Dynamics of the Inflationary Scalar Field

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I. Introduction

We have known for a long time that the Hot Big Bang picture is basically the correct description of the universe. However, the model has some issues that need to be resolved: the flatness problem, that the universe is flat to less than one percent; the horizon problem, that the universe is isotropic to less than a thousandth of a percent; and the magnetic monopole problem, that there are no magnetic monopoles to less than one in ten thousandths of a percent. Inflation is the only contender for an explanation of why these numbers are so. To date, it has not been verified, but its predictions continue to match the data at incredible accuracy.

If the universe is dominated by a scalar field ϕ with equation of state $w = -1$ then the scale factor increases exponentially with a nearly constant Hubble parameter. If the number of e-folds is greater than 60 then all three problems can be solved. In this paper we will present numerical results investigating inflation with a quadratic potential

$$V(\phi) = \frac{m^2\phi^2}{2}. \quad (1)$$

We will compute the standard results and determine if they agree with the most recent measurements from Planck. Furthermore we will investigate the generation of primordial gravitational waves and non-Gaussianities in initial density fluctuations.

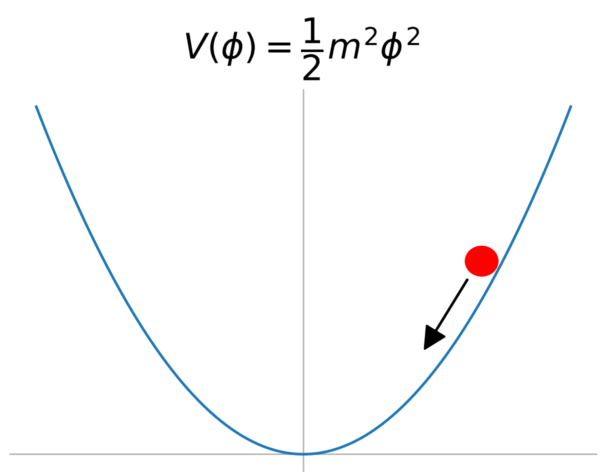

Figure 1: A classical picture of inflation. The "ball" (the inflaton field ϕ) rolls down the "hill" (the potential $V(\phi)$). The arrow denotes the velocity of the ball.

II. Setup

It is well known in classical field theory that the Euler-Lagrange equation for a scalar particle subject to a potential $V(\phi)$ is

$$\partial_\mu \partial^\mu \phi + V'(\phi) = 0 \tag{2}$$

where $'$ denotes a derivative with respect to ϕ. Fluctuations in the field are by assumption very small and can be ignored, so $\partial_i \partial^i = 0$. Furthermore, in an expanding universe, we must promote the partial derivatives to covariant derivatives, giving rise to a friction term $3H\dot{\phi}$, where H is the Hubble parameter and the dot denotes a derivative with respect to time. Postulating the inflaton to be a scalar field thus immediately leads us to the equation of motion for inflation

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0. \tag{3}$$

I. Dimensionless Rescaling

In order to rewrite this equation in dimensionless units, we define $\widetilde{\phi} \equiv \phi / M_{PL}$ and $u \equiv \ln a$. Then, using the fact that $\frac{d}{dt} = H\frac{d}{du}$, we find that equation (3) becomes

$$\begin{aligned} H\frac{d}{du}\left(H\frac{d\widetilde{\phi}}{du}\right) + 3H^2\frac{d\widetilde{\phi}}{du} + \frac{V'(\phi)}{M_{PL}} &= 0 \\ \implies H^2\frac{d^2\widetilde{\phi}}{du^2} + H\left(\frac{d\widetilde{\phi}}{du}\right)\left(\frac{dH}{du}\right) + 3H^2\frac{d\widetilde{\phi}}{du} + \frac{V'(\phi)}{M_{PL}} &= 0 \end{aligned} \tag{4}$$

We make use of

$$\begin{aligned} H &= \sqrt{\frac{8\pi G}{3}\left(V + \frac{H^2}{2}\left(\frac{d\widetilde{\phi}}{du}\right)^2\right)} \\ \implies H &= \sqrt{\frac{\frac{8\pi G}{3}V}{1 - \frac{1}{2}\left(\frac{d\widetilde{\phi}}{du}\right)^2}} \end{aligned} \tag{5}$$

where we note that H depends on u through V and $\frac{d\widetilde{\phi}}{du}$.

