Not all is lost: sorting and stable matching

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Summary

It is well-known that in the absence of transfers, stable matchings are positive assortative when agents’ preferences are strictly monotonic in each other’s attributes and attributes are commonly known. Instead, monotonicity is consistent with the existence of stable matchings that exhibit negative sorting when the attributes on one side of the market are private information. This paper sheds light on the scope of this consistency, showing that in most monotonic markets not every agent can know that a stable matching is negative, but not positive, assortative.

Keywords: incomplete information, sorting, stable matching.

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1 Introduction

It is well-known that in matching markets without transfers in which agents’ preferences are strictly increasing in the attribute or type of those on the other side, and all types are commonly known, stable matching are positively assortative (PAM) (see, e.g., Becker (1973)). Interestingly, however, Bikhchandani (2017) analyzes markets in which the type of agents on one side, say workers, is private information, and shows that, within monotonic environments, there can be (incomplete-information) stable matchings that are negatively assortative (NAM).

The notion of incomplete-information stability studied by Bikhchandani (2017) adapts the one introduced by Liu et al. (2014) to markets without transferable utilities, and so presumes that firms use their information “cautiously” when evaluating any blocking opportunity; namely, firms are informed of the type of their own worker and are willing to participate in a block with another worker only if they know that the type of the blocking worker is strictly better than the type of their own worker, after they have drawn every possible inference consistent with the common observation that no block takes place.

Given the epistemically permissive nature of incomplete-information stability, the existence of incomplete-information stable matchings exhibiting negative sorting should come as no surprise, even within monotonic environments. On the contrary, the element of surprise should come, this paper argues, from the following observation: in most monotonic environments, the negative assorted nature of an incomplete-information stable matching cannot be known by all firms. Let me elaborate.

When assortativeness concerns the set of matched agents, Proposition 1
shows that if, within a monotonic environment, an incomplete-information stable and nonempty outcome prescribes that only one worker is matched to the lowest-matched-firm type (LMFT), then common knowledge that the outcome is NAM, among the set of firms that have the highest-matched type, implies that some of those firms must consider possible that some of those firms considers possible that....the outcome is both NAM and PAM.\footnote{Nonempty outcomes are those in which at least one agent is matched.}

Intuitively, NAM and monotonicity jointly imply that if a complete-information block exists, then one must be formed by one of the highest-type matched firms and one of the highest-type matched workers. The assumed negative assortative nature of the allocation implies that some of those workers is matched to a firm with the lowest-matched type, but LMFT entails, in fact, that only one such worker exists. If the outcome is indeed incomplete-information stable, then at least one of the highest-matched-type blocking firms must consider possible a state at which it forms a complete-information block with \textit{no} worker at all. By monotonicity, the type of the blocking worker must be lower at this new state. If the outcome is once again NAM, but not PAM, then LMFT guarantees that another firm with the same type than the original blocking firm must form a complete-information block with the same blocking worker, and so the same argument can be repeated, lowering the type of the blocking worker even further. Eventually, a state at which the outcome is PAM must be reached. Whenever LMFT fails, the “anchor” it provides to carry out this inductive argument needs not exist, and so a “loop” in the sequence of states required by incomplete-information stability can be created, with each state in the sequence exhibiting NAM, but not PAM (Example 2).
Proposition 1 implies that it cannot be commonly known, among all agents, that a nonempty LMFT outcome is NAM, but not PAM (Corollary 1), but also delivers a very sharp prediction in a large subset of monotonic markets and LMFT outcomes: if an LMFT outcome exhibits, in addition, only one worker matched to the highest-matched-firm type (HMFT) and such a firm knows that the outcome is NAM, then the firm must also consider possible that the outcome is PAM (Corollary 2). Since the unitary bounds on the number of firms required by LMFT and HMFT concern only the “end tails” of the distribution of firms’ types, mutual knowledge that an incomplete-information stable allocation is NAM, but not PAM, is impossible in most monotonic markets.

The rest of the paper is organized as follows. Section 2 describes the environment and Section 3 the notions of complete- and incomplete-information stability. Section 4 contains the main result of the paper, along with its immediate corollaries. Section 5 shows that the content of Proposition 1 can be extended to require sorting of all agents, matched or not, and discusses the role of the common knowledge assumption embedded in Proposition 1.

2 The Environment

Let \( J = \{1, \ldots, |J|\} \subseteq \mathbb{N} \) and \( I = \{1, \ldots, |I|\} \subseteq \mathbb{N} \) be the finite sets of firms and workers in a one-to-one labor market without transfers. I will write \( i \in I \) for an individual worker and \( j \in J \) for an individual firm. The finite sets of workers and firms are, respectively, \( W := \{1, \ldots, K\} \subseteq \mathbb{N} \) and \( F := \{1, \ldots, L\} \subseteq \mathbb{N} \). I denote by \( w \in W^{|I|} \) a vector of workers’ types.
and by $f \in F^{\mid J\mid}$ a vector of firms’ types. Firms’ types are assumed to be commonly known among workers and firms, and so a vector $f$ of firms’ types will be fixed throughout. Thus, $f_j$ will denote the fixed type of firm $j$. Instead, workers are assumed to have private information about their types. Thus, I will refer to any $w \in W^{\mid I\mid}$ as a state, and write $w_i$ for the type of worker $i$.

I let $u_i(w_i, f_j)$ denote worker $i$’s utility whenever she is of type $w_i$ and is matched to firm $j$ of type $f_j$. Similarly, $v_j(w_i, f_j)$ represents firm $j$’s utility when it is of type $f_j$ and is matched to a worker of type $w_i$. The dependence of $u_i$ on $f_j$ and of $v_j$ on $w_i$ captures the idea that types describe unobservable characteristics, such as productivity. An environment is a collection $(I, J, W, F, \{u_i\}_{i \in I}, \{v_j\}_{j \in J})$.

I shall focus throughout the paper on the class of monotonic environments; namely, those satisfying the following assumption:

**Assumption 1.** An environment is monotonic if $u_i(w_i, f_j)$ is strictly increasing in $f_j$ for every $i$ and every $w_i$ and $v_j(w_i, f_j)$ is strictly increasing in $w_i$ for every $j$ and every $f_j$.

Assumption 1 is standard; it asks for the utility of every agent to be strictly increasing on the type of the agent on the other side of the market. Notice, however, that Assumption 1 does not require utilities to increase (either weakly or strictly) with respect to an agent’s own type.

