

# The Extended Press-Schechter Formalism

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## I. INTRODUCTION

In inflationary models, quantum fluctuations of the inflaton field give rise to fluctuations in the energy density. The amplitude of these fluctuations is approximately  $10^{-5}$ , as measured by Planck. These fluctuations have since evolved into the structures we see today, in a way that can be described by your particular cosmological theory (namely,  $\Lambda$ CDM). The theory predicts statistical properties of the distributions of objects on the sky, which are described by an infinite number of correlation functions. The correlation functions relevant to our discussion is the one-point correlation function, i.e., the number density of objects on the sky, and the two-point correlation function. The smallest objects possible to describe by cosmology are galaxy clusters, which are the objects you should have in mind during this talk. In this talk we will develop a formalism to predict the mass function  $n(M)$ , where  $n$  is the number density of objects mass  $M$ .

## II. EVOLUTION OF DENSITY PERTURBATIONS

### A. General Case

Accumulations of mass such as galaxy clusters constitute what are called overdensities; other areas constitute underdensities. Formally, we define an overdensity by

$$\delta(x) = \frac{\rho(x) - \bar{\rho}}{\bar{\rho}} \quad (1)$$

where  $\rho(x)$  is the density at point  $x = (t, \mathbf{x})$ , and  $\bar{\rho}$  is the average density over all space. Inflationary theory predicts that the distribution of these overdensities is initially Gaussian with standard deviation  $10^{-5}$ . To describe the evolution, we treat the clusters as particles in a fluid. Defining the covariant derivative for a fluid

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla, \quad (2)$$

where the velocity of the fluid flow is  $\mathbf{u}$ , we have the continuity (mass conservation) equation

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{u}) = 0. \quad (3)$$

We finally have the Euler equation for conservation of momentum

$$\frac{D\mathbf{u}}{Dt} = -\frac{\nabla p}{\rho} - \nabla\Phi \quad (4)$$

where  $p$  is the pressure of the fluid and  $\Phi$  is the gravitational potential satisfying the Poisson equation

$$\nabla^2\Phi = 4\pi G\rho. \quad (5)$$

By converting to comoving coordinates  $\mathbf{x} = a(t)\mathbf{r}$  and ridding ourselves of  $\rho$  in favor of  $\delta$ , we have

$$\frac{\partial\delta}{\partial t} + \frac{1+\delta}{a}\nabla \cdot \mathbf{v} = 0 \quad (6)$$

$$\frac{\partial\mathbf{v}}{\partial t} + \frac{\dot{a}}{a}\mathbf{v} + \frac{1}{a}(\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla\Phi}{a} - \frac{\nabla p}{a\bar{\rho}(1+\delta)} \quad (7)$$

$$\nabla^2\Phi = 4\pi G\bar{\rho}a^2\delta \quad (8)$$

where the space derivatives are now understood to be taken with respect to comoving coordinates. For small density perturbations—the regime we will consider—we can ignore terms quadratic in  $\mathbf{v}$ .

By considering the thermodynamic properties of the fluid, we can rewrite the  $\nabla p/\rho$  term to obtain

$$\frac{\partial\mathbf{v}}{\partial t} + \frac{\dot{a}}{a}\mathbf{v} = -\frac{\nabla\Phi}{a} - \frac{c_s^2}{a}\nabla\delta - \frac{2T}{3a}\nabla S \quad (9)$$

where  $c_s$  is the sound speed in the medium, and  $T$  and  $S$  are the local temperature and entropy, respectively. Finally, taking a spatial derivative of this expression and combining the previous equations, we arrive at

$$\frac{\partial^2\delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial\delta}{\partial t} = 4\pi G\bar{\rho}\delta + \frac{c_s^2}{a^2}\nabla^2\delta + \frac{2}{3}\frac{T}{a^2}\nabla^2 S \quad (10)$$

Switching to Fourier space yields

$$\frac{\partial^2\delta_{\mathbf{k}}}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial\delta_{\mathbf{k}}}{\partial t} = \left(4\pi G\bar{\rho} - \frac{k^2 c_s^2}{a^2}\right)\delta_{\mathbf{k}} - \frac{2}{3}\frac{T}{a^2}k^2 S_{\mathbf{k}} \quad (11)$$

Current measurements constrain the initial density perturbations to be isentropic to less than 1%, so we will take  $S_{\mathbf{k}} = 0$ .

