We study optimal procurement mechanisms for a financially constrained buyer subject to ex post incentive compatibility and individual rationality. In our setup, a buyer who has a fixed budget wishes to purchase a homogeneous product up to a maximum demand amount. We characterize the optimal mechanism when the budget constraint always binds and when the maximum demand constraint always binds. When either constraint can bind, we characterize the optimal mechanism for the case of two suppliers under regularity assumptions on the cost distribution. We also show that these optimal mechanisms have straightforward dynamic implementations.

Introduction

Although the theory of optimal procurement is well understood when the buyer is not financially constrained, challenges remain when dealing with a budget-constrained buyer. We show that in a symmetric environment, the optimal deterministic, ex post incentive compatible, ex post individual rationality mechanism for a budget-constrained buyer has a simple form that can easily be implemented through a reverse auction. The restrictions we place on the mechanism allow us to focus on mechanisms that are relatively straightforward to communicate and implement and for bidders to understand. Roughly speaking, bidding starts at the level of the budget and continues downward until all remaining active bidders can be accommodated within the budget. The mechanism we identify has the appealing features that it is robust in the sense that bidders have dominant strategies (the strategic problem of each participant is just a sequence of in-out decisions and conjectures about the behavior of others do not matter) and it is simple in the sense of having equilibrium with truthful bidding, only paying winners, and not involving random awards. As we show, allowing random allocations can increase buyer surplus, but our numerical calculations suggest not much is sacrificed by restricting attention to deterministic allocations.


Setup

A buyer faces a set $S = \{1, \ldots, S\}$ of suppliers that can produce any amount $q_s$ of a homogeneous good at a constant and privately known unit cost $c_s$ drawn from twice continuously differentiable cdf $F$ with density $f$ that is strictly positive on the support $[c_L, c_H]$, where $0 < c_L < c_H$. The
buyer can commit to any feasible trading mechanism and aims to maximize its expected surplus. By the revelation principle, any equilibrium of any feasible trading mechanism can be implemented by a direct revelation mechanism, which in this case is a set of functions \( q_s : [c_L, c_H]^S \to \mathbb{R} \) and \( m_s : [c_L, c_H]^S \to \mathbb{R} \) for \( s \in \mathcal{S} \) that satisfy individual rationality and incentive compatibility.

We restrict attention to mechanisms that have equilibria in dominant strategies. Thus, we impose both incentive compatibility and individual rationality ex post.

In addition, a feasible mechanism \((q, m)\) must satisfy: \( \forall (c_s, c_{-s}) \in [c_L, c_H]^S \), the budget constraint,

\[
\sum_{s \in \mathcal{S}} m(c_s, c_{-s}) \leq B, \quad (1)
\]

the buyer’s demand constraint,

\[
\sum_{s \in \mathcal{S}} q(c_s, c_{-s}) \leq D, \quad (2)
\]

and the requirement of nonnegative quantities for all \( s \in \mathcal{S} \),

\[
q(c_s, c_{-s}) \geq 0. \quad (3)
\]

The buyer’s problem is to select a mechanism \((q, m)\) that maximizes its expected surplus

\[
\int_{[c_L, c_H]^S} \left( \sum_{s \in \mathcal{S}} [v \cdot q(c_s, c_{-s}) - m(c_s, c_{-s})] \right) \prod_{s \in \mathcal{S}} f(c_s) dc_s \quad (4)
\]

subject to ex post incentive compatibility, ex post individual rationality, the budget constraint, the buyer’s demand constraint, and the requirement of nonnegative quantities.

**Key results**

The format of the optimal mechanism depends on the position of the “normalized budget” \( b \equiv B/D \) relative to the support of cost distribution \([c_L, c_H]\). There are three cases, corresponding to large budgets, small budgets, and intermediate budgets. The case of a large budget corresponds to the usual case.

