Effective Field Theories

Zachary Johnson, Ben Sheff, Chris Dessert

April 16, 2018

1 Goal

Our goal will essentially be in getting to β function for non-abelian gauge theories (like QCD) using the effective action and background field method. This method involves shifting our fields such that we have a fixed, classical background field on top of which we compute quantum corrections. At the end of this computation, we will explain some intuitive arguments for why non-abelian theories create the possibility of asymptotic freedom.

2 Lecture Notes

2.1 Background Method

As a reference, I write the original Lagrangian, covariant derivative definition, and field strength.

$$\mathcal{L} = -\frac{1}{4g^2}(F^a_{\mu\nu})^2 + \bar{\psi} i \not{D} \psi \tag{1}$$

$$D_\mu = \partial_\mu - i t^a A^a_\mu \tag{2}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu \tag{3}$$

Where a runs over the gauge fields. In QCD, or SU(3), there are 8 gluons. The fields, under $infinitesimal$ gauge transformations, become

$$\delta A^a_\mu = \partial_\mu \alpha^a + f^{abc} A^b_\mu \alpha^c = (D_\mu \alpha)^a \tag{4}$$

$$\delta F^a_{\mu\nu} = -f^{abc} \alpha^b F^c_{\mu\nu} \tag{5}$$

$$\delta\psi = i\alpha^a t^a \psi \tag{6}$$

Having defined these we can now start to talk about the background field method. The general procedure is that we take our original field and break it up into two pieces: one is a fixed (not integrated in the path integral) field, the background field, and the other is the quantized dynamical gauge field. We write this in terms of the quantum field, $\mathcal{A}$, and the background field, A, as

$$A^a_\mu \to A^a_\mu + \mathcal{A}^a_\mu \tag{7}$$

We then redefine our old covariant derivatives and field strengths as classical.

$$D_\mu \to D_\mu - i\mathcal{A}^a_\mu t^a \tag{8}$$

$$F^a_{\mu\nu} \to +D_\mu \mathcal{A}^a_\nu - D_\mu \mathcal{A}^a_\mu + f^{abc} \mathcal{A}^b_\mu \mathcal{A}^c_\nu \tag{9}$$

Here you might care to note that this t^a is in any representation. For the real, adjoint representation that the gauge fields and ghosts are in we have $t^a \to (t^a)_{bc} = f^{abc}$.

This gives us the new gauge-fixed Lagrangian

$$\mathcal{L} = -\frac{1}{4g^2}(F^a_{\mu\nu} + D_\mu \mathcal{A}^a_\nu - D_\mu \mathcal{A}^a_\mu + f^{abc}\mathcal{A}^b_\mu \mathcal{A}^c_\nu)^2 + \bar{\psi}\left(i\not{D} + \not{\mathcal{A}}^a t^a\right)\psi \tag{10}$$

$$- \frac{1}{2g^2}(D^\mu \mathcal{A}_\mu)^2 - \bar{c}^a \left((D^2)^{ac} + f^{abc}(D^\mu \mathcal{A}_\mu)^b\right) c^c \tag{11}$$

2.2 One-Loop Effective Action

Now we want to find the true vacuum of the theory, and then add quantum corrections on top of that. Using the effective action comes down to creating an A (or ϕ_{cl}) s.t.the following are satisfied

$$A(x) = \langle\Omega|A(x)|\Omega\rangle_J \tag{12}$$

$$\frac{\delta\Gamma[A]}{\delta A(x)} = -J(x) \tag{13}$$

Then we expand for small background fluctuations

$$\mathcal{L}[A+\mathcal{A}] = \mathcal{L}[A] + \mathcal{L}'[A]\mathcal{A} + \frac{1}{2}\mathcal{L}''[A]\mathcal{A}^2 + \cdots \tag{14}$$