II. Numerical Input

It is far easier to numerically solve two first-order differential equations than the second-order differential equations. To this end, we define $\widetilde{\psi} = \frac{d\widetilde{\phi}}{du}$ to rewrite (4) as the system of equations

$$\begin{cases} \widetilde{\psi} = \frac{d\widetilde{\phi}}{du} \\ \frac{d\widetilde{\psi}}{du} = -\frac{1}{H}\frac{dH}{du}\widetilde{\psi} - 3\widetilde{\psi} - \frac{V'(\phi)}{H^2 M_{PL}} \end{cases} \tag{6}$$

but noting we need to expand the derivative of H

$$\frac{dH}{du} = \frac{dH}{d\widetilde{\phi}}\widetilde{\psi} + \frac{dH}{d\widetilde{\psi}}\frac{d\widetilde{\psi}}{du} \tag{7}$$

This finally yields a nonlinear set of equations

$$\begin{cases} \widetilde{\psi} = \dfrac{d\widetilde{\phi}}{du} \\ \dfrac{d\widetilde{\psi}}{du} = -\dfrac{1}{1 + \dfrac{1}{H}\dfrac{dH}{d\widetilde{\psi}}\widetilde{\psi}}\left[\left(\dfrac{1}{H}\dfrac{dH}{d\widetilde{\phi}}\widetilde{\psi} + 3\right)\widetilde{\psi} + \dfrac{V'(\phi)}{H^2 M_{PL}}\right] \end{cases} \tag{8}$$

For simplicity, we will evaluate the derivatives of H numerically rather than analytically.

III. Quadratic Potential

Having completed the setup of the equations in full generality, we now explicitly take the quadratic potential (1), yielding

$$V'(\phi) = \frac{d}{d\phi}\left(\frac{1}{2}m^2\phi^2\right) = m^2\phi = m^2 M_{PL}\widetilde{\phi} \tag{9}$$

and so we find that our system of differential equations applied to the quadratic potential is

$$\begin{cases} \widetilde{\psi} = \dfrac{d\widetilde{\phi}}{du} \\ \dfrac{d\widetilde{\psi}}{du} = -\dfrac{1}{1 + \dfrac{1}{H}\dfrac{dH}{d\widetilde{\psi}}\widetilde{\psi}}\left[\left(\dfrac{1}{H}\dfrac{dH}{d\widetilde{\phi}}\widetilde{\psi} + 3\right)\widetilde{\psi} + \dfrac{m^2}{H^2}\widetilde{\phi}\right] \end{cases} \tag{10}$$

where the Hubble parameter is

$$H = \sqrt{\frac{\frac{4\pi}{3}m^2\widetilde{\phi}^2}{1 - \frac{1}{2}\left(\frac{d\widetilde{\phi}}{du}\right)^2}}. \tag{11}$$

Here we took as a definition $G = \frac{1}{M_{PL}^2}$. We take as our initial conditions $m = 10^{-6}M_{PL}$, $\phi_0 = 17M_{PL}$ (equivalently, $\widetilde{\phi}_0 = 17$), and zero initial velocity. Note that H has the same units as m so the term with $\frac{m^2}{H^2}$ in equation (10) can simply be input into the code as $m = 10^{-6}$—it does not depend on m.

III. Results

In this section, all values will be plotted against the number of e-folds before the end of inflation. Thus, time actually runs to the left.

I. The Inflaton

In Figure 2 we present the value of the inflaton field as it rolls down its potential. It is clear that it spends a lot of time near the initial point, corresponding to the fact that during inflation the Hubble parameter is large and so there is a lot of friction due to the expansion of the universe.