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2More generally, utilities might depend on the “identity” of the assignment; namely, on the indices $i$ and $j$. This dependence would model situations where some observable characteristics, encapsulated by each agent’s index, are of value to firms and workers.
2.1 Allocations and outcomes

An allocation is a one-to-one map $\mu : I \cup J \rightarrow I \cup J$, where $\mu(i)$ denotes the firm assigned to worker $i$, $\mu(j)$ denotes the worker assigned to firm $j$, and $\mu(i) = j$ if and only if $\mu(j) = i$. I write $\mu_i$ and $\mu_j$ instead of $\mu(i)$ and $\mu(j)$. Moreover, I write $\mu_i = \emptyset$ instead of $\mu_i = i$ whenever $i$ is unassigned (and similarly for any $j$). To deal with unmatched agents, I introduce a “dummy” type 0 and let $f_{\mu_i} = 0 = w_{\mu_j}$ whenever $\mu_i = \mu_j = \emptyset$.

Without loss, I assume that $u_i(w_i, 0) = v_j(0, f_j) = 0$ for every $i$, every $j$, every $w$, and every $f$. An outcome is a triplet $(\mu, w, f)$. An outcome $(\mu, w, f)$ is nonempty if there exists $i$ with $\mu_i \neq \emptyset$.

2.2 Information

To capture firms’ uncertainty about workers’ types, I make firms’ knowledge explicit; namely, I assume that each firm $j$ possesses a partition $\Pi_j$ over some subset of $W^{|I|}$ (see Chen & Hu (2019) for a similar formulation.) I denote by $\Pi_j(w)$ the cell of firm’s $j$ partition that contains $w$. This cell describes the set of states that firm $j$ considers possible when the actual state is $w$. I will denote by $\mathcal{M}$ the meet associated with the partitions in $\Pi = \{\Pi_j\}_{j \in J}$, and refer to $\Pi$ as an information structure. There is complete information when $\Pi_j(w) = \{w\}$ for every $j$ and every $w$.

The following well-known definition will be useful later on:

**Definition 1.** Fix any information structure $\Pi$. Firm $j$ knows the event $E \subseteq W^{|I|}$ at $w$ iff $\Pi_j(w) \subseteq E$. An event $E \subseteq W^{|I|}$ is commonly known by all firms, at $w$, iff $\mathcal{M}(w) \subseteq E$.

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3Since $\emptyset < 1$, this ensures that this dummy type is strictly lower than every possible type of any firm and worker.
3 Stability

3.1 Complete-information stability

An outcome \((\mu, w, f)\) is individually rational if \(u_i(w_i, f_{\mu_i}) \geq 0\) for every \(i\) and \(v_j(w_{\mu_j}, f_j) \geq 0\) for every \(j\). Let \(\sum^0\) denote the set of individually rational outcomes, and write \(\sum^0(\mu, f)\) for the set of states at which \(\mu\) is individually rational.

An outcome \((\mu, w, f) \in \sum^0\) is complete-information blocked by a pair \((i, j)\) if \(u_i(w_i, f_j) > u_i(w_i, f_{\mu_i})\) and \(v_j(w_{\mu_j}, f_j) > v_j(w_{\mu_j}, f_j)\).

An outcome \((\mu, w, f)\) is complete-information stable if \((\mu, w, f) \in \sum^0\) and \((\mu, w, f)\) is not complete-information blocked. Let \(S\) denote the set of complete-information stable outcomes, and write \(S(\mu, f)\) for the set of states at which \(\mu\) is complete-information stable.

3.2 Incomplete-information stability

Since I allow for arbitrary information structures, the following blocking concept is a generalization of that proposed by Bikhchandani (2017), which, in turn, adapts the one introduced by Liu et al. (2014) to markets without transfers.

Definition 2. Fix an information structure \(\Pi\), and any nonempty set \(X \subseteq \sum^0\). An outcome \((\mu, w, f) \in X\) is \(X\)-blocked if there is a pair \((i, j)\) such that:

1. \(u_i(w_i, f_j) > u_i(w_i, f_{\mu_i})\); and

2. for every \(w' \in \Pi_j(w) \cap X\) such that \(u_i(w'_i, f_j) > u_i(w'_i, f_{\mu_i})\) and \(w'_j = w_{\mu_j}\), we have...
\[ v_j(w'_i, f_j) > v_j(w_{\mu_j}, f_j). \] (1)

To get an intuitive grasp, let \( X = \sum^0 \). Then, a firm forms a \( \sum^0 \)-block with a given worker if and only if the firm *knows*, after accounting for the individually rational nature of the allocation, the type of its own worker, and the worker’s willingness to participate, that they form a complete-information block.

Define, for every \( k \geq 1 \):

\[
\sum_k := \{ (\mu, w, f) : (\mu, w, f) \in \sum^{k-1} \text{ and } (\mu, w, f) \text{ is not } \sum^{k-1} \text{-blocked} \}.
\]

\[ \sum := \bigcap_{k \geq 0} \sum_k \] describes the set of all outcomes that are incomplete-information stable in the sense of Liu *et al.* (2014) when transfers are not available.\(^4\) I write \( \sum(\mu, f) \) for the set of states at which \( \mu \) is incomplete-information stable, \( \sum(w, f) \) for the set of allocations that are incomplete-information stable at \( (w, f) \), and \( \sum(\mu) \) for the set of pairs \( (w, f) \) at which \( \mu \) is incomplete-information stable. The following result, due to Bikhchandani (2017), says that an incomplete-information stable matching exists at every state.

**Lemma 1.** If \( (\mu, w, f) \in S \), then \( (\mu, w, f) \in \sum \). Thus, \( \sum(w, f) \neq \emptyset \) for every \( (w, f) \).\(^5\)

### 3.3 Common knowledge and stability

Consider a group of workers and firms, a commonly known vector \( f \), and a given information structure \( \Pi \) that describes each firm’s knowledge about

\(^4\)Notice that \( \sum \) fixes an information structure. This dependence will be omitted, however, since an information structure will be fixed throughout.

