1. Introduction

- The 2-torus \( T^2 \), familiarly known as a donut, is the 2-dimensional manifold formed by gluing opposite sides of a parallelogram together. Mathematically, we define it by \( T^2 = \mathbb{R}^2 / \mathbb{Z}^2 \).

- One can measure the distance between two points \( p, q \) on \( T^2 \) by first "unrolling" the torus, then measuring the Euclidean distance between \( p \) and \( q \) in the resulting parallelogram. This method of measuring distance gives \( T^2 \) a flat metric, essentially meaning that locally the geometry of the torus looks like the geometry of the familiar euclidean plane.

- Say two tori are "equivalent" if one can be smoothly transformed into the other while preserving the distances between points. How many different unmarked lattices, hence nonisometric tori, are there?

- In this project, we studied the connections between two methods of parametrizing all lattice bases \( \{ \text{are spanned by} \} \) lattices determined by \( \tau \), and lattices in \( \mathbb{R}^2 \) that are identical up to rotation and uniform scaling.

2. The Upper Half Plane and Euclidean Lattices

The upper half plane \( \mathbb{H}^2 \) is defined to be \( \{ z \in \mathbb{C} : \text{Re}(z) > 0 \} \). The group \( \text{SL}_2(\mathbb{R}) = \{ M \in \text{GL}_2(\mathbb{R}) : \det(M) = 1 \} \) acts on \( \mathbb{H}^2 \) via fractional linear transformations

\[
\begin{pmatrix} u & \overline{v} \\ v & \overline{u} \end{pmatrix} : z \mapsto \frac{az + b}{cz + d},
\]

This action is transitive: for any point \( \tau = x + iy \in \mathbb{H}^2 \), the transformation

\[
M_\tau = \begin{pmatrix} \sqrt{y} & -1 \\ 0 & \sqrt{y} \end{pmatrix}
\]

maps the point \( \tau \) to the point \( \iota \). The columns of \( M_\tau \) determine a lattice in \( \mathbb{R}^2 \). As \( \det M_\tau = 1 \), the fundamental parallelogram \( \mathbb{P} \) of the resulting lattice will have area 1. Thus, we identify \( \tau \) with the torus that results from gluing the opposite sides of \( \mathbb{P} \).

Example 1: \( i \) versus \( 1 + i \). The lattices determined by \( i \) and \( 1 + i \) are spanned by \( \{(1,0)^t, (0,1)^t\} \) and \( \{(1,0)^t, (1,1)^t\} \), respectively. While the lattice bases are different, the lattice points they determine are identical. We conclude that \( i \) and \( i + 1 \) determine the same torus up to isometry.

To handle this double-counting, we identify points \( \tau, p \in \mathbb{H}^2 \) if they determine the same lattice up to choice of basis. The action of \( \text{SL}_2(\mathbb{Z}) \) sends lattice bases to lattice bases of the same lattice. So, we map a point \( \tau \) by the action of \( \text{SL}_2(\mathbb{Z}) \) to find a sole representative for each unmarked lattice, hence nonisometric torus.

3. Results

How our program works:

- The user clicks a point \( p \) in the upper half-plane. Using the \( \text{SL}_2(\mathbb{Z}) \) action, a shadow point \( q \) is generated in the fundamental domain \( \mathcal{F} = \{ z \in \mathbb{C} : -1/2 \leq \text{Re}(z) \leq 1/2, |z| \geq 1 \} \).

Generation of the shadow point

It is well-known that \( \text{SL}_2(\mathbb{Z}) = \langle S, T \rangle \) where \( S \) and \( T \) are the fractional linear transformations

\[
S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\]

These transformations can be combined to build all elements of \( \text{SL}_2(\mathbb{Z}) \). We use an algorithm similar to the Euclidean algorithm for division to reverse-engineer the decomposition of the \( A \in \text{SL}_2(\mathbb{Z}) \) such that \( AP = q \).

Theorem. The following procedure, given \( p \in \mathbb{H}^2 \), will return \( A \in \text{SL}_2(\mathbb{Z}) \) such that \( Ap = q \):

1. Apply \( T \) to \( p \) until \( -1/2 \leq W(p) \leq 1/2 \). Update the value of \( p \).
2. Apply \( S \) to invert \( p \) about the unit circle. Update the value of \( p \). Note that \( 3p \) strictly increases under this inversion.
3. Repeat (1) and (2) until the \( |z| \) moves \( p \) above the unit circle.
4. Apply \( T \) until \( p \) sits in \( \mathcal{F} \).

As there are only finitely many regions in the tiling with real part between \(-1/2 \) and \( 1/2 \) and imaginary part bounded below by some \( k > 0 \), this procedure will terminate.

4. Visualizing Other Spaces of Geometric Structures

The ideas used in this project can be used to parametrize non-Euclidean metrics on a variety of genus \( g \) surfaces. For example, how many hyperbolic metrics exist on a sphere with three disks removed, commonly referred to as a pair of pants? We are not restricted to hyperbolic geometry: we can consider convex projective structures as well. In this setting, cut a pair of pants into two triangles and unfold it onto a convex subset of the projective plane in a way analogous to unrolling a torus. We can then define a metric on this convex subset of the projective plane to define a metric on the pair of pants. William Goldman has come up with a system of 8 coordinates which completely parametrizes the space of all convex projective structures on a pair of pants. In order to better understand these coordinates, we created a visualization tool which generates tilings of a convex subset of the projective plane given a choice of these coordinates. These tilings allow us to visualize the metric induced on the pair of pants (see Figure 4). Because this space has 8 parameters, convex projective structures can vary in complicated ways (see Figure 5), offering many opportunities for further investigation.

Visualizing Structures on the Torus and Pair of Pants.

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Although \( p \) and the shadow point \( q \) are in the same \( \text{SL}_2(\mathbb{Z}) \)-orbit, their associated lattices are only equivalent up to rotation. Why? The action of changing bases via \( \text{SL}_2(\mathbb{Z}) \) induces a rotation on the tangent space to the point \( p \). As we think of two lattices to be equivalent if they differ by rotation, we apply a rotation \( R \) which fixes \( A \) so that \( \det(A) = \pm 1 \). See Figure 2.

Explicitly, we have the equation

\[
M_\tau(p) = (R \circ M_q \circ A)(p),
\]

and we wish to solve for \( R \) such that we may correct the incidental rotation. An application of the chain rule yields

\[
R(p) = \frac{1}{|z|^2} \text{ad}(A(p)),
\]

so we can make the proper adjustment to the lattice associated with \( M_q \).

Figure 3: Rotation is introduced by the \( L \) transformation, which we cancel by post-composing with \( R \) so that the diagram "commutes."

Figure 4: A convex subset of the projective plane can be tiled by triangles parameterized by Goldman’s coordinates.

Figure 5: Choosing different values for the parameters can result in significant distortions of the tiling.

References