Note however, that this first order term in $\mathcal{A}$ would give it a VEV. We require the entirety of the VEV to be in the classical field, so this term is zero. Only the classical piece, and 2nd order and higher quantum effects are left.

$$e^{i\Gamma[A]} \equiv \int D\mathcal{A}Dc \, \exp\left[i\int d^4x\mathcal{L}[A+\mathcal{A}]\right] \tag{15}$$

Since quadratic order is leading order, we want to rewrite the Lagrangian as

$$e^{i\Gamma[A]} \equiv e^{i\int d^4x\mathcal{L}[A]}\int D\mathcal{A}Dc \, \exp\left[i\int d^4x(\mathcal{A}^a_\mu\Delta^{ab}_{G,1}\mathcal{A}^b_\mu + \bar{c}^a\Delta^{ab}_{G,0}c^b)\right] \tag{16}$$

Where G denotes the adjoint representation(or transforming like gluons). This is important because we know how to integrate quadratic exponents, they are generalized Gaussian integrals. Thus

$$e^{i\Gamma[A]} \equiv e^{i\int d^4x\mathcal{L}[A]}(\det\Delta_{G,1})^{-1/2}(\det\Delta_{G,0})^{+1}(\det\Delta)^{-1/2} \tag{17}$$

The deltas depend *only* on the classical field A. The power of the determinant depends on whether it is grassmannian or not. I also have simply thrown in the fermion term for us. Now I calculate the gluon $\Delta_{G,1}$. Expanding out all terms in $\mathcal{L}$ quadratic in $\mathcal{A}$

$$\mathcal{L}_\mathcal{A} = -\frac{1}{2g^2}\left[\frac{1}{2}(D_\mu\mathcal{A}^a_\nu - D_\mu\mathcal{A}^a_\mu)^2 + F^{a\mu\nu}f^{abc}\mathcal{A}^b_\mu\mathcal{A}^c_\nu + (D^\mu\mathcal{A}_\mu)^a(D^\nu\mathcal{A}_\nu)^a\right] \tag{18}$$

$$= -\frac{1}{2g^2}\left[(D_\mu\mathcal{A}^a_\nu D^\mu\mathcal{A}^{a\nu} - D_\mu\mathcal{A}^a_\nu D^\nu\mathcal{A}^{a\mu} + D^\mu\mathcal{A}^a_\mu D^\nu\mathcal{A}^a_\nu) + F^{a\mu\nu}f^{abc}\mathcal{A}^b_\mu\mathcal{A}^c_\nu\right] \tag{19}$$

Quickly notice that $D^T = -D$, or explicitly

$$(D_\mu\mathcal{A}_\nu)^a = (\partial_\mu\delta^{ac} + f^{abc}A^b_\mu)\mathcal{A}^c \tag{20}$$

$$= -\mathcal{A}^c_\nu(\partial_\mu\delta^{ca} + f^{cba}A^b_\mu) = -(\mathcal{A}_\nu D_\mu)^a \tag{21}$$

This allows us to "integrate by parts". We get

$$\mathcal{L}_\mathcal{A} = -\frac{1}{2g^2}\left[\mathcal{A}^a_\mu((-D_\mu D^\mu)^{ab}g^{\mu\nu} + (D^\nu D^\mu)^{ab} - (D^\mu D^\nu)^{ab})\mathcal{A}^b_\nu - \mathcal{A}^a_\mu f^{abc}F^{b\mu\nu}\mathcal{A}^c_\nu\right] \tag{22}$$