**Proposition 0** If \( c_H \leq b \) and the virtual cost function is nonincreasing and nonnegative, then the optimal mechanism is the second-price auction:

\[
q^{SPA}(c_s, c_{-s}) \equiv D \cdot 1_{[c_s < c_{(2)}]} \quad \text{and} \quad m^{SPA}(c_s, c_{-s}) \equiv c_{(2)} \cdot D \cdot 1_{[c_s < c_{(2)}]}.
\]

With a small budget, the SPA mechanism is no longer feasible. However, the buyer’s problem can again be solved using familiar mechanism design techniques. Specifically, we can use the envelope theorem to eliminate \( q \) (instead of \( m \), as it is normally done) from the problem.

**Proposition 1** If \( b \leq c_L \) and the function \( \psi(c) \equiv v \left( \frac{1}{c} - \frac{1}{c(2)} \cdot \frac{F(c)}{f(c)} \right) - 1 \) is nonincreasing and nonnegative, then the optimal mechanism is the second-unit-price auction:

\[
q^{SUPA}(c_s, c_{-s}) \equiv B \cdot 1_{[c_s < c_{(2)}]} \quad \text{and} \quad m^{SUPA}(c_s, c_{-s}) \equiv B \cdot 1_{[c_s < c_{(2)}]}.
\]

The case of an intermediate budget is more challenging. In light of the previous two propositions, it is reasonable to conjecture that in this case the buyer’s problem is solved, at least for some cost
distributions, by a hybrid mechanism that coincides with the SUPA when \( b < c(2) \) and with the SPA otherwise. The next result negates this conjecture.

**Theorem 1** If \( c_L < b < c_H \), there is no twice continuously differentiable cost distribution for which the hybrid mechanism is optimal for the buyer.

For the case with two suppliers we show that, under regularity conditions on the cost distribution, the buyer’s problem is solved by a “semi-split award” (SSA) mechanism. Any such mechanism is characterized by two thresholds, \( c_0 \in (b, c_H] \) and \( c_\tau \equiv \frac{c_L b}{2c_0 - b} \), and the quantity and payment for a supplier with cost \( x \), when the opponent has cost \( y \), are given by:

\[
q_{SSA}(x, y; c_0) = \begin{cases} 
\frac{B}{y}, & \text{if } \max\{x, c_0\} < y \\
D, & \text{if } x < c_\tau < y < c_0 \\
\frac{B}{2} \cdot \frac{1}{c_0}, & \text{if } c_\tau \leq x, y \leq c_0 \\
D, & \text{if } x < y < c_\tau \\
0, & \text{otherwise}
\end{cases}
\]

\[
m_{SSA}(x, y; c_0) = \begin{cases} 
B, & \text{if } \max\{x, c_0\} < y \\
\frac{B}{2}, & \text{if } c_\tau \leq x, y \leq c_0 \\
\frac{B}{2}, & \text{if } c_\tau \leq x, y \leq c_0 \\
0, & \text{otherwise}
\end{cases}
\]

**Theorem 2** Assuming two suppliers, there exists a class of cost distributions for which the SSA mechanism is optimal for the buyer.

The class of cost distributions referred to in Theorem 2 includes the uniform distribution and the power distribution with density \( f(x) \propto x^{-2} \).

**Dynamic implementation**

Clearly the second-price auction and second-unit-price auction have dynamic implementations. It is perhaps less obvious that the SSA mechanism has a straightforward dynamic implementation. For the case of two suppliers, a descending clock price can be used, with bidders choosing when to exit. If the clock stops (i.e., one bidder exits) at a price above \( c_0 \), then the remaining bidder supplies quantity \( B \) divided by the final clock price and is paid \( B \). If a bidder exits at a clock price below \( c_0 \) but above \( c_\tau \), then the clock continues until either the other bidder exits or the clock price reaches \( c_\tau \). If the other bidder exits prior to the clock price reaching \( c_\tau \), then each bidder supplies quantity \( \frac{B}{(2c_0)} \) and is paid \( B/2 \), but if the clock price reaches \( c_\tau \), then the remaining active bidder supplies quantity \( D \) and is paid \( B \). Finally, if no bidder exits until the clock price is less than or equal to \( c_\tau \), then the final active bidder supplies quantity \( D \) and is paid \( D \) multiplied by the final clock price.
References