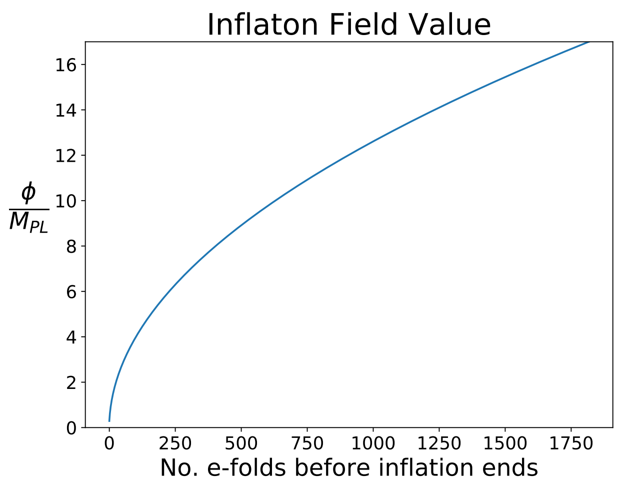

Figure 2: Numerical results for the time evolution of the inflaton during inflation for the quadratic potential.

```
#Definitions
G = 1 #1/Mpl^2
m = 10**-6 #Mpl
def H(phi,psi):
    return np.sqrt(8*np.pi*G/3*(1./2*m**2*phi**2))/np.sqrt(1-1./2*psi**2)
def dHdphi(phi,psi):
    eps = 10**-5
    return (H(phi+eps,psi)-H(phi,psi))/eps
def dHdpsi(phi,psi):
    eps = 10**-5
    return (H(phi,psi+eps)-H(phi,psi))/eps

#Calculate derivatives
def d_dlnus(var, lna, params):
    phi, psi = var    # unpack current values of the variables
    H0, m = params     # unpack parameters
    derivs = [psi,    # derivative of phi
             -1./(1+1/H(phi,psi)*dHdpsi(phi,psi)*psi) #derivative of psi
             *((1/H(phi,psi)*dHdphi(phi,psi)*psi + 3)*psi+m**2/H(phi,psi)**2*phi)]
    return derivs
```

```
#Setup
phi0 = 17
psi0 = 0
H0 = H(phi0,psi0)
params = [H0, m]
var0 = [phi0, psi0]

#Calculate over this u=lna
lnaStop = 2000
num_points = 10000
lna = np.linspace(0, lnaStop, num_points)

#Call solver
sol = odeint(d_dlnus, var0, lna, args=(params,))

#Solutions
phi = sol[:,0]
```

II. Slow Roll Parameters

For inflation to take place, the field needs to be in "slow-roll," $\dot{\phi}^2/2 \ll V(\phi)$. It can be shown that this is equivalent to

$$\epsilon = \frac{M_{PL}^2}{16\pi}\left(\frac{V'}{V}\right)^2 \ll 1 \tag{12}$$

$$\eta = \frac{M_{PL}^2}{8\pi}\left(\frac{V''}{V}\right) \ll 1 \tag{13}$$

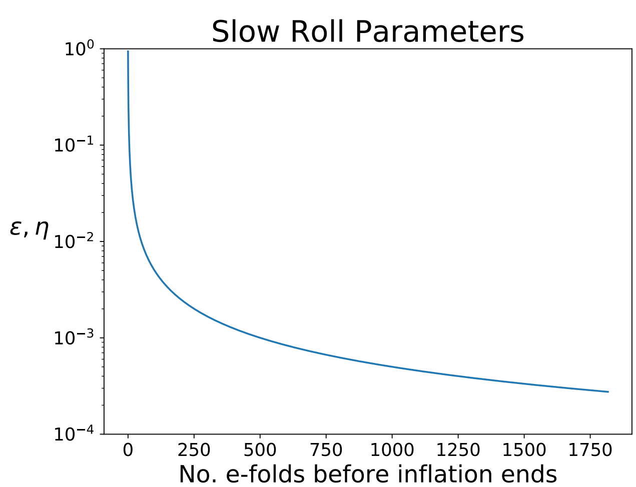

Figure 3: Numerical results for the time evolution of the slow roll parameters during inflation. For the quadratic potential, $\epsilon = \eta$.

