1 Introduction

Buyers often face budget constraints that restrict their ability to pay for the goods that they want to purchase. Yet, with some notable exceptions including Laffont and Robert (1996), Pai and Vohra (2014), Malakhov and Vohra (2008) and Che and Gale (1998) the literature on mechanism design and auctions had for most part focussed on the situations where budget constraints are absent.

In this paper we characterize the optimal mechanism in an environment where the buyers have private values and commonly known and asymmetric/unequal budgets. With unequal budgets, our problem is that of asymmetric optimal mechanism design, which is significantly more complex than a mechanism in which all participants are ex-ante symmetric. In a symmetric situation a mechanism designer has to construct a single allocation profile which is offered to every participant. Yet, in the asymmetric environment, such as the one we study, the designer has to design ex-ante asymmetric allocation profiles, one for each buyer, and do so in a consistent way.

The optimal mechanism belongs to one of two classes. When the budget differences are small, the optimal mechanism -called “top-auction”- discriminates only between high-valuation buyers for whom the budget constraint is binding. It is characterized by a common threshold valuation $\bar{x}_t$ at which the budget constraint of each bidder becomes binding, so any bidder with valuation exceeding $\bar{x}_t$ pays a transfer equal to her budget, and the probability of getting the good is the same for all types of a particular bidder in $[\bar{x}_t, 1]$. All low valuations buyers are treated symmetrically despite budget differences.

When the budgets are sufficiently different, the “top auction” is no longer feasible. The optimal mechanism for this case - “budget-handicap auction”- discriminates in favor of buyers with small budgets when the valuations are low, and in favor of buyers with larger budgets when the valuations are high. The threshold valuations at which the budget constraints are binding differ across buyers, with richer buyers having higher thresholds. Importantly, in the budget-handicap auction the seller also discriminates between buyers with low values. Below their respective thresholds, a buyer with a lower budget gets the good with a higher probability than the buyer with a higher budget. The buyers with lower budgets also have lower reserve prices. Thus, the seller handicaps the buyers with high budgets at low values, which introduces an additional inefficiency compared to the top auction. However, this “handicapping” of the high-budget bidders creates more competition for them from low-budget bidders. This allows the seller to extract more surplus from the former.
2 Model

A single seller wants to sell one unit of the good to \( n \) bidders. Bidder \( i \) has privately known value \( x_i \) for the good drawn from the common knowledge distribution \( F(.) \) with support \([0, 1]\) and with density function \( f(.) \). Bidder \( i \) has budget \( m_i \), so that his payment in the mechanism can never exceed \( m_i \). The budgets are commonly known and will be assumed to be sufficiently small.

Assumption 1 Increasing Hazard rate:

\[
\frac{f(x)}{1 - F(x)} \text{ is increasing in } x \text{ for all } x \in [0, 1]
\]  

Bidder \( i \) with valuation \( x_i \) gets a payoff equal to \( x_i q_i - t_i \) if she gets the good with probability \( q_i \) and makes a payment \( t_i \) to the seller. The seller has zero value for the good, so her payoff is the sum of the payments that she receives from the buyers, \( \sum_{i=1}^{n} t_i \). All the bidders and the seller are risk-neutral. The seller has all bargaining power and acts as a mechanism designer to maximize her expected payoff from the mechanism. By the Revelation principle (\( \square \)) we restrict attention to direct truthful mechanisms which specify the probabilities of trading and the payments as the functions of the buyers’ announced valuations.

Next, let us establish the following simple but useful result defining valuation thresholds:

Lemma 1 Let \( \bar{x}_i \in [0, 1] \) be defined as follows:

\[
\bar{x}_i = \sup\{x_i \in [0, 1] | t_i(x_i) < m_i\}
\]

If \( \bar{x}_i < 1 \), then \( t_i(x_i) = m_i \) for all \( x_i \in [\bar{x}_i, 1] \)

3 Main Results: Top and Budget-Handicap Auctions

First, let us define the top auction. Let \( \bar{x}^t \) be the unique solution to the following equation:

\[
\sum_{i=1, \ldots, n} m_i = \bar{x}^t \frac{1 - F(\bar{x}^t)^n}{1 - F(\bar{x}^t)} - n \int_{r_t: r_t = 1 - F(\bar{x}^t) - \frac{(1 - F(\bar{x}^t))^2}{1 - F(\bar{x}^t) + 2 f(\bar{x}^t)}} F(s)^{n-1} ds
\]