However, interestingly, if you remember, we defined $F^a_{\mu\nu}t^a \equiv i[D_\mu, D_\nu]$. Thus we can combine two of these terms to

$$\mathcal{L}_\mathcal{A} = -\frac{1}{2g^2}\mathcal{A}^a_\mu\left[(-D^2)^{ac}g^{\mu\nu} - 2f^{abc}F^{b\mu\nu}\right]\mathcal{A}^c_\nu \tag{23}$$

$$= -\frac{1}{2g^2}\mathcal{A}^a_\mu\left[(-D^2)^{ac}g^{\mu\nu} + 2(if^{abc})(iF^b_{\rho\sigma}g^{\rho\mu}g^{\sigma\nu})\right]\mathcal{A}^c_\nu \tag{24}$$

$$\tag{25}$$

But this is really interesting, since we can use the generator of lorentz transformations

$$(J^{\rho\sigma})_{\alpha\beta} = i(\delta^\rho_\alpha\delta^\sigma_\beta - \delta^\sigma_\alpha\delta^\rho_\beta) \tag{26}$$

to finally get Dirac's magnetic dipole interaction!

$$\mathcal{L}_{\mathcal{A}} = -\frac{1}{2g^2}\mathcal{A}^a_\mu \left[(-D^2)^{ac}g^{\mu\nu} + 2(\frac{1}{2}F^b_{\rho\sigma}J^{\rho\sigma})^{\mu\nu}(t^b_G)^{ac}\right]\mathcal{A}^c_\nu \tag{27}$$

This is our magnetic dipole interaction, complete with g=2. We can then rewrite the general expression as

$$\Delta_{r,j} = -D^2 + 2(\frac{1}{2}F^b_{\rho\sigma}J^{\rho\sigma})t^b \tag{28}$$

Now we will use this to compute the determinants in eqn. (17).

3 Computing the Functional Determinants

3.1 Expanding the Effective Action

We can rewrite the effective action for the classical fields as

$$e^{i\Gamma[A]} = \exp\left[i\int d^4x\left(-\frac{1}{4g^2}(F^a_{\mu\nu})^2 + \mathcal{L}_{ct}\right) - \frac{1}{2}\log\det\Delta_{G,1} + \log\det\Delta_{G,0} + \frac{N_f}{2}\log\det\Delta_{r,1/2}\right] \tag{29}$$

so it is clear that the corrections to the effective action come in the form of the log determinants of the general covariant background field d'Alembertian for each quantum field. If we expand these d'Alembertians, we must get a gauge-invariant quantity. To lowest order in A^a_μ, we must have

$$\log\det\Delta_{r,j} = i\int d^4x\left(-\frac{1}{4}\mathbf{C}_{r,j}(F^a_{\mu\nu})^2\right) \tag{30}$$

where $\mathbf{C}_{r,j}$ is some constant factor that differs for each field, depending on its representation r and spin j. When we add all the contributions, we will find

$$e^{i\Gamma[A]} \to \exp\left[i\int d^4x\left(-\frac{1}{4}\left(\frac{1}{g^2} + \frac{1}{2}\mathbf{C}_{G,1} - \mathbf{C}_{G,0} - \frac{N_f}{2}\mathbf{C}_{r,1/2}\right)(F^a_{\mu\nu})^2\right)\right] \tag{31}$$

This looks suspiciously like a running coupling. In fact since these corrections are coming in at lowest order to the effective action, we should expect that they are equivalent to a one-loop computation. Specifically, in the framework of Wilsonian renormalization, we would expect that corrections to the non-Abelian "charge" comes from the low energy (i.e., classical) modes propagating with corrections due to the high-energy degrees of freedom we are integrating out (i.e., quantum gauge fields, ghosts, and fermions). The relevant Feynman diagrams are then corrections to the gauge boson propagator.