It is worth noting that for the quadratic potential $\epsilon = \eta = \dfrac{1}{4\pi\tilde{\phi}^2}$. In Figure 3 we plot our results for the evolution of these parameters during inflation.

```
#Define Slow Roll Parameters
eps = 1./(4*np.pi*phi**2)
eta = eps
```

III. Number of e-folds

By definition, inflation ends when $\epsilon = 1$ or $\eta = 1$. We can then define the number of e-folds of inflation

$$N = \ln\left(\frac{a_f}{a_i}\right) \tag{14}$$

where for convenience we have defined in the code $a_i = 1$ (an arbitrary rescaling) and we define a_f as the scale factor at the first time $\epsilon \geq 1$. Our code finds that

$$\boxed{N = 1816.} \tag{15}$$

Inflation only requires 60 e-folds to solve the problems of the Hot Big Bang model, so our model is consistent.

```
#Find where inflation stops
#I've normalized such that a = 1 when inflation starts
#a_init = 1
arg_crit = np.argmax(eps>1)
#N = np.log(a_final/a_init), with which our normalization is simply
N = lna[arg_crit]
```

IV. Spectral Index

To have a universe without excessive mass fluctuations on extreme scales the power spectrum should be a power law $P(k) = Ak^n$ where n is the spectral index. To a good approximation, inflation generically predicts

$$n = 1 - 6\epsilon + 2\eta + \mathcal{O}(\epsilon^2, \eta^2). \tag{16}$$

where this is evaluated 60 e-folds before the end of inflation. Up to the higher order terms, our code predicts

$$\boxed{n = 0.967.} \tag{17}$$

Figure 4 we plot our results for the evolution of the spectral index during inflation.

```
#Find spectral index
n = 1 - 6*eps + 2*eta
n_60 = np.interp(N-60,lna,n)
```

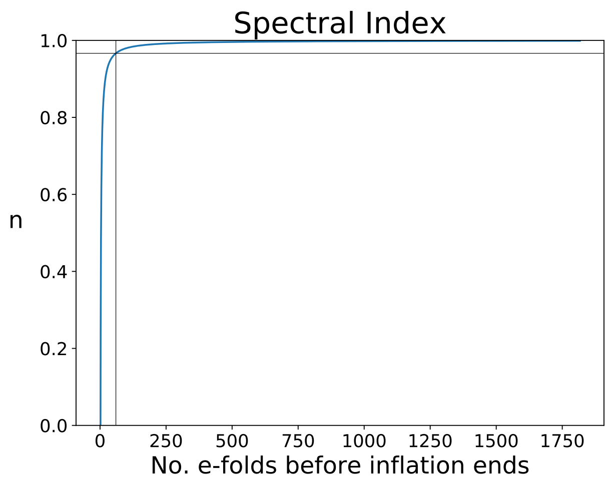

Figure 4: Numerical results for the time evolution of the spectral index during quadratic inflation. The vertical line denotes 60 e-folds before inflation ends, which is important because this is when the current cosmological parameters were imprinted. The horizontal line denotes the value of the spectral index at this time, $n = 0.967$.