\(^5\)Lemma 1 is true for any information structure, and follows immediately from Proposition 1 in Liu *et al.* (2014).
workers’ types. Once a state $w$ is realized and an individually rational, commonly observed allocation $\mu$ is in place, the common observation that no block takes place leads every firm to refine its knowledge about workers’ types by ruling out states in which some firm considers possible that some firm considers possible that...some firm knows that it forms a block with a given worker. That is, for any information structure $\Pi$ and any $\mu$ such that $\sum(\mu) \neq \emptyset$, firms’ refined knowledge at any $w \in \sum(\mu, f)$ is described by a finer information structure $\Pi^\mu$ where, for every $j$,

$$\Pi^\mu_j(w) := \Pi_j(w) \cap \sum(\mu, f).$$

In words: $\Pi^\mu_j(w)$ captures $j$’s information, after it has made all of the possible inferences that are consistent with the common observation that no firm-worker pair blocks. Let $M^\mu$ denote the meet of $\Pi^\mu$. By construction, $M^\mu(w) := M(w) \cap \sum(\mu, f)$ contains every state in $M(w)$ except those at which some firm considers possible that some firm considers possible that...some firm knows that it forms a block with a given worker. Thus, $M^\mu(w)$ describes the smallest common knowledge event among all firms, at $w$, after they have drawn every possible inference consistent with the incomplete-information stability hypothesis.

### 4 Main result

The following two subsections describe the class of outcomes the main result will apply to. The last subsection presents the main result.
4.1 Assortative outcomes

Definition 3. An outcome \((\mu, w, f)\) is positively assortative (PAM) if, for every \(i, i'\) such that \(\mu_i \neq \emptyset\) and \(\mu_{i'} \neq \emptyset\), we have \(w_i > w_{i'} \Rightarrow f_{\mu_i} \geq f_{\mu_{i'}}\).

Negative assortativeness (NAM) can be defined analogously, replacing \(f_{\mu_i} \geq f_{\mu_{i'}}\) by \(f_{\mu_i} \leq f_{\mu_{i'}}\). Notice that these definitions of PAM and NAM account only for matched agents.\(^6\) This is standard in the literature, but Section 5.1 offers a discussion of the case in which one requires positive and negative sorting of all agents.

I will write \(\mathcal{P}\) and \(\mathcal{N}\) for the set of outcomes that are, respectively, positively and negatively assortative. Thus, \(\mathcal{P}(\mu, f)\) and \(\mathcal{N}(\mu, f)\) will denote the set of states at which \(\mu\) is positive and negative assortative, respectively.\(^7\)

Two comments are in order. First, notice that \(\mathcal{P}(\mu, f) \cap \mathcal{N}(\mu, f) \neq \emptyset\) for every \(\mu\), as every nonempty allocation is both positively and negatively assortative at every state at which either every matched worker or every matched firm has the same type.\(^8\) Second, complete-information stable outcomes are positively assortative in monotonic environments; i.e., \(S \subseteq \mathcal{P}\).

4.2 Lowest-matched-firm-type outcomes

For any pair \((\mu, f)\) where \(\mu\) is nonempty, let \(\mathcal{J}(\mu, f)\) denote the set of firms holding the lowest type in \(f\), among those that are matched in \(\mu\); i.e.:

\[\mathcal{J}(\mu, f) := \{j : f_j = \min\{f_{j'} : \mu_{j'} \neq \emptyset\}\}\]

\(^6\)This is why asking for “every \(j, j'\) such that \(\mu_j \neq \emptyset\) and \(\mu_{j'} \neq \emptyset\), \(f_j > f_{j'} \Rightarrow w_{\mu_j} \geq w_{\mu_{j'}}\)” in PAM is redundant.

\(^7\)Recall that a vector \(f\) is fixed throughout, so that a state is only a description of workers’ types.

\(^8\)Notice that empty outcomes are both PAM and NAM since, by convention, \(f_{\mu_i} = 0\) whenever \(\mu_i = \emptyset\).
Definition 4. A nonempty outcome \((\mu, w, f)\) satisfies the lowest-matched-firm-type (LMFT) property if \(J(\mu, f)\) is a singleton.

In words, LMFT requires that there is only one matched firm holding the lowest-matched type; i.e., the lowest type in \(f\), among all matched firms. Notice that LMFT is a joint condition on \(\mu\) and \(f\), both commonly known among workers and firms, but does not depend on the state.

I view LMFT as a weak requirement, as it concerns only the “left end tail” of the distribution of firms’ types, among those that are matched. In particular, LMFT is weaker than requiring that there are no two firms with the lowest type in \(f\), as it concerns only the set of matched firms, and much weaker than requiring different firms to be of different types.\(^9\)

4.3 Sorting and common knowledge

This section describes the main finding of the paper. To do so, however, a few more symbols are necessary.

For any pair \((\mu, f)\) where \(\mu\) is nonempty, let \(\bar{J}(\mu, f)\) denote the set of firms holding the highest type in \(f\), among those that are matched in \(\mu\); i.e.:
\[
\bar{J}(\mu, f) := \{ j : f_j = \max \{ f_\tilde{j} : \tilde{j} \in \mu \neq \emptyset \} \}.
\]

For any nonempty outcome \((\mu, w, f)\) \(\in \sum\), I will write \(M_{\bar{J}(\mu, f)}^\mu(w)\) for the meet of the partitions in \(\{\Pi_j^\mu(w)\}_{j \in \bar{J}(\mu, f)}\). Thus, \(M_{\bar{J}(\mu, f)}^\mu(w)\) is the smallest common knowledge event, at \(w\), among firms in \(\bar{J}(\mu, f)\).

The following is the main result of the paper:

Proposition 1. Suppose that Assumption 1 holds, and fix any nonempty outcome \((\mu, w, f)\) \(\in \sum\). If \((\mu, w, f)\) satisfies the LMFT property, then

\(^9\)In fact, LMFT does not even rule out nonempty outcomes at which every firm has the same type.
\[
\mathcal{M}_{\bar{J}(\mu,f)}^\mu(w) \subseteq \mathcal{N}(\mu, f) \Rightarrow \mathcal{M}_{\bar{J}(\mu,f)}^\mu(w) \cap \mathcal{P}(\mu, f) \neq \emptyset.
\]

Notice that \(\mathcal{M}_{\bar{J}(\mu,f)}^\mu(w) \subseteq \mathcal{N}(\mu, f)\) formalizes the statement “it is common knowledge among firms in \(\bar{J}(\mu, f)\), at \(w\), that \(\mu\) is negatively assortative.” On the other hand, \(\mathcal{M}_{\bar{J}(\mu,f)}^\mu(w) \cap \mathcal{P}(\mu, f) \neq \emptyset\) formalizes the statement “it is not common knowledge among firms in \(\bar{J}\), at \(w\), that \(\mu\) is not positively assortative.”