### B. Jean's Length

Notice that this equation is a simple harmonic oscillator, so we can define a “frequency”:

$$\omega^2 = \left(\frac{k^2 c_s^2}{a^2} - 4\pi G\bar{\rho}\right) = (k^2 - k_J^2)c_s^2 \quad (12)$$

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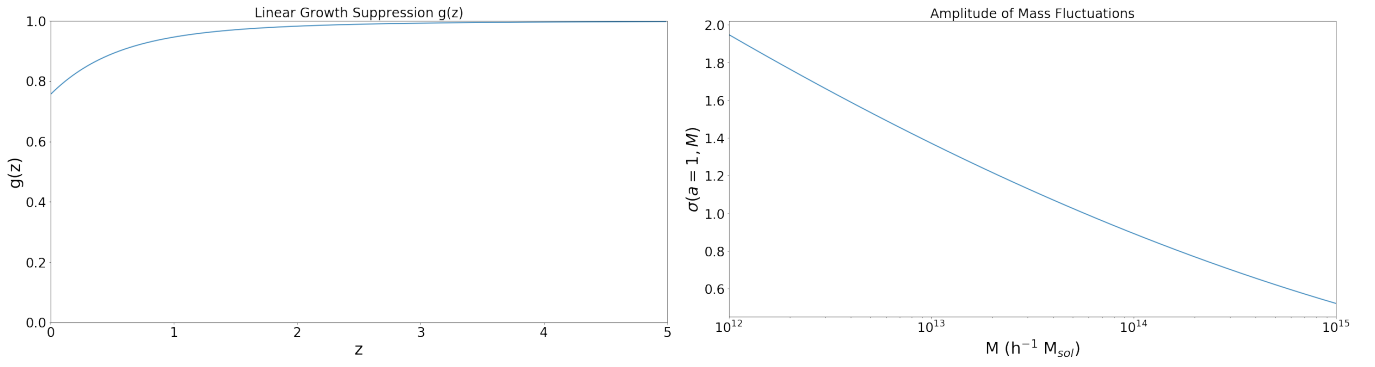


Fig. 1. On the left, the growth suppression factor  $g(z)$  is plotted. On the right, the amplitude of mass fluctuations  $\sigma$  at the present day is plotted.

yielding

$$\frac{\partial^2 \delta_{\mathbf{k}}}{\partial t^2} + 2\frac{\dot{a}}{a} \frac{\partial \delta_{\mathbf{k}}}{\partial t} = -\omega^2 \quad (13)$$

where

$$\lambda_J = \frac{2\pi a}{k_J} = c_s \sqrt{\frac{\pi}{G\bar{\rho}}} \quad (14)$$

is the Jean's length. We see that for  $\omega^2 > 0$  ( $\lambda < \lambda_J$ ) the modes do not collapse to form structures— $\delta_{\mathbf{k}}$  is oscillating, while for  $\omega^2 < 0$  ( $\lambda > \lambda_J$ ) the modes do collapse. Expressed in another way, only objects larger than the Jeans mass  $M_J$  can form, where

$$M_J = \frac{4\pi}{3} \left(\frac{\lambda_J}{2}\right)^3 \bar{\rho}. \quad (15)$$

To give some numerical values, before recombination  $M_J \approx 10^{19} M_\odot$  while after  $M_J \approx 10^5 M_\odot$ . Therefore, before recombination no structure could form, and afterwards we began to see growth.

### C. Late Universe Evolution

In the late universe, on small scales compared to the particle horizon, our master equation reads

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\rho_M(t)\delta = 0. \quad (16)$$

We are interested in solving this differential equation for  $\Lambda$ CDM. To develop some intuition, let's do some simple cases first.