V. Density Perturbations

Due to the quantum properties of the inflaton field, we expect that there will be small spatial fluctuations in ϕ giving rise to terms modifying equation (3). It can be shown that the variance of the resulting density perturbations is

$$\delta_H^2 = \frac{512\pi}{75 M_{PL}} \left(\frac{V^3}{V'^2} \right). \tag{18}$$

These fluctuations correspond to matter fluctuations, and will eventually collapse and form galaxies. Our code yields

$$\boxed{\delta_H = 1.57 \times 10^{-5}} \tag{19}$$

Figure 5 we plot our results for the evolution of the amplitude of density fluctuations during inflation.

```
#Find density perturbations
delta_H = np.sqrt(512*np.pi/(75*8)*m**2*phi**4)

#60 e-folds before inflation
delta_60 = np.interp(N-60,lna,delta_H)
```

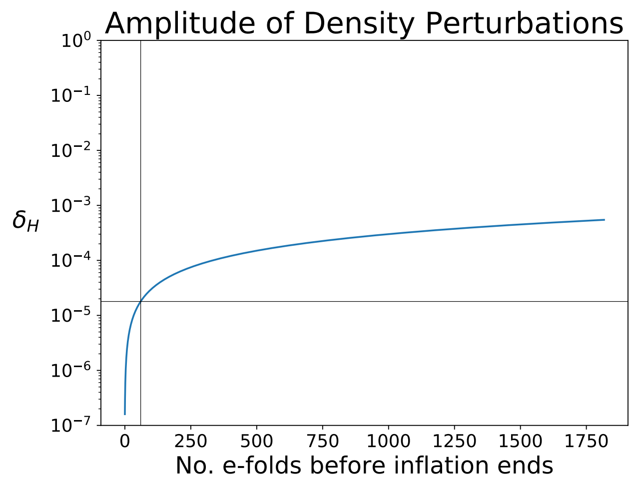

Figure 5: Numerical results for the generation of density perturbations during quadratic inflation. The vertical line denotes 60 e-folds before inflation ends, which is important because this is when the current cosmological parameters were imprinted. The horizontal line denotes the amplitude of density perturbations at this time, $\delta_H = 1.57 \times 10^{-5}$.

VI. Consistency with *Planck* measurements

The latest results from *Planck* measure $n = 0.968 \pm 0.006$. They measure $\delta_H \approx 10^{-5}$. This model of inflation is in good agreement with these measurements.

To obtain inflation with $N \geq 60$, we require $\phi_0 > 3.16$. Clearly, as ϕ_0 increases, the number of e-folds of inflation also increases. Figure 6 shows this explicitly. For small ϕ_0, we don't get 60 e-folds of inflation, so n is not even defined—the model does not solve the problems it is supposed to. For all ϕ_0 not already ruled out by the $N \geq 60$ constraint, n is consistent with the *Planck* measurements. This is shown in Figure 7.

```
#Definitions
G = 1 #1/Mpl^2
m = 10**-6 #Mpl
def H(phi,psi):
    return np.sqrt(8*np.pi*G/3*(1./2*m**2*phi**2))/np.sqrt(1-1./2*psi**2)
def dHdphi(phi,psi):
    eps = 10**-5
    return (H(phi+eps,psi)-H(phi,psi))/eps
def dHdpsi(phi,psi):
    eps = 10**-5
    return (H(phi,psi+eps)-H(phi,psi))/eps

#Calculate derivatives
def d_dlnus(var, lna, params):
    phi, psi = var     # unpack current values of the variables
    H0, m = params      # unpack parameters
    derivs = [psi,     # derivative of phi
            -1./(1+1/H(phi,psi)*dHdpsi(phi,psi)*psi) #derivative of psi
            *((1/H(phi,psi)*dHdphi(phi,psi)*psi + 3)*psi+m**2/H(phi,psi)**2*phi)]
    return derivs
```

