (0.5,0.5) if $z > 1/3$).

The other type of possible pair is ($q \in (0,1)$, player 1 randomize).

3. The first two pairs obviously won’t survive no profitable deviation condition check if $z < 1/3$. If $z > 1/3$, ($q = 1$, player 1 always offers (0.5,0.5), player 2 accepts an offer of (0.5,0.5)) is a PBE.

For the third pair to satisfy the condition, we first note that player 2 must play a mixing strategy with probability of 1/2 to accept a (0.5,0.5) offer (to make sure player 1 randomize is optimal). And moreover, to make player 2’s randomize optimal, we require $p = 1/3$ which means $q^*$ should solve the equation below

$$\frac{1}{3} = \frac{z}{z + (1 - z)q^*}$$

This gives us one PBE: (belief is $q^*$, player 1 plays $q^*$, player 2 plays a mixing strategy with probability of 1/2 to accept).