Anticipating that these diagrams are logarithmically divergent (by checking the degree of superficial divergence, for example), we write

$$\mathbf{C}_{r,j} = c_{r,j}\log\frac{\Lambda^2}{k^2} + \text{finite} \tag{32}$$

where k is the energy scale of the background field. After imposing renormalization conditions we have

$$\mathbf{C}_{r,j} = c_{r,j}\log\frac{M^2}{k^2} + \text{finite} \tag{33}$$

We can see this leads to a running coupling

$$\frac{1}{g^2(k)} = \frac{1}{g^2} + \left(\frac{1}{2}c_{G,1} - c_{G,0} - \frac{N_f}{2}c_{r,1/2}\right)\log\frac{M^2}{k^2} \tag{34}$$

$$g^2(k) = \frac{g^2}{1 - \left(\frac{1}{2}c_{G,1} - c_{G,0} - \frac{N_f}{2}c_{r,1/2}\right) g^2 \log \frac{k^2}{M^2}} \tag{35}$$

Using a result of one of our homeworks, we find the β function for this theory in terms of the undetermined coefficients

$$\beta(g) = \left(\frac{1}{2}c_{G,1} - c_{G,0} - \frac{N_f}{2}c_{r,1/2}\right) g^3 \tag{36}$$

3.2 Computation of the β Function

Now we must actually calculate these $c_{r,j}$. Since fermions act similarly in non-Abelian theories and in QED, we will again ignore fermions to focus on the truly non-Abelian aspects, gluon self-interactions and ghosts. Recall from the definition of the generalized d'Alembertian

$$\begin{aligned}
\Delta_{r,j} &= -\mathcal{D}^2 + 2(\frac{1}{2}F^a_{\rho\sigma}\mathcal{J}^{\rho\sigma})t^a && (37)\\
&= -(\partial_\mu - iA^a_\mu t^a)^2 + 2(\frac{1}{2}F^a_{\rho\sigma}\mathcal{J}^{\rho\sigma})t^a && (38)\\
&= -\partial^2 + i[\partial^\mu A^a_\mu t^a + A^a_\mu t^a \partial^\mu] + A^{a\mu}t^a A^b_\mu t^b + 2(\frac{1}{2}F^a_{\rho\sigma}\mathcal{J}^{\rho\sigma})t^a && (39)\\
&= -\partial^2 + \Delta^{(1)} + \Delta^{(2)} + \Delta^{(\mathcal{J})} && (40)
\end{aligned}$$

where the superscript (1) signifies this term is $\mathcal{O}(A^1)$, (2) signifies $\mathcal{O}(A^2)$, and $(\mathcal{J})$ signifies $\mathcal{O}(\mathcal{J})$. Since we are actually interested in the log det, using the identity $\log\det = \mathrm{tr}\log$,

$$\begin{aligned}
\log\det\Delta_{r,j} &= \log\det\left[-\partial^2 + \Delta^{(1)} + \Delta^{(2)} + \Delta^{(\mathcal{J})}\right] && (41)\\
&= \log\det\left[-\partial^2\right] + \mathrm{tr}\log[1 + (-\partial^2)^{-1}(\Delta^{(1)} + \Delta^{(2)} + \Delta^{(\mathcal{J})})] && (42)\\
&= \log\det\left[-\partial^2\right] + \mathrm{tr}\left[(-\partial^2)^{-1}(\Delta^{(1)} + \Delta^{(2)} + \Delta^{(\mathcal{J})})\right] && (43)\\
&\quad - \frac{1}{2}\mathrm{tr}\left[(-\partial^2)^{-1}(\Delta^{(1)} + \Delta^{(2)} + \Delta^{(\mathcal{J})})(-\partial^2)^{-1}(\Delta^{(1)} + \Delta^{(2)} + \Delta^{(\mathcal{J})})\right] && (44)
\end{aligned}$$