For a matter dominated universe, known as an Einstein de Sitter (EdS) universe, the scale factor  $a(t) \propto t^{2/3}$  so  $H = 2/(3t)$ . The solutions are

$$\delta(t) = Ca(t) + Da^{-3/2}(t) \quad (17)$$

where we have a growing mode and a decaying mode. Clearly we're interested in the growing solutions. For a  $\Lambda$  dominated universe,  $H(t) = H_0$  and we obtain the solution

$$\delta(t) = E + Fe^{-2Ht}. \quad (18)$$

In  $\Lambda$ CDM, we have  $\Omega_{M,0} = 0.3$  and  $\Omega_{\Lambda,0} = 0.7$ . Qualitatively, we can understand that since dark energy is only becoming more dominant, all the structure that is going to form has formed—structure will only decay from now on. To understand this scenario at a quantitative level, let's rescale  $\delta$  to that of today by introducing the growth function

$$D(a) = \frac{\delta(a)}{\delta(1)}. \quad (19)$$

Furthermore, define the growth suppression factor

$$g(a) = \frac{D(a)}{a} \quad (20)$$

which factors out the growth that one would see in EdS. For EdS clearly  $g(a) = 1$ , while there is less growth with more  $\Lambda$ , so for  $\Lambda$ CDM we would generically expect  $g(a) < 1$ . With these definitions, we can show that  $g(a)$  satisfies the following differential equation:

$$2\frac{d^2 g}{d \ln a^2} + (5 - 6\Omega_\Lambda(a))\frac{dg}{d \ln a} + 6\Omega_\Lambda(a)g = 0 \quad (21)$$

This equation has the accurate analytic approximate solution

$$g(a) = \frac{5}{2} \frac{\Omega_M}{\Omega_M^{4/7} - \Omega_\Lambda + \left(1 + \frac{1}{2}\Omega_M\right) \left(1 + \frac{1}{70}\Omega_\Lambda\right)}. \quad (22)$$

## III. THE PRESS-SCHECHTER MASS FUNCTION

### A. The Power Spectrum

The two point correlation function of overdensities, by isotropy and homogeneity, can be written

$$\langle \delta_{\mathbf{k}} \delta_{\mathbf{k}'} \rangle = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') P(k) \quad (23)$$

where  $P(k)$  is the power spectrum. It tells you the average value of  $\delta_{\mathbf{k}}^2$  throughout the universe. We can take the Fourier