The proof of Proposition 1 can be found in the appendix, but a rough intuition of its content goes as follows: starting from any outcome satisfying the stated assumptions and “moving through the meet” of the partitions of all matched firms holding the highest-matched type to justify their unwillingness to participate in a complete-information block with the worker of the firm with the lowest-matched type, one must eventually reach a state at which either none of the firms with the highest-matched type is involved in a complete-information block—at which point the outcome is complete-information stable, and therefore PAM—or every worker has the lowest possible type—at which point the outcome is complete-information stable, and therefore PAM.

Two comments about Proposition 1 are in order. First, Proposition 1 does not say that if an LMFT outcome is incomplete-information stable, then there cannot be common knowledge that it is not PAM. Instead, it says that such nonexistence of common knowledge arises when, in particular, there is common knowledge that the outcome is NAM. Put another way, Proposition 1 implies that if for some LMFT and incomplete-information stable outcome there is common knowledge among firms that the outcome is not PAM, then some firm must consider possible that some firm considers possible that...the outcome exhibits “mix” sorting. Second, neither LMFT
nor the required existence of common knowledge of NAM can be dispensed with. The latter is illustrated in Section 5.2, and the former in Example 1 below.

Two corollaries of Proposition 1 are immediate. The first one states that the content of Proposition 1 is also true for the set of all firms.

**Corollary 1.** Suppose that Assumption 1 holds, and fix any nonempty outcome \((\mu, w, f) \in \sum\). If \((\mu, w, f)\) satisfies the LMFT property, then

\[
\mathcal{M}^\mu(w) \subseteq \mathcal{N}(\mu, f) \Rightarrow \mathcal{M}^\mu(w) \cap \mathcal{P}(\mu, f) \neq \emptyset.
\]

Corollary 1 is true simply because, at every state, every event that is commonly known among all firms must also be commonly known among any subgroup of them.

The second corollary of Proposition 1 concerns nonempty LMFT outcomes that, in addition, exhibit only one matched firm holding the highest-matched type. This last condition is defined as follows:

**Definition 5.** A nonempty outcome \((\mu, w, f)\) satisfies the highest-matched-firm-type (HMFT) property if \(\bar{J}(\mu, f)\) is a singleton.

Notice that HMFT imposes a unitary bound on the “right end tail” of \(f\), but only among those firms that are matched. For outcomes that satisfy both LMFT and HMFT, the content of Proposition 1 is quite sharp:

**Corollary 2.** Suppose that Assumption 1 holds, and fix any nonempty outcome \((\mu, w, f) \in \sum\). If \((\mu, w, f)\) satisfies the LMFT property and \(\bar{J}(\mu, f) = \{j\}\), then

\[
\Pi_{j}^\mu(w) \subseteq \mathcal{N}(\mu, f) \Rightarrow \Pi_{j}^\mu(w) \cap \mathcal{P}(\mu, f) \neq \emptyset.
\]
In words, \( j \) cannot know that an outcome satisfying the stated assumptions is NAM without also considering possible that it is PAM. To the extent that both LMFT and HMFT are not demanding conditions, Corollary 2 implies that in a large number of monotonic markets it cannot be mutually known that an incomplete-information stable outcome is NAM, but not of PAM.

**Example 1.** There are three workers and three firms, \( I = \{i_1, i_2, i_3\} \) and \( J = \{j_1, j_2, j_3\} \). The utility of every \( i \) is given by \( u_i(w_i, f_j) = f_j \) and the utility of every \( j \) by \( v_j(w_i, f_j) = w_i \). Thus, the environment is monotonic. Suppose that \( W = \{1, \ldots, 4\} \) and \( F = \{2, 3\} \) are the sets of types for workers and firms, where \( f_{j_1} = f_{j_2} = 2 \), and \( f_{j_3} = 3 \). Consider, in particular, the profiles \( w \) and \( w' \), where

\[
\begin{align*}
w_{i_2} &= w_{i_3} = 2, \text{ and } w_{i_1} = 4, \text{ and} \\
w'_{i_1} &= w'_{i_3} = 2, \text{ and } w'_{i_2} = 4.
\end{align*}
\]

Imagine the allocation \( \mu \) that assigns \( i_1 \) to \( j_1 \), \( i_2 \) to \( j_2 \) and \( i_3 \) to \( j_3 \). Consider now an information structure that prescribes:

\[
\Pi_{j_1}(w) = \Pi_{j_2}(w) = \{w\} \\
\Pi_{j_1}(w') = \Pi_{j_2}(w) = \{w'\}, \text{ and} \\
\Pi_{j_3}(w) = \Pi_{j_3}(w') = \{w, w'\}.
\]

Clearly, \( \{w, w'\} \subseteq \sum(\mu, f) \cap \mathcal{N}(\mu, f) \) and \( \{w, w'\} \cap S(\mu, f) = \emptyset \). Hence, \( \{w, w'\} \cap \mathcal{P}(\mu, f) = \emptyset \). Since \( \Pi_j^\mu(w) = \Pi_j(w) \) and \( \Pi_j^\mu(w') = \Pi_j(w') \) for every \( j \), there is common knowledge among all firms that \( \mu \) is both NAM and not PAM. Finally, notice that both \((\mu, w, f)\) and \((\mu, w', f)\) are nonempty, but neither satisfies LMFT—because there are two firms, \( j_1 \) and \( j_2 \), with type 2, the lowest-matched type.
Since the outcome in Example 1 satisfies both LMFT and HMFT, Example 1 illustrates that LMFT can be dispensed with neither in Proposition 1 nor in Corollary 2.

5 Discussion

5.1 Sorting all agents

Extending the content of Proposition 1 to require sorting of all agents, matched or not, runs into two difficulties, one new, the other well-known. The latter is that complete-information stable outcomes need not exhibit positive sorting in monotonic environments if high-type agents find some types on the other side unacceptable. This observation carries over to incomplete-information environments, and points to the “need” of assuming that every type is acceptable. Interestingly, however, adding this assumption to the set of conditions in Proposition 1 does not suffice. This is because of the second difficulty mentioned above, driven purely by the presence of incomplete information. Plainly, assuming that every type is acceptable ensures—when monotonicity is in place—that “all firms of high type” are matched in any incomplete-information stable outcome, as pointed by Bikhchandani (2017), but does not require the same for high-type workers. As a consequence, there can be incomplete-information stable—and LMFT—outcomes involving two or more unmatched workers with high-types that jeopardize the desired result (see Example 2 below.) What this suggests, and Proposition 2 establishes below, is that when assortativeness concerns the set of all agents, the desired result is only guaranteed for outcomes that satisfy LMFT if no worker is unmatched, but otherwise exhibit,
at most, one of them.