The term $\log\det\left[-\partial^2\right]$ becomes an irrelevant constant when we normalize our path integral, but the other terms are real perturbations that yield amplitudes. The terms linear in A^a_μ vanish by gauge invariance or using the fact that they always come with a matrix t^a whose trace is zero. Thus to lowest order, we need to compute the quadratic terms in A^a_μ. These terms either have two powers of $\Delta^{(1)}$, one power of $\Delta^{(2)}$, or two powers of $\Delta^{(\mathcal{J})}$. The terms with one power of $\Delta^{(\mathcal{J})}$ are eliminated by $\mathrm{tr}\,\mathcal{J}^{\rho\sigma} = 0$, so there are no cross terms at this order to worry about. There are thus three classes of contributions that we will attribute to three classes of diagram:

$$\begin{aligned}
\log\det\Delta_{r,j} \to &-\frac{1}{2}\mathrm{tr}\left[(-\partial^2)^{-1}\Delta^{(1)}(-\partial^2)^{-1}\Delta^{(1)}\right] && (45)\\
&+ \mathrm{tr}\left[(-\partial^2)^{-1}\Delta^{(2)}\right] && (46)\\
&- \frac{1}{2}\mathrm{tr}\left[(-\partial^2)^{-1}\Delta^{(\mathcal{J})}(-\partial^2)^{-1}\Delta^{(\mathcal{J})}\right] && (47)
\end{aligned}$$

The terms together will generally organize themselves as Eq. (30), which we can write in Fourier space at this order as

$$\log\det\Delta_{r,j} = \frac{i}{2}\int \frac{d^4k}{(2\pi)^4} A^a_\mu(-k)A^a_\nu(k)(k^2 g^{\mu\nu} - k^\mu k^\nu)\mathbf{C}_{r,j} \tag{48}$$

Due to time constraints, I'll just write down the sum of the first two contributions, which organize themselves a similar form:

$$-\frac{1}{2}\,\mathrm{tr}\Big[(-\partial^2)^{-1}\Delta^{(1)}(-\partial^2)^{-1}\Delta^{(1)}\Big]+\mathrm{tr}\Big[(-\partial^2)^{-1}\Delta^{(2)}\Big]= \tag{49}$$

$$\frac{i}{2}\int\frac{d^4k}{(2\pi)^4}A^a_\mu(-k)A^a_\nu(k)(k^2g^{\mu\nu}-k^\mu k^\nu)\left[-\frac{C(r)d(j)}{3(4\pi)^2}\Gamma[2-\frac{d}{2}]\right] \tag{50}$$

where $C(r)$ is the index of the representation r and $d(j)$ is 1 for scalars and 4 for spinors and vectors, which come in from the identity $\mathrm{tr}\big[t^at^b\big]=C(r)d(j)\delta^{ab}$. These contributions come from the first two diagrams in Figure 16.10 in Peskin and Schroeder.

However we'll evaluate the third contribution containing the magnetic moment term. Remember that the trace is not only over gauge and spin indices, but also sums over the eigenvalues of the operator, which we can take to be in position space. This yields a delta function which we leave implicit.

$$-\frac{1}{2}\,\mathrm{tr}\Big[(-\partial^2)^{-1}\Delta^{(\mathcal{J})}(-\partial^2)^{-1}\Delta^{(\mathcal{J})}\Big] \tag{51}$$

$$=-\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4p}{(2\pi)^4}A^{a\mu}A^{a\nu}\frac{1}{p^2}\frac{1}{(p+k)^2}\,\mathrm{tr}(2ik_\rho g_{\mu\sigma}\mathcal{J}^{\rho\sigma})t^a(-2ik_\alpha g_{\nu\beta}\mathcal{J}^{\alpha\beta})t^b \tag{52}$$

This term, we can see, looks like it's coming from a diagram with a different type of vertex—the last diagram in Figure 16.10.