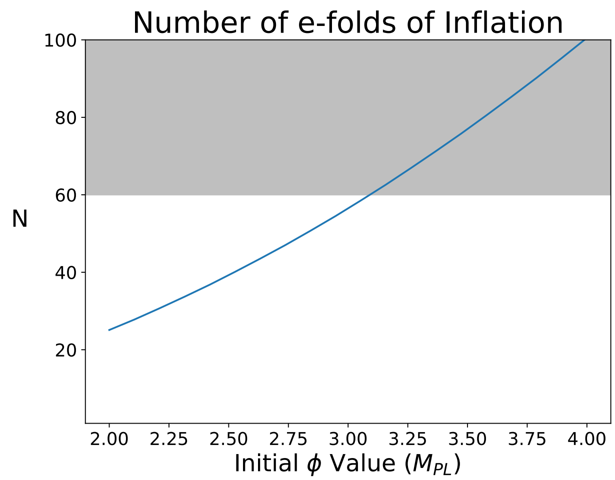

Figure 6: The number of e-folds of inflation plotted against initial inflaton field value. The shaded region denotes $N \geq 60$, where inflation solves the problems it was proposed to solve. We see that we have $N \geq 60$ for $\phi_0 \geq 3.16\ M_{PL}$. From there, N increases without bound.

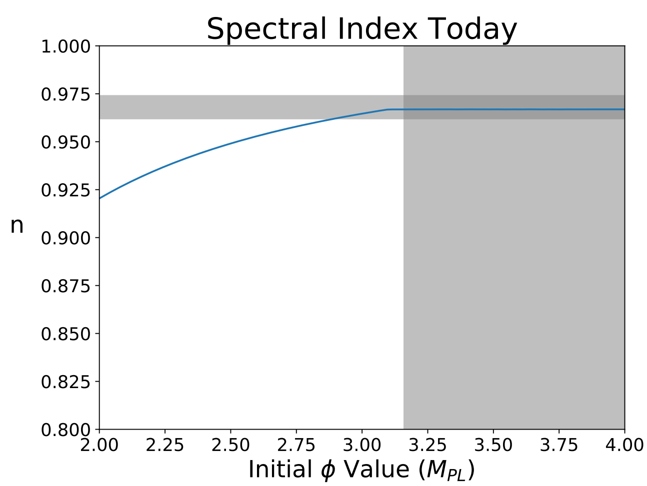

Figure 7: Spectral index 60 e-folds before inflation ended plotted against initial inflaton field value. The vertical shaded region denotes the *Planck* allowed region for n. The horizontal shaded region denotes the region in which our code delivers 60 e-folds. For values of ϕ_0 that are not ruled out by N requirements, the spectral index is within the required region. As ϕ_0 increases, n stays in the allowed region to at least $\phi_0 = 1000\ M_{PL}$.

```
#Calculate over this u=lna
lnaStart = 0
lnaStop = 1000
num_points = 10000
lna = np.linspace(lnaStart, lnaStop, num_points)

for i, phi0 in enumerate(phi_range):
    #Setup
    psi0 = 0
    H0 = H(phi0,psi0)
    params = [H0, m]
    var0 = [phi0, psi0]

    #Call solver
    sol = odeint(d_dlnus, var0, lna, args=(params,))

    #Solutions
    phi = sol[:,0]

    eps = 1./(4*np.pi*phi**2)
    eta = eps

    #Find where inflation stops
    arg_crit = np.argmax(eps>1)
    N = lna[arg_crit]
    Nefs[i] = N

    #Find n
    n = 1 - 6*eps + 2*eta
    n_60 = np.interp(N-60,lna,n)
    ns[i] = n_60
```

IV. Project Extensions

I. Tensor Spectral Index

As in the case of the matter power spectrum, there is a tensor power spectrum that is generically predicted to be of the form Ak^{n_t} where n_t is the tensor spectral index. In terms of the slow roll parameters, we have

$$n_t = -2\epsilon \tag{20}$$

where of course this should be evaluated 60 e-folds before inflation ends. Our code finds

$$\boxed{n_t = -0.017} \tag{21}$$

```
#Find tensor spectral index
n_t = -2*eps_60
```