5.1.1 Strongly assortative outcomes

Definition 6. \((\mu, w, f)\) is strongly positively assortative (SPAM) if:

1. For every \(i, i'\), \(w_i > w_{i'} \Rightarrow f_{\mu_i} \geq f_{\mu_{i'}}\) and
2. For every \(j, j'\), \(f_j > f_{j'} \Rightarrow w_{\mu_j} \geq w_{\mu_{j'}}\).

Strong negative assortativeness (SNAM) is defined analoguously, replacing \(f_{\mu_i} \geq f_{\mu_{i'}}\) by \(f_{\mu_i} \leq f_{\mu_{i'}}\), and \(w_{\mu_j} \geq w_{\mu_{j'}}\) by \(w_{\mu_j} \leq w_{\mu_{j'}}\). Notice that, unlike PAM, SPAM requires positive sorting among all workers and firms, matched or not. The same distinction applies to NAM and SNAM. Thus, SPAM and SNAM are stronger than PAM and NAM, respectively.

I will write \(P\) and \(N\) for the set of outcomes that are, respectively, strongly positively and strongly negatively assortative. Thus, \(P(\mu, f)\) and \(N(\mu, f)\) will denote the set of states at which \(\mu\) is strongly positive and strongly negative assortative, respectively. Notice that \(P(\mu, f) \cap N(\mu, f) \neq \emptyset\) for every \(\mu\), as every nonempty allocation is both strongly positively and strongly negatively assortative at every state at which either every worker or every firm has the same type.\(^{10}\)

5.1.2 Lowest-firm-type outcomes

As mentioned above, LMFT is not sufficient for the result in Proposition 1 to be true when one replaces PAM by SPAM and NAM by SNAM in its statement, even if every agent finds every type on the other side to be acceptable. The reason, as explained above, is that incomplete-information

\(^{10}\)Notice that empty outcomes are both SPAM and SNAM since, by convention, \(f_{\mu_i} = 0\) whenever \(\mu_i = \emptyset\), and \(w_{\mu_j} = 0\) whenever \(\mu_j = \emptyset\).
stable and LMFT outcomes are compatible with the existence of more than one unmatched worker. This is illustrated by Example 2.

Example 2. Suppose that $I = \{i_1, i_2, i_3\}$ and $J = \{j\}$. Let $W = \{1, 2\}$ and $F = \{1\}$. The utility of every $i$ is given by $u_i(w_i, f_j) = f_j$ and the utility of every $j$ by $v_j(w_i, f_j) = w_i$. Thus, the environment is monotonic and every type is acceptable by every agent. Take any allocation $\mu$ in which some worker is assigned to the firm. Without loss of generality, let this worker be $i_1$. Consider, in particular, the profiles $w$ and $w'$, where

$$w_{i_1} = w_{i_2} = 1, \text{ and } w_{i_3} = 2,$$

$$w'_{i_1} = w'_{i_3} = 1, \text{ and } w'_{i_2} = 2.$$

Consider now an information structure that prescribes:

$$\Pi_{j_1}(w) = \Pi_{j_1}(w') = \{w, w'\}.$$

Clearly, $\{w, w'\} \subseteq \sum(\mu, f) \cap N(\mu, f)$ and $\{w, w'\} \cap P(\mu, f) = \emptyset$. Thus, it is common knowledge that the allocation is NAM, but not PAM. Moreover, both $(\mu, w, f)$ and $(\mu, w', f)$ are nonempty and satisfy both LMFT and HMFT.

It turns out that what is needed to “restore” the desired result, when sorting involves all agents, is the following condition:

Definition 7. A nonempty outcome $(\mu, w, f)$ satisfies the lowest-firm-type (LFT) property if

$$|\arg\min_{i'} f_{\mu, i'}| = 1.$$

To see the difference between LMFT and LFT it is convenient to notice that a nonempty outcome is LMFT if and only if $|\arg\min_{i', \mu_i \neq \emptyset} f_{\mu, i'}| = 1$. 

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Thus, LMT looks for the minimum firm-type among all workers, including those that are unmatched. Intuitively, a nonempty outcome $(\mu, w, f)$ satisfies LFT if there is at most one “firm” holding the lowest-firm-type, including 0, the type of the “empty firm.” Thus, in outcomes that satisfy LFT there can be at most one unmatched worker, but if no unmatched worker exists, no two matched firms can have the lowest-matched type in $f$, as required by LMFT. In other words, LFT and LMFT are independent properties, but they coincide with one another whenever no unmatched worker exists.\textsuperscript{11}

5.1.3 Individually rational environments

The next assumption implies that every matching is individually rational:

**Assumption 2.** An environment is individually rational if $u_i(w_i, f_j) \geq 0$ and $v_j(w_i, f_j) \geq 0$ for every $i$, every $j$, every $w_i$, and every $f_j$.

Assumption 2, which implies that every type is acceptable to every agent, is present in Bikhchandani (2017). Notice that under Assumptions 1 and 2, complete-information stable outcomes are SPAM; i.e., $S \subseteq P$.

5.1.4 Sorting and common knowledge

For any $f$, let $\hat{J}(f)$ denote the set of firms holding the highest type in $f$; i.e.,

$$\hat{J}(f) := \{j : f_j = \max\{f_j\}\}.$$

Like $\bar{J}$, $\hat{J}$ depends on the vector of firms’ types $f$. Unlike $\bar{J}$, however, $\hat{J}$ does not depend on the allocation. Thus, the difference between $\bar{J}$ and $\hat{J}$

\textsuperscript{11}Notice that LFT does not rule out nonempty outcomes at which every firm has the same type. Indeed, the number of matched firms with the lowest-type becomes irrelevant whenever there is an unmatched worker.
\( \hat{J} \) is that the latter considers the set of all firms, not only those that are matched.

For any nonempty outcome \((\mu, w, f) \in \sum\), I will write \( \mathcal{M}_j^\mu \) for the meet of the partitions in \( \{\Pi_j^\mu(w)\}_{j \in \hat{J}(f)} \). Thus, \( \mathcal{M}_j^\mu(w) \) is the smallest common knowledge event, at \( w \), among firms in \( \hat{J}(f) \).