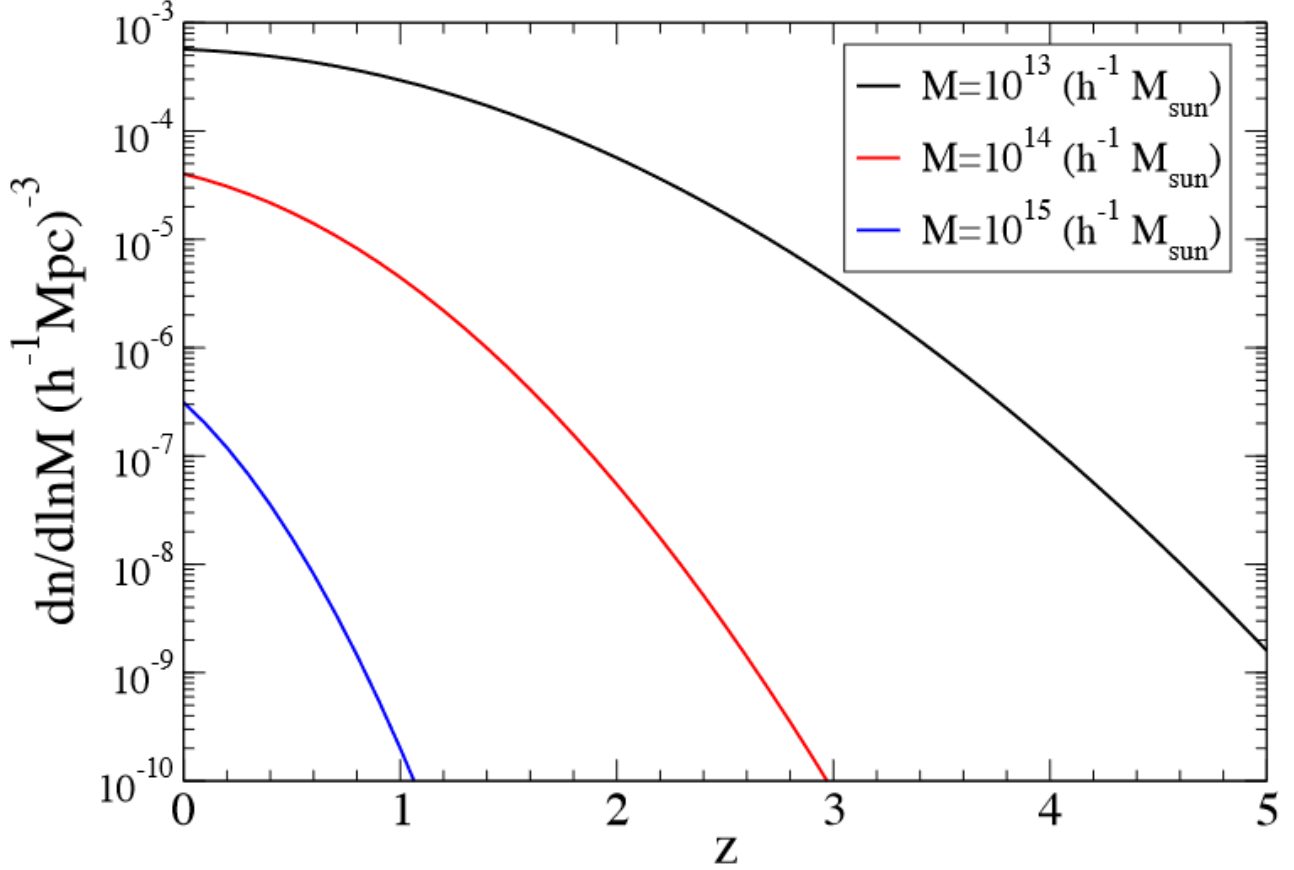


Fig. 2. The Press-Schechter mass function is plotted for several values of the masses as a function of redshift.

transform to obtain the real-space two-point correlation function

$$\xi(r) = \frac{1}{(2\pi)^3} \int P(k) e^{-i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{k} \quad (24)$$

To find the variance of mass fluctuations, we want to evaluate this at  $r = 0$ , which tells you the average value of  $\delta^2(\mathbf{x})$  throughout the universe. This yields

$$\xi(0) = \sigma^2 = \int_0^\infty \Delta^2(k) d \ln k. \quad (25)$$

where

$$\Delta^2(k) = \frac{k^3 P(k)}{2\pi^2} \quad (26)$$

is the logarithmic band power.

By a lengthy computation, one obtains

$$\Delta^2(k, a) = \frac{4}{25} \frac{A}{\Omega_{M,0}^2} \left( \frac{k}{k_{piv}} \right)^{n-1} \left( \frac{k}{H_0} \right)^4 (ag(a))^2 T^2(k) \quad (27)$$

where  $A = 2.43 \times 10^{-9}$ ,  $k_{piv} = .005 \text{ Mpc}^{-1}$ , and  $T(k)$

is the transfer function that describes the stunted growth of the modes entering the horizon during radiation domination. Notice that this means  $\sigma^2$  is UV divergent.

### B. Window Functions

This divergence is due to the fact that the very short modes are grainy (and cosmology is not applicable on such small scales, anyway). To fix this issue we can define the “top-hat” window function (or filter)

$$W(r, R) = \frac{3}{4\pi R^3} \Theta(R - r) \quad (28)$$

where  $\Theta(x)$  is the Heaviside function. The window function smooths the overdensity at the scale  $R$ , like so:

$$\delta(\mathbf{r}, R) = \int W(\mathbf{r} - \mathbf{r}', R) \delta(\mathbf{r}') d\mathbf{r}' \quad (29)$$