Definition 1 A “top auction” for \( n \) bidders with budgets \( m_1, \ldots, m_n \), with \( m_i \geq m_{i+1} \) for all \( i = 1, \ldots, n-1 \), is a mechanism with a common threshold \( \bar{x}^t = \bar{x}_1 = \ldots = \bar{x}_n \) uniquely solving \( \square \), reservation values \( r_1 = \ldots = r_n = r_t \) defined by \( r_t = \frac{1 - F(\bar{x}^t) - \frac{(1 - F(\bar{x}^t))^2}{1 - F(\bar{x}^t) + 2 f(\bar{x}^t)}}{f(\bar{x}^t)} \), and trading probabilities \( q_i(x_i) = F(x_i)^{n-1} \) for all \( x_i \in [r, \bar{x}^t] \) and \( q_i(\bar{x}^t) \) satisfying:

\[
m_i = \bar{x}^t q_i(\bar{x}^t) - \int_{r_t}^{\bar{x}^t} F(s)^{n-1} ds
\]

\[
\sum_{i=1, \ldots, n} q_i(\bar{x}^t) = \frac{1 - F(\bar{x}^t)^n}{1 - F(\bar{x}^t)}
\]
Our first result shows that the “top auction” is optimal whenever it is feasible. The feasibility condition (6) is given below.

**Theorem 1** Suppose that for a profile of bidders with budgets $m_1, ..., m_n$, with $m_i \geq m_{i+1}$ for all $i = 1, ..., n-1$, the threshold $\bar{x}^t$ uniquely solving (3) is such that $\bar{x}^t < 1$.

The unique optimal mechanism is a “top auction” with a common threshold $\bar{x}^t$ if and only if for every $k = 1, 2, ..., n-1$ we have:

$$\frac{m_1 + \ldots + m_k}{k} - \frac{m_{k+1} + \ldots + m_n}{n-k} \leq \bar{x}^t \left( \frac{1 - F(\bar{x}^t)^k}{k(1 - F(\bar{x}^t))} - F(\bar{x}^t)^k \frac{1 - F(\bar{x}^t)^{n-k}}{(n-k)(1 - F(\bar{x}^t))} \right).$$ (6)

The distinguishing feature of the top auction is that it allocates the good efficiently when the buyers’ valuations lie in $[r, \bar{x}]$. The only additional inefficiency compared to the standard optimal auction happens at the “top”: when several buyers have valuations above $\bar{x}$, the good is allocated randomly among them, with probabilities increasing in their budgets.

However, when the feasibility condition for the top auction (6) fails, the seller has to use additional tools to discriminate between the bidders and, in particular, set different thresholds for them. Naturally, lower-budget bidders have lower thresholds, although there may still exist some clusters of bidders sharing the same threshold. The richer bidders with valuations above their higher thresholds have higher probabilities of trading and pay higher transfers than poorer bidders with valuations above their lower thresholds. Specifically, we have:

**Theorem 2** Suppose that (6) fails for some $k$. Then the optimal auction is a “budget handicap auction” which is uniquely defined by a vector of threshold values $(\bar{x}_1, ..., \bar{x}_n)$ s.t. $\bar{x}_i \geq \bar{x}_{i+1}$ for all $i \in \{1, ..., n-1\}$, with strict inequality for at least some $i$.

Significantly, there is another type of price discrimination in the “budget handicap auction”: a poorer bidder with a low value has a higher probability of trading and pays a higher transfer than a richer bidder with the same value. The details are available in the original manuscript (Bulatov and Severinov 2016). This additional type of price discrimination increases inefficiency, but is unavoidable when budget differences are sufficiently large. The most challenging part in computing the optimal “budget handicap” auction is to determine which groups of bidders constitute clusters with common thresholds.

As a final result, we establish that the seller prefers an equal allocation of budgets across bidders.
Lemma 2 Suppose that the aggregate budget of all bidders is fixed i.e. \( \sum_i m_i = M \), where \( M \) is sufficiently small. (In particular, \( M \leq np^m \) where \( p^m \) is a monopoly price i.e., \( p^m = \arg \max_p F(p)(1-p) \).

Then the seller gets a maximal payoff in the optimal mechanism when all bidders’ budgets are equal i.e., \( m_i = \frac{M}{n} \) for all \( i = 1, \ldots, n \).

References