The following result extends Proposition 1:

**Proposition 2.** Suppose that Assumptions 1 and 2 hold, and fix any outcome \((\mu, w, f) \in \sum\). If \((\mu, w, f) \) satisfies the LFT property, then

\[
\mathcal{M}_j^\mu(w) \subseteq N(\mu, f) \Rightarrow \mathcal{M}_j^\mu(w) \cap P(\mu, f) \neq \emptyset.
\]

The proof can be found in the Appendix, but its logic mimics that of Proposition 1. In particular, Example 2 above illustrates that LFT is necessary for Proposition 2 to be true for every outcome satisfying each of its assumptions. Notice that Proposition 2 is stronger than Proposition 2.\(^{12}\)

In particular, \( \mathcal{M}_{j(f)}^\mu(w) \subseteq N(\mu, f) \) is true whenever \( \mathcal{M}_{j(f)}(w) \subseteq N(\mu, f) \) holds, because \( \hat{J}(\mu, f) \subseteq \hat{J}(f) \) follows from the assumptions in Proposition 2. Moreover, implications similar to those described by Corollaries 1 and 2 can be obtained. I state them for completion.\(^{13}\)

**Corollary 3.** Suppose that Assumptions 1 and 2 hold, and fix any outcome \((\mu, w, f) \in \sum\). If \((\mu, w, f) \) satisfies the LFT property, then

\[
\mathcal{M}^\mu(w) \subseteq N(\mu, f) \Rightarrow \mathcal{M}^\mu(w) \cap P(\mu, f) \neq \emptyset.
\]

\(^{12}\)Example 2 illustrates that Proposition 1 does not imply Proposition 2.

\(^{13}\)Notice that Proposition 2 does not seem to focus on the class of nonempty outcomes. This is not the case, however, because Assumptions 1 and 2 jointly imply that every incomplete-information stable outcome is nonempty (see Proposition 1 in Bikhchandani (2017)). This observation will carry over to Corollaries 3 and 4.
The second corollary of Proposition 2 concerns nonempty LFT outcomes that, in addition, exhibit only one firm holding the highest type. This last condition is defined as follows:

**Definition 8.** A nonempty outcome \((\mu, w, f)\) satisfies the highest-firm-type (HFT) property if \(\hat{J}(f)\) is a singleton.

The difference between HMFT and HFT is that the latter looks for the maximum-type firms, among all firms in \(f\), matched or not. Thus, unlike HMFT, HFT depends only on \(f\), not on the allocation.\(^{14}\) As before, for outcomes that satisfy both LFT and HFT, the content of Proposition 2 is quite sharp:

**Corollary 4.** Suppose that Assumptions 1 and 2 hold, and fix any outcome \((\mu, w, f) \in \sum\). If \((\mu, w, f)\) satisfies the LFT property and \(\hat{J}(f) = \{j\}\), then

\[
\Pi_j^\mu(w) \subseteq N(\mu, f) \Rightarrow \Pi_j^\mu(w) \cap P(\mu, f) \neq \emptyset.
\]

To the extent that both LFT and HFT are not demanding conditions, Corollary 4 implies, once again, that in a large number of monotonic markets it cannot be mutually known that an incomplete-information stable outcome is NAM, but not of PAM.\(^{15}\)

### 5.2 Common knowledge

Is Proposition 1 true if one retains the hypothesis that the given outcome is NAM, but drops the requirement that such hypothesis is commonly known? \(^{14}\)Notice that a nonempty outcome is HMFT if and only if \(|\arg \max_{i' : \mu_i \neq 0} f_{i'}| = 1\), whereas HFT is equivalent to \(|\arg \max_i f_{\mu_i'}| = 1\).

\(^{15}\)Notice that Example 2 illustrates that LFT is also needed for Corollary 2, as the outcome in that example satisfies HFT.
Suppose that \( W \) and firms, where \( f \) utility of every \( j \) is “yes.”

In other words, is the “common knowledge part” in Proposition 1 essential for its content to be true? The following example illustrates that the answer is “yes.”

**Example 3.** There are four workers and four firms, \( I = \{i_1, i_2, i_3, i_4\} \) and \( J = \{j_1, j_2, j_3, j_4\} \). The utility of every \( i \) is given by \( u_i(w_i, f_i) = f_i \) and the utility of every \( j \) by \( v_j(w_i, f_j) = w_i \). Thus, the environment is monotonic.

Suppose that \( W = F = \{1, ..., 4\} = \{1, ..., 4\} \) is the set of types for workers and firms, where \( f_{j_1} = 1, f_{j_2} = 2, f_{j_3} = 3, \) and \( f_{j_4} = 4 \). Consider the four states \( \{w^1_1, w^2_1, w^3_1, w^4_1\} \), where

\[
\begin{align*}
  w^1_{i_1} &= w^1_{i_2} = w^1_{i_3} = 4, \text{ and } w^1_{i_4} = 3, \\
  w^2_{i_1} &= w^2_{i_2} = w^2_{i_3} = 3, \text{ and } w^2_{i_4} = 2, \\
  w^3_{i_1} &= w^3_{i_2} = 2, w^3_{i_3} = 1, \text{ and } w^3_{i_4} = 3, \text{ and } \\
  w^4_{i_1} &= w^4_{i_2} = 1, w^4_{i_3} = 4, \text{ and } w^4_{i_4} = 3. 
\end{align*}
\]

Imagine the allocation \( \mu \) that assigns \( i_1 \) to \( j_1 \), \( i_2 \) to \( j_2 \), \( i_3 \) to \( j_3 \), and \( i_4 \) to \( j_4 \). Consider an information structure that prescribes:

\[
\begin{align*}
\Pi_{j_1}(w^r) &= \{w^r\}, \text{ for } r \in \{1, ..., 4\}, \\
\Pi_{j_2}(w^1) &= \{w^1\}, \Pi_{j_2}(w^2) = \{w^2\}, \text{ and } \Pi_{j_2}(w^3) = \Pi_{j_2}(w^4) = \{w^3, w^4\}, \\
\Pi_{j_3}(w^1) &= \{w^1\}, \Pi_{j_3}(w^4) = \{w^4\}, \text{ and } \Pi_{j_3}(w^2) = \Pi_{j_3}(w^3) = \{w^2, w^3\}, \text{ and } \\
\Pi_{j_4}(w^1) &= \Pi_{j_4}(w^2) = \{w^1, w^2\}, \text{ and } \Pi_{j_4}(w^3) = \Pi_{j_4}(w^4) = \{w^3, w^4\}. 
\end{align*}
\]

It is not hard to check that \( \{w^1, w^2, w^3, w^4\} \subseteq \sum(\mu, f) \). Thus, \( \Pi^r_j(w^r) = \Pi_j(w^r) \) for every \( r \in \{1, ..., 4\} \) and every \( j \), so that \( M^{\Pi^r_j}(w^1) = \{w^1, w^2, w^3, w^4\} \).