In Fourier space this results in

$$\delta_{\mathbf{k}}(R) = W(k, R) \delta_{\mathbf{k}} \quad (30)$$

where

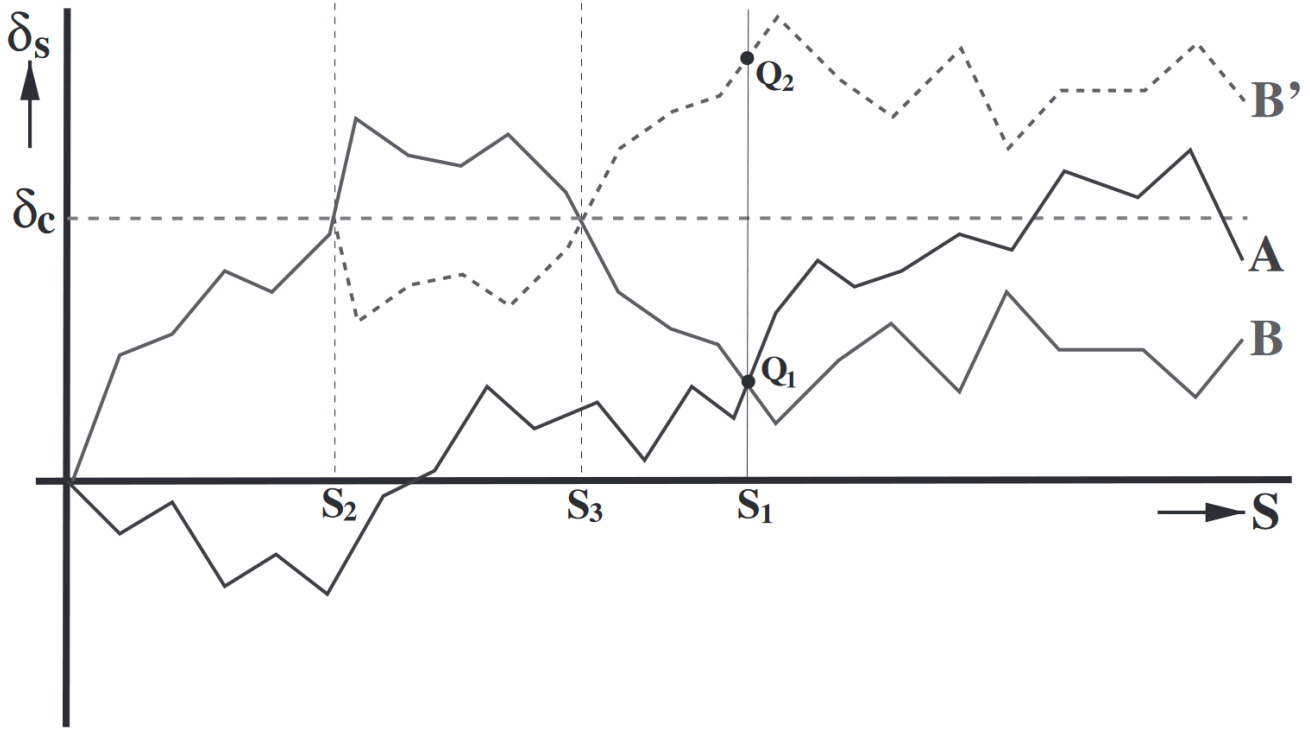


Fig. 3. Markovian random walk trajectories in  $(S, \delta(S))$  space [2].

$$W(k, R) = \frac{3j_1(kR)}{kR}. \quad (31)$$

This modifies our rms amplitude of mass fluctuations to be

$$\sigma^2(R) = \int_0^\infty \Delta^2(k) W^2(k, R) d \ln k. \quad (32)$$

where we have now smoothed over a scale  $R$ .

In fact the useful filter for this talk is the “k-space top-hat” or “sharp k-space” filter

$$W(kR) = \Theta(-kR) \quad (33)$$

which we will use from now on. Since it is more natural to use mass when discussing halos, we redefine the scale in terms of the mass

$$M = 6\pi^2 \rho_M R^3 \quad (34)$$

enclosed on that scale. This mass seems a bit strange because of a technical issue with the definition of the window function.

