Discrete homotopy theory and cubical sets

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Outline

1. Origins
2. Discrete homotopy theory
3. Two applications
4. A cubical set
Original motivation

- Represent socio-technical complex systems as simplicial complexes $K$, possibly with dynamical information attached.
- Identify “$q$-clusters” and “$q$-holes,” i.e. well-connected regions and connectivity gaps in dimension $q$.
- $q$-holes can represent structural deficiencies in the system.
- Method: assign an object to $K$ (for us, a group) measuring combinatorial connectedness in each dimension.
Connectivity graphs

- $K$ a simplicial complex
- Let $\Gamma_q(K)$ denote the $q$-connectivity graph of $K$
  - Vertices: maximal simplices $\sigma \in K$ of dimension $\geq q$
  - Edge between $\sigma$ and $\tau$ if they share a $q$-face

- $q$-holes are chordless cycles of length $\geq 5$ in $\Gamma_q(K)$
- Can detect these combinatorially using homotopical ideas
Graph maps and grids

- **A graph map** $f : G \to H$ is a function
  - $f : V(G) \to V(H)$
  - $u \sim v \Rightarrow f(u) \sim f(v)$ or $f(u) = f(v)$
- Let $\mathbb{Z}^n$ denote the infinite $n$-dimensional grid graph
- We want graph maps $f : \mathbb{Z}^n \to \Gamma_q(K)$ with “finite support” (constant outside finite set)

![Graph maps and grids diagram](image-url)
A discrete homotopy consists of
- Finite sequence of graphs maps $f_i : \mathbb{Z}^n \rightarrow \Gamma_q(K)$ with finite support
- For all $i$ and $v \in V$ we have $f_i(v) \sim f_{i+1}(v)$ or $f_i(v) = f_{i+1}(v)$

$\Gamma_1(K)$

\[ f_1 \quad f_2 \quad f_3 \]
Discrete homotopy groups

- Fix a base vertex \( \sigma_0 \in \Gamma_q(K) \)
- Discrete homotopy defines an equivalence relation on graph maps \( f : \mathbb{Z}^n \rightarrow \Gamma_q(K) \) based at \( \sigma_0 \) (\( f \equiv \sigma_0 \) outside finite set)
- Can define a product on discrete homotopy classes:

\[
\begin{bmatrix}
\cdot \\
\end{bmatrix} \cdot 
\begin{bmatrix}
\cdot \\
\end{bmatrix} = 
\begin{bmatrix}
\cdot \\
\end{bmatrix}
\]

Definition-Theorem (Barcelo–Kramer–Laubenbacher–Weaver 2001)

The **discrete homotopy groups** are the groups \( A_n^q(K, \sigma_0) \) whose elements are discrete homotopy classes of graph maps \( \mathbb{Z}^n \rightarrow \Gamma_q(K) \) based at \( \sigma_0 \) and whose products are defined as above.
While $A_1(K, \sigma_0)$ detects chordless $\geq 5$-cycles in $\Gamma_q(K)$, it ignores 3- and 4-cycles.

- Highlights the cubical nature of the discrete homotopy groups
- Can contract a discrete loop around the 4-cycle in two steps:
Examples

- If $\Delta$ is a simplex, then $A^q_n(\Delta, \sigma_0)$ is trivial for all $q$, $n > 0$ and $\sigma_0$
- If $n > 1$, then $A^q_n(K, \sigma_0)$ is abelian
- $A^q_1(K, \sigma_0)$ detects $q$-holes of length $\geq 5$, but not of length $\leq 4$:
  
  $K = \begin{array} \text{pentagon} \\
  \end{array}$
  
  $A^q_1(K, \sigma_0) \cong \begin{cases} 
  \mathbb{Z} & \text{if } q = 1 \\
  1 & \text{if } q = 0, 2
  \end{cases}$

  $L = \begin{array} \text{square} \\
  \end{array}$
  
  $A^q_1(L, \tau_0) \cong 1$ if $q = 0, 1, 2$

- Suppress the base point $\sigma_0$ when $\Gamma_q(K)$ is connected
Proposition (Barcelo–Kramer–Laubenbacher–Weaver 2001)

Let $X^q(K)$ be the CW complex obtained by attaching a 2-cell to every 3- and 4-cycle of $\Gamma_q(K)$. Then $A_1^q(K, \sigma_0) \cong \pi_1(X^q(K), \sigma_0)$.

Special case: Graphs

For (connected) graphs $K = G$, we can define discrete homotopy groups $A_n(G)$ directly by using graph maps $\mathbb{Z}^n \rightarrow G$ instead of $\mathbb{Z}^n \rightarrow \Gamma_0(G)$.

Theorem (L. 2020)

For each $n$, there is an infinite family of graphs $G$ for which $A_n(G)$ is nontrivial. These are the only known examples of nontrivial higher discrete homotopy groups in the literature.
Many ideas from classical topology can be meaningfully ported to the discrete setting:

- Discrete Seifert-van Kampen theorem
- Relative discrete homotopy groups and long exact sequences
- Accompanying homology theory for metric spaces, called **discrete singular cubical homology**
  - Satisfies discrete versions of Eilenberg-Steenrod axioms (plays nice with discrete homotopy)
  - Discrete Hurewicz theorem in dimension 1 (first homology group is abelianization of discrete fundamental group)
- Spectral sequences
Application: Subspace arrangements

- $W$ a finite real reflection group of rank $n$
- $\Sigma(W)$ the Coxeter complex of type $W$
- $\mathcal{W}_{n,k}$ the arrangement of fixed subspaces of all rank-$(k - 1)$ irreducible parabolic subgroups of $W$ (interesting when $k \geq 3$)
- $\mathcal{W}_{n,k}$ generalizes Coxeter arrangements ($k = 2$) and $k$-equal arrangements ($W = A_n$)

Theorem (Barcelo–Severs–White 2011)

Let $U(\mathcal{W}_{n,k})$ denote the complement of $\mathcal{W}_{n,k}$. Then

$$\pi_1(U(\mathcal{W}_{n,k})) \cong A_1^{n-k+1}(\Sigma(W)).$$
$W$ admits a presentation with generating set $S$ and relations

1. $s^2 = 1$ for all $s \in S$
2. $st = ts$ for all $s, t \in S$ with $m(s, t) = 2$
3. $sts = tst$ for all $s, t \in S$ with $m(s, t) = 3$

... 

**Theorem (Rephrasing of Brieskorn 1971)**

The fundamental group of the complement of the complexification of $\mathcal{W}_{n,2}$ is given by the above generators and relations, minus relation $\text{4}$. 

**Theorem (Rephrasing of Barcelo–Severs–White 2011)**

The fundamental group $\pi_1(U(\mathcal{W}_{n,3}))$ is given by the above generators with only the relations $\text{1}$ and $\text{2}$. 
Application: Group theory

- $A_n(G)$ definition requires vertices of $G$ to be distance $\leq 1$ apart; can require only distance $\leq r$ to get generalization $A_{n,r}(G)$
- Let $F_S$ denote free group (finite rank) with normal subgroup $N$ and $\overline{S}$ the image of $S$ in $F_S/N$
- Can recover $N$ from $F_S/N$ using homotopy of Cayley graph:
  \[ N \cong \pi_1(\text{Cay}(F_S/N, \overline{S})) \]
- Discrete homotopy can do the same for any finitely presented group

Theorem (Delabie–Khukhro 2020)

Let $G = \langle S \mid R \rangle$ be a finitely presented group with identity $e$ and normal subgroup $N$. There is a value of $r$ depending only on $S$ and $R$ such that $N \cong A_{1,r}(\text{Cay}(G/N, \overline{S}), e)$. 
What do we want?

- Consider only graphs $K = G$ from now on
- We understand concretely what $A_1(G)$ computes:

$$A_1 \equiv \pi_1$$

- Can we achieve a similar understanding of higher homotopy groups?

Goal

Construct a topological space $X$ such that $A_n(G) \cong \pi_n(X)$ for all $n$. 
Let $Q_n \subset \mathbb{Z}^n$ be induced by all vertices with all coordinates 0 or 1.

- $Q_0$
- $Q_1$
- $Q_2$
- $Q_3$

Fix a graph $G$ whose discrete homotopy groups we are interested in.

Let $M_n(G) = \text{Hom}(Q_n, G)$ (graph maps from the $n$-cube to $G$).

We will define face and degeneracy maps for $M_\bullet(G)$. 
The cubical set

For $i = 1, \ldots, n$ and $\varepsilon = 0, 1$ define

$$a_{i,\varepsilon}(n) : Q_{n-1} \to Q_n$$
$$(x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_{i-1}, \varepsilon, x_{i+1}, \ldots, x_{n-1})$$

$$b_i(n) : Q_n \to Q_{n-1}$$
$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n),$$

Recall that $M_n(G) = \text{Hom}(Q_n, G)$. There are induced maps

$$\alpha_{i,\varepsilon}(n) : M_n(G) \to M_{n-1}(G)$$
and

$$\beta_i(n) : M_{n-1}(G) \to M_n(G)$$

We obtain a cubical set $M_\bullet(G) : \square \to \text{Set}$ with face maps $\alpha_{i,\varepsilon}$ and degeneracy maps $\beta_i$. 
Theorem (Babson–Barcelo–de Longueville–Laubenbacher 2006)

Let $X(G)$ denote the geometric realization of $M\cdot(G)$. If a certain cubical approximation property* holds, then for all $n$ we have

$$A_n(G) \cong \pi_n(X(G)).$$

The asterisk: Proposed cubical approximation theorem

Let $X$ be a cubical set and $f : I^n \to |X|$ a continuous map such that $f|_{\partial I^n}$ is cubical. There exists a cubical subdivision $D^n$ of $I^n$ and a cubical map $f' : D^n \to |X|$ such that $f \simeq f'$ and $f|_{\partial D^n} = f'|_{\partial D^n}$.

While this statement seems plausible, no one has been able to prove it or find it in the literature!
Big questions (I am not a topologist 😊)

- Does the cubical approximation theorem hold?
- The CW complex $X(G)$ is infinite dimensional in general. Can we find a finite-dimensional deformation retract?
- Can we use the (conditional) fact that $A_n(G) \cong \pi_n(X(G))$ to directly find nontrivial $A_n(G)$ for $n \geq 2$?
- Using the theorem, can the tools of classical homotopy theory be leveraged to prove discrete versions of other famous theorems in topology? (Hurewicz for higher dimensions, Dold–Thom, etc.)
Thank you!
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