Since $\mathcal{J}^{\mu\nu}$ is antisymmetric in μ and ν, Lorentz invariance allows us to define the coefficient $C(j)$ as

$$\mathrm{tr}\big[\mathcal{J}^{\rho\sigma}\mathcal{J}^{\alpha\beta}\big]=(g^{\rho\alpha}g^{\sigma\beta}-g^{\rho\beta}g^{\sigma\alpha})C(j) \tag{53}$$

and by working through the algebra you can show $C(j)=2j$. Then (24) can be evaluated as

$$-\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4p}{(2\pi)^4}A^{a\mu}A^{a\nu}\frac{1}{p^2}\frac{1}{(p+k)^2}(k^2g^{\mu\nu}-k^\mu k^\nu)4C(r)C(j) \tag{54}$$

$$=\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}A^{a\mu}A^{a\nu}(k^2g^{\mu\nu}-k^\mu k^\nu)\left[-i\frac{4C(r)C(j)}{(4\pi)^2}\Gamma[2-\frac{d}{2}]\right] \tag{55}$$

Remembering the $c_{r,j}$ is the coefficient of the divergence, we can immediately write down

$$c_{r,j}=\frac{1}{(4\pi)^2}[\frac{1}{3}d(j)-4C(j)]C(r)=\frac{C(r)}{(4\pi)^2}\begin{cases}+1/3 & \text{spin } 0\\ -8/3, & \text{spin } 1/2\\ -20/3, & \text{spin } 1\end{cases} \tag{56}$$

Now we can plug these into the β function! Remembering that bosons and ghosts are in the adjoint representation G and fermions are in the fundamental representation r, we have

$$\beta(g)=\frac{g^3}{(4\pi)^2}\left(\frac{1}{2}\left(-\frac{20}{3}\right)C(G)-\frac{1}{3}C(G)-\frac{N_f}{2}\left(-\frac{8}{3}\right)C(r)\right) \tag{57}$$

$$=-\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}C(G)-\frac{4}{3}N_fC(r)\right) \tag{58}$$

We see that for sufficiently few fermions, $\beta(g)<0$, so we get asymptotic freedom.

4 Appendix

4.0.1 Faddeev-Popov

We introduce

$$1 = \int \mathcal{D}\alpha \delta(G^a(A)) \text{det}(\frac{\delta G^a}{\delta \alpha}) \tag{59}$$

Where we decide to fix the gauge of the quantum field in modified Lorenz gauge

$$G^a(A) = (D^\mu \mathcal{A}_\mu)^a - w^a \tag{60}$$

This gives us the new gauge-fixed Lagrangian

$$\mathcal{L} = -\frac{1}{4g^2}(F^a_{\mu\nu} + D_\mu \mathcal{A}^a_\nu - D_\mu \mathcal{A}^a_\mu + f^{abc}\mathcal{A}^b_\mu \mathcal{A}^c_\nu)^2 + \bar{\psi}\left(i\not{D} + \not{\mathcal{A}}^a t^a\right)\psi \tag{61}$$

$$-\frac{1}{2g^2}(D^\mu \mathcal{A}_\mu)^2 - \bar{c}^a\left((D^2)^{ac} + f^{abc}(D^\mu \mathcal{A}_\mu)^b\right)c^c \tag{62}$$

The first two terms are simply the old Lagrangian terms, but with F and D redefined in terms of *only* the classical field A. Looking back at eqn.(59), I point out that the δ gives the gauge-fixing $D^\mu \mathcal{A}_\mu$ term, and the determinant can be written as a fermionic (technically just Grassmannian) Gaussian integral, given exactly as the term with ghosts c^a in it.

4.0.2 Surviving Gauge Invariance

Amid our gauge fixing $(D^\mu A_\mu)^a - \omega^a = 0$, we have a remaining *local* gauge symmetry left of...

$$\delta A^a_\mu = (D_\mu \beta)^a \tag{63}$$

$$\delta \mathcal{A}^a_\mu = -f^{abc}\beta^b \mathcal{A}^c_\mu \tag{64}$$

$$\delta \psi = i\beta^a t^a \psi \tag{65}$$

$$\delta c = -f^{abc}\beta^b c^c \tag{66}$$

This means that even after gauge-fixing, we are left with local gauge transformations for all of our fields. In particular, the classical field has exactly the usual gauge transformation left. Ghosts transform like gauge fields because they have to cancel gauge degrees of freedom.