While \( w^1 \in \mathcal{N}(\mu, f) \), \( M^{\Pi^r_j}(w^1) \subseteq \mathcal{N}(\mu, f) \), since \( \{w^1, w^2, w^3\} \cap \mathcal{N}(\mu, f) = \emptyset \). Clearly, \( M^{\Pi^r_j}(w^1) \subseteq \mathcal{P}(\mu, f) \) either. Since \( w^1 \) satisfies the LMFT property, the desired observation follows. \( \square \)
Appendix

Proof of Proposition 1

Proof. Fix any nonempty outcome \((\mu, w, f) \in \sum\) that satisfies the LMFT property, and suppose that it is commonly known among firms in \(\bar{\mathcal{J}}\), at \(w\), that \(\mu\) is NAM; i.e., that \(\mathcal{M}_\mathcal{J}(w) \subseteq \mathcal{N}(\mu, f)\).\(^{16}\)

For any \(w' \in \mathcal{M}_{\mathcal{J}(\mu, f)}(w)\), define \(\bar{k}_{w'}\) and \(\underline{k}_{w'}\) to be, respectively, the highest and lowest worker-type, in \(w'\), among all matched workers; i.e.,

\[
\bar{k}_{w'} := \max \{ w'_i : \mu_i' \neq \emptyset \} \quad \text{and} \quad \underline{k}_{w'} := \min \{ w'_i : \mu_i' \neq \emptyset \}.
\]

Since \((\mu, w, f)\) satisfies the LMFT property, let \(\mathcal{J}(\mu, f) = \{ j \}\) and \(i = \mu_j\).

Notice that if \((\mu, w', f) \in S\) for some \(w' \in \mathcal{M}(w)\), then \((\mu, w', f) \in \mathcal{P}\), so that the desired result follows.

LMFT guarantees that it is commonly known, at \(w\), among firms in \(\bar{\mathcal{J}}(\mu, f)\), that \(i\) has the highest type, among all matched workers. This is shown next:

Claim 1: \(w'_i = \bar{k}_{w'}\) for every \(w' \in \mathcal{M}_{\mathcal{J}(\mu, f)}(w)\).

Proof. Suppose, instead, that \(w'_i < \bar{k}_{w'}\) in some \(w' \in \mathcal{M}_{\mathcal{J}(\mu, f)}(w)\). By construction, there must then be some \(i'\) with \(w'_{i'} = \bar{k}_{w'}\) such that \(\mu_{i'} \neq \emptyset\).

We would then have \(w'_{i'} > w'_i\). Since \((\mu, w', f) \in \mathcal{N}\) by hypothesis, however, it would then follow that \(f_{\mu_{i'}} \leq f_{\mu_i} = f_j\). Yet this would contradict that \((\mu, w, f)\) satisfies the LMFT property. □

Let \(\bar{l}\) be the highest firm-type, in \(f\), among all matched firms; i.e.,

\[
\bar{l} := \max \{ f_{j'} : \mu_{j'} \neq \emptyset \}.
\]

\(^{16}\)Recall that \(\mathcal{J}(\mu, f) := \{ j : f_j = \max \{ f_{j'} : \mu_{j'} \neq \emptyset \} \}\) and that \(\mathcal{M}_{\mathcal{J}(\mu, f)}(w)\) denotes the meet of the partitions in \(\{ \Pi_j \}_{j \in \mathcal{J}(\mu, f)}\).
Thus, $f_j' = \bar{l}$ for every $j' \in \bar{J}(\mu, f)$. The following claim shows that it is common knowledge among firms in $\bar{J}(\mu, f)$, at $w$, that some firm in $\bar{J}(\mu, f)$ is matched to a worker with the highest type, among those that are matched; i.e.,

**Claim 2:** In every $w' \in M_{\bar{J}(\mu, f)}^\mu(w)$, there exists $\hat{j} \in \bar{J}(\mu, f)$ such that $w'_{\hat{j}} = \underline{k}^{w'}$.

**Proof.** By definition, some matched firm with type $l$ exists. Suppose, contrary to hypothesis, that there exists $w' \in M_{\bar{J}(\mu, f)}^\mu(w)$ at which no firm $j' \in \bar{J}(\mu, f)$ is matched to a worker of type $k_{\hat{j}}w'$. By construction, there exists a worker $i'$ of type $k_{\hat{j}}w'$ such that $\mu_{i'} \neq \emptyset$. By hypothesis, moreover, $f_{\mu_{i'}} < \bar{l}$. But since $\mu_{j'} \neq \emptyset$ for every $j' \in \bar{J}(\mu, f)$, we have $f_{j'} > f_{\mu_{i'}}$ and $w'_{\mu_{i'}} > w'_{j'} = \underline{k}^{w'}$, for every $j' \in \bar{J}(\mu, f)$, contradicting that $(\mu, w', f) \in \mathcal{N}$.

I show the desired result by induction, using Claims 1 and 2. If there exists some $w' \in M_{\bar{J}(\mu, f)}^\mu(w)$ such that $\bar{k}_{w'} = 1$, then $\underline{k}_{w'} = \underline{k}^{w'}$. But then, the very definitions of $\bar{k}_{w'}$ and $\underline{k}^{w'}$ imply that $w'_i = \underline{k}^{w'}$ for every $i$ such that $\mu_i \neq \emptyset$. Hence, $(\mu, w', f) \in \mathcal{P}$, as desired.