### C. The Press-Schechter Mass Function

In 1974, Press and Schechter developed a formalism for determining the number density of clusters by postulating that the probability of collapse can be computed by examining the amplitude of mass fluctuations on that scale. To

this end, they assumed that at some point  $\mathbf{x}$ , the probability that  $\delta(\mathbf{x}, M) > \delta_c$  for some critical value  $\delta_c$  is equal to the fraction of mass contained in halos with mass greater than  $M$ . For spherical collapse,  $\delta_c \approx 1.686$ .

The swapping of  $R$  in favor of  $M$  leads to a subtle issue. We can consider some mass associated with a peak density  $\delta_1(\mathbf{x}) = \delta(\mathbf{x}, R_1)$  as also associated with  $\delta_2(\mathbf{x}) = \delta(\mathbf{x}, R_2)$  for  $R_2 > R_1$ . If  $\delta_2 < \delta_1$  then we are okay; we can consider the mass as part of both, because  $\delta(\mathbf{x})$  will first collapse on the scale  $R_1$  and then on  $R_2$ . This is just the statement that mass slowly accumulates to form larger and larger structures. However, if  $\delta_1 < \delta_2$  then we get what is referred to as the “cloud-in-cloud” problem. We are counting an object that has already been included in a larger mass, and should not include it in our count for  $R_1$ .

Forging ahead despite this issue, the mass fraction of collapsed objects is

$$F(> M) = \frac{1}{\sqrt{2\pi}\sigma(M)} \int_{\delta_c}^\infty \exp\left(-\frac{\delta^2}{2\sigma^2(M)}\right) d\delta \quad (35)$$

$$= \frac{1}{2} \operatorname{erfc}\left(\frac{\delta_c}{\sqrt{2}\sigma(M)}\right) \quad (36)$$

Note that as  $M \rightarrow 0$  ( $\sigma(M) \rightarrow \infty$ ), the fraction of collapsed objects is  $1/2$ , i.e. only half all mass is contained in collapsed objects. This mass is the mass that was initially in overdense regions; the underdense mass does not collapse in this formalism. We would like it to, because eventually it should end up accreted in some overdense region. To fix

this issue, Press and Schechter multiplied the mass fraction by a factor of two, but offered no substantial justification.

We now have the tools to write down the mass function. The comoving number density of collapsed objects at a given mass is the mass fraction of collapsed objects times the comoving density

$$\begin{aligned} n(M)dM &= \frac{\rho_{M,0}}{M} \frac{\partial F(> M)}{\partial M} dM \\ &= \sqrt{\frac{2}{\pi}} \frac{\rho_{M,0}}{M^2} \frac{\delta_c}{\sigma} \left| \frac{d \ln \sigma}{d \ln M} \right| \exp\left(-\frac{\delta_c^2}{2\sigma^2(M)}\right) dM \end{aligned} \quad (37)$$

This is the Press-Schechter mass function. This function accurately describes our universe to the 50% level, which is actually incredible for a number of reasons. First, remember our assumption of small density perturbations/linear theory. But halos are clearly nonlinear objects, because  $\delta_c > 1!$  Second, we assumed that the perturbations are distributed in a Gaussian manner around the mean. But this cannot be true, because  $\delta$  is bounded below by  $-1$ . So the fact that this function is accurate at all is quite amazing.

So what have we learned from this formula? The number density of halos falls exponentially with increasing mass. Larger scales  $R$  or, equivalently, larger masses  $M$ , are less common in the universe due to the drop in  $\sigma$  making it less likely that any patch that size has an overdensity  $\delta > \delta_c$ .

#### IV. THE EXTENDED PRESS-SCHECHESTER FORMALISM

##### A. The Excursion Set

The Extended Press-Schechter Formalism gives a natural explanation for this factor of two we inserted into our mass function earlier. For conciseness, we let  $S \equiv \sigma^2(M)$  be our mass variable, where  $S$  is monotonically decreasing in  $M$ . To each point we can associate a trajectory  $\delta(S)$  in  $(S, \delta(S))$  space, where  $\delta(S)$  is the overdensity smoothed with a window function of mass  $S$ . Note the overdensity in real space is

$$\delta(\mathbf{x}, k_c) = \int d^3\mathbf{k} W(\mathbf{k}R) \delta_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} = \int_{k < k_c} d^3\mathbf{k} \delta_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \quad (39)$$