II. Generation of Gravitational Waves

Just like the quantum nature of the inflaton gave rise to density perturbations in the matter (scalar) power spectrum, it can also generate density perturbations in spacetime (the tensor power spectrum). It can be shown that the ratio of these spectra, the tensor-to-scalar ratio, is

$$r = 16\epsilon. \tag{22}$$

Our analysis yields the result

$$\boxed{r = 0.13.} \tag{23}$$

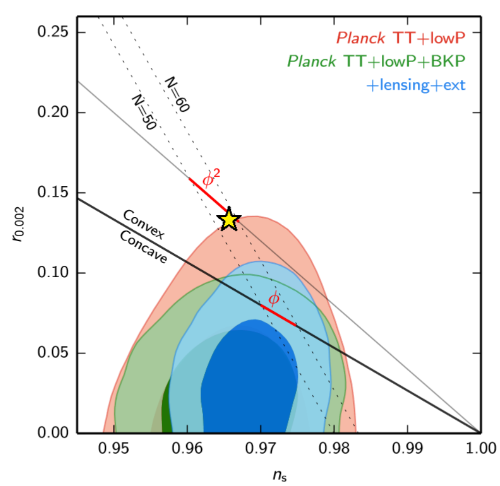

Figure 8: *Planck* results in the (n,r) plane. Here our notations differ, n_s corresponds to the scalar spectral index n and $r_{0.002}$ corresponds to the tensor-to-scalar ratio r. The current constraints are plotted in red, green, or blue, depending on how much data is used. The star denotes our model's point in parameter space, (n,r) = (0.13, 0.967). We see it matches with the *Planck* results, as it lies on the quadratic potential line.

This is actually excluded by the most recent *Planck* result, $r < 0.11$, which can be seen in Figure 8. This is the reason that power-law potentials are now disfavored. This is too bad—if equation (1) was the true model for inflation, *Planck* would have found primordial gravitational waves!

```
#Find tensor-to-scalar ratio
r = 16*eps_60
```

III. Energy Scale of Inflation

The tensor to scalar ratio is proportional to the energy scale of inflation,

$$V^{1/4} = \left(\frac{r}{0.01}\right)^{1/4} \times 10^{16} \text{ GeV} \tag{24}$$

If we substitute in our value for r, we obtain

$$\boxed{V^{1/4} = 1.9 \times 10^{16} \text{ GeV}} \tag{25}$$

Of course, we can also calculate this directly, by finding the value of ϕ 60 e-folds before the end of inflation. This yields

$$\phi_{60} = 1.8 \times 10^{16} \text{ GeV} \tag{26}$$

which happily is fairly consistent.

```
#Find energy scale of inflation
#Theoretical scale
Vth = (r/.01)**(1./4)*10**16

#Computed scale
phi_60 = np.interp(N-60,lna,phi)
Vcp = (0.5*m**2*phi_60**2*(1.22*10**19)**4)**(1./4)
```

IV. Non-Gaussianities

To a good approximation, the density fluctuations produced by inflation are Gaussian, which is in fact what we have assumed so far. Now we investigate the departures from Gaussianity that arise from inflationary models. These departures arise, in part, because general relativity is a non-linear theory. A good place to start is the bispectrum (the three-point function). The bispectrum is zero for Gaussian perturbations, and generically could be nonzero for departures. The corrections we expect to come in at quadratic order, and theory predicts the perturbations to be of the form

$$\zeta(x) = \zeta_G(x) - \frac{3}{5} f_{NL} \left(\zeta_G(x)^2 - \langle \zeta_G(x)^2 \rangle\right) \tag{27}$$

where ζ_G is Gaussian with standard deviation $\sigma = \delta_H$ and ζ is the "true" distribution, containing the departure from Gaussianity. The factor of 3/5 comes from defining the perturbations in terms of gravitational potentials, and the f_{NL} factor describes the strength of the nonlinearities. For single-field inflation, it can be shown that