Suppose, as induction hypothesis, that the desired result is true if $\bar{k}_{w'} \leq t$ in any $w' \in M_{\bar{J}(\mu, f)}^\mu(w)$. Take, then, any $w' \in M_{\bar{J}(\mu, f)}^\mu(w)$ and assume that $\bar{k}_{w'} = t + 1$. If $\underline{k}^{w'} = t + 1$, then we are again done. Thus, assume that $\underline{k}^{w'} < t + 1$, and take any $\hat{j} \in \bar{J}(\mu, f)$ such that $w'_{\hat{j}} = \underline{k}^{w'}$. Since $(\mu, w', f) \in \sum$, monotonicity entails that there must be some $w'' \in \Pi_j^\mu(w') \subseteq M_{\bar{J}(\mu, f)}^\mu(w') = M_{\bar{J}(\mu, f)}^\mu(w)$ such that $w''_i \leq w'_{\hat{j}} = \underline{k}^{w'} < t + 1$. But then, $w''_i \leq t$. Hence, the induction hypothesis delivers the desired result. \hfill $\Box$
Proof of Proposition 2

The proof of Proposition 2 will make use of the following two definitions. An allocation \( \mu \) is \textbf{maximal} if \( |I| \leq |J| \) entails \( \mu_i \neq \emptyset \) for every \( i \) and \( |J| \leq |I| \) entails \( \mu_j \neq \emptyset \) for every \( j \). That is, maximal allocations are those in which every agent on the “short” side of the market is matched to an agent on the other side. An allocation \( \mu \) \textbf{matches all firms of high types} if \( \mu_j \neq \emptyset \) implies that \( \mu_{j'} \neq \emptyset \) for every \( j' \) such that \( f_{j'} > f_j \).

The following remark is a consequence of Assumptions 1 and 2, as highlighted by Bikhchandani (2017).

**Remark 1.** If Assumptions 1 and 2 hold, then \( (\mu, w, f) \in \sum \) implies that \( \mu \) is both maximal and matches all firms of high types.

I use Remark 1 to prove Proposition 2.

**Proof.** Fix any outcome \( (\mu, w, f) \in \sum \) that satisfies the LFT property, and suppose that it is commonly known among firms in \( \hat{J}(f) \), at \( w \), that \( \mu \) is NAM; i.e., \( \mathcal{M}_{\hat{J}(f)}^\mu(w) \subseteq N(\mu, f) \). By Remark 1, \( (\mu, w, f) \) is nonempty. Notice, further, that if \( (\mu, w', f) \in S \) for some \( w' \in M_{\hat{J}(f)}^\mu(w) \), then the desired result follows because \( S \subseteq P \). Hence, assume that \( (\mu, w', f) \notin S \) for every \( w' \in M_{\hat{J}(f)}^\mu(w) \).

The next claim shows that no firm in \( \hat{J}(f) \) can be unmatched.

**Claim 1:** \( \mu_{j'} \neq \emptyset \) for every \( j' \in \hat{J}(f) \).

**Proof.** Suppose not. That is, suppose that there is some \( j' \in \hat{J}(f) \) such that \( \mu_{j'} = \emptyset \). Since \( \mu \) is maximal and matches all firms of high types, it

\[^{17}\text{Recall that } \hat{J}(f) := \{j : f_j = \max\{f_j\}\} \text{ and that } \mathcal{M}_{\hat{J}(f)}^\mu \text{ denotes the meet of the partitions in } \{\Pi_j^\mu\}_{j \in \hat{J}(f)}; \text{ i.e., the smallest common knowledge event, at } w, \text{ among firms in } \hat{J}(f).\]
would then follow that there is some \( j' \in \tilde{J}(f) \) such that \( \mu_{j'} \neq \emptyset \) but \( \mu_j = \emptyset \) for every \( j \) with \( f_j < f_{j'} \). But then, \( (\mu, w', f) \in S \) for every \( w' \in \mathcal{M}_j^\mu(f)(w) \), contradicting our working hypothesis.

Notice that Claim 1 implies that \( \tilde{J}(f) = \tilde{J}(\mu, f) \). If \( \mu_i \neq \emptyset \), then \( \mu_{i'} \neq \emptyset \) for every \( i' \in I \), and so LFT coincides with LMFT. But then, the desired result follows from Proposition 1. To see why, suppose that that is not the case. Since no worker is unmatched and Proposition 1 guarantees that there is some \( w' \in \mathcal{M}_{\tilde{J}(\mu, f)}(w) = \mathcal{M}_{\tilde{J}(f)}(w) \) such that \( (\mu, w', f) \in P \), it must be that there exist firms \( j' \) and \( j'' \), where either \( \mu_{j'} = \emptyset \) or \( \mu_{j''} = \emptyset \), such that \( f_{j'} > f_{j''} \) but \( w_{\mu_{j'}} < w_{\mu_{j''}} \). Clearly, we must have \( \mu_{j''} \neq \emptyset \).

But since \( \mu \) matches all firms of high types, it follows that \( \mu_{j'} \neq \emptyset \), a contradiction. Hence, if \( \mu_i \neq \emptyset \) there must be some \( w' \in \mathcal{M}_{\tilde{J}(f)}^\mu(w) \) such that \( (\mu, w', f) \in P \).

Suppose now that \( \mu_i = \emptyset \). The argument for this case mimics the argument in Proposition 1. To start, notice that the fact that \( (\mu, w, f) \) satisfies property LFT implies that it is commonly known, at \( w \), among firms in \( \tilde{J}(f) \), that \( i \) has the highest-worker type. To see this, define, for any \( w' \in \mathcal{M}_{\tilde{J}(f)}^\mu(w) \), the numbers \( \bar{k}_{w'} \) and \( \underline{k}_{w'} \) to be, respectively, the highest and lowest worker-type, in \( w' \); i.e.,

\[
\bar{k}_{w'} := \max\{w'_{i'}\} \quad \text{and} \quad \underline{k}_{w'} := \min\{w'_{i'}\}.
\]

Since \( (\mu, w, f) \) satisfies the LFT property, let \( i = \arg\min_{i'} f_{\mu_{i'}} \).

**Claim 2:** \( w'_i = \bar{k}_{w'} \) for every \( w' \in \mathcal{M}_{\tilde{J}(f)}^\mu(w) \).

**Proof.** Suppose, instead, that \( w'_i < \bar{k}_{w'} \) in some \( w' \in \mathcal{M}_{\tilde{J}(f)}^\mu(w) \). By construction, there must then be some \( i' \) with \( w'_{i'} = \bar{k}_{w'} \). Since \( \mu_i = \emptyset \), but
$(\mu, w, f)$ satisfies the LFT property, it follows that $f_{\mu_i} = 0$ but $f_{\mu_i'} > 0$. But then, $w_{\mu_i'} > w_i'$ and $f_{\mu_i'} > f_{\mu_i} = 0$, contradicting that $(\mu, w', f) \in N$. \hfill \Box

Let $\bar{l}$ be the highest firm-type, in $f$; i.e., $\bar{l} := max\{f_{