where  $k_c = 1/R$ . The reason we use this filter is that it has the property that the change  $\Delta\delta(\mathbf{x}, k_c)$  due to an increase in the filter argument  $k_c \rightarrow k_c + \Delta k_c$  is a Gaussian random variable with variance

$$\langle (\Delta\delta(\mathbf{x}, k_c))^2 \rangle = \langle (\delta(\mathbf{x}, k_c + \Delta k_c) - \delta(\mathbf{x}, k_c))^2 \rangle \quad (40)$$

$$= \int_{k_c}^{k_c + \Delta k_c} \Delta^2(k) d \ln k \quad (41)$$

$$= \sigma^2(k_c + \Delta k_c) - \sigma^2(k_c) \quad (42)$$

which is independent of the original value of  $\delta(\mathbf{x}, k_c)$ —satisfying the criteria for a Markovian random walk. Furthermore, as  $S \rightarrow 0$ ,  $\delta \rightarrow 0$ .

Consider the mass scale  $M_1(S_1)$ . The ansatz of the Press-Schechter formalism applied to this scenario implies that the

fraction of trajectories with  $\delta > \delta_c$  at  $S_1$  equals the fraction of mass in collapsed objects with  $M > M_1$ . Thus, trajectory  $B$  is not part of a collapsed object with  $M > M_1$ . However, it is part of a collapsed object with  $M > M_3 > M_1$ , yielding a contradiction. The ansatz does not consider all of the Markovian trajectories (like  $B$ ), only those which, once reaching  $\delta_c$ , stay above. However, due to the properties of a Markovian random walk, trajectory  $B$  is just as likely as  $B'$ . So each trajectory that the ansatz accounts for has a corresponding trajectory that mirrors it after first upcrossing (like  $B'$ ). Therefore, we modify the ansatz to the following: the probability that  $\delta > \delta_c$  is the same as twice the fraction of mass contained in halos with mass greater than  $M$ . This resolves the question of the factor of two initially introduced to account for the underdensities collapsing.

To derive the Press-Schechter mass function from the extended formalism, consider the fraction of trajectories with their first upcrossing after  $S_1$  (see trajectory  $A$ ). According to our (new) ansatz, these trajectories correspond to mass elements in collapsed objects of mass  $M < M_1$ . To obtain this fraction, we need the fraction of trajectories with  $\delta(S_1) < \delta_c$  but we must also subtract out the trajectories like  $B$ . Since  $P(B) = P(B')$ , we can write this fraction as

$$F(< S_1) = \int_{-\infty}^{\delta_c} (P(\delta, S_1) - P(2\delta_c - \delta, S_1)) d\delta \quad (43)$$

$$= \frac{1}{\sqrt{2\pi}} \left[ \int_{-\infty}^{\delta_c/\sqrt{S_1}} dx e^{-x^2/2} - \int_{-\delta_c/\sqrt{S_1}}^{\infty} dx e^{-x^2/2} \right] \quad (44)$$

This fraction is the fraction of masses included in halos with  $S > S_1$ , i.e.,  $M < M_1$ . Because the trajectory is a random walk, every trajectory will cross the barrier as  $M \rightarrow 0$  ( $S \rightarrow \infty$ ), so this fraction will go to zero. We wanted the fraction of masses included in halos with  $M > M_1$ , that is,  $F(> M) = 1 - F(< S_1)$ , so now we have correctly accounted for this factor of two:

$$n(M)dM = \frac{\rho_{M,0}}{M} \frac{\partial F(> M)}{\partial M} dM \quad (45)$$

$$= \sqrt{\frac{2}{\pi}} \frac{\rho_{M,0}}{M^2} \frac{\delta_c}{\sigma} \left| \frac{d \ln \sigma}{d \ln M} \right| \exp\left(-\frac{\delta_c^2}{2\sigma^2(M)}\right) dM \quad (46)$$

#### REFERENCES

- [1] D. Huterer, *Cosmology I: The Early Universe Lecture Notes* (2018).
- [2] H. Mo, F. van den Bosch, and S. White, *Galaxy Formation and Evolution* (Cambridge University Press, Cambridge, 2010).