$$f_{NL} = \frac{5}{12} (n - 1) \tag{28}$$

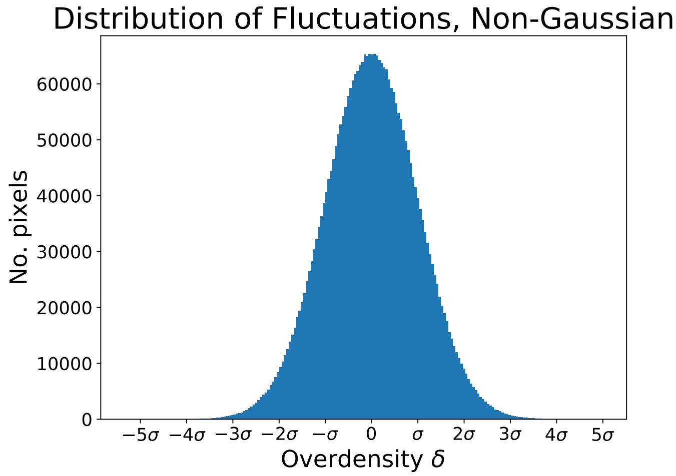

Figure 9: Monte Carlo results for the non-Gaussian distribution (27). The x-axis plots overdensity $\delta = \dfrac{\rho - \bar{\rho}}{\bar{\rho}}$. The Gaussian distribution clearly dominates, and the non-Gaussianities cannot be seen. Note the approximate scale is $\sigma = \delta_H = 1.57 \times 10^{-5}$.

Our code finds

$$\boxed{f_{NL} = 0.014} \tag{29}$$

I have performed Monte Carlo analysis on the resulting distribution. I sampled the distribution (27) 12×512^2 times, the number of pixels on a *nside* = 512 HEALPY map that might be used for research purposes. By eye, of course, the corrections are not perceptible (and *Planck* found no evidence for them, either!). The distribution is shown in Figure 9. By examining the quadratic corrections, we can understand why they are so hard to find. These are shown in Figure 10. The average shift in overdensity $\delta = \dfrac{\rho - \bar{\rho}}{\bar{\rho}}$ due to these non-Gaussianities is $\frac{3}{5} f_{NL} \sigma^2$, where ρ is the density of the pixel and $\bar{\rho}$ is the average density over the sky. The ratio between this average shift and σ is $\frac{3}{5} f_{NL} \sigma = 1.3 \times 10^{-7}$. Finding these non-Gaussianities is clearly a Herculean task, but success in this area would be one of the most incredible achievements of the next century. I eagerly await the results of the next-generation CMB experiments.

```
#Define a gaussian and the perturbation
sigma = delta_60
#Mean is zero
num = 12*512**2 #number of draws = npix on nside = 512 HEALPY map
ZetaG = np.random.normal(size=num)*sigma
#Our full non-Gaussianity
Zeta = ZetaG + 3./5*fNL*(np.square(ZetaG)-sigma**2)
```

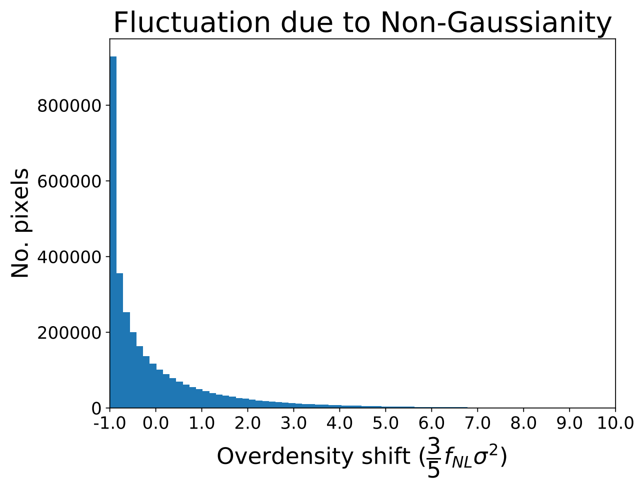

Figure 10: The shift in overdensity due to the non-Gaussian term in equation (27). These shifts are order 10^7 times smaller than σ, the standard deviation of the Gaussian distribution. This explains why they are so difficult to find.

References

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[4] D. Huterer. Inflationary seed of density fluctuations. 7:75–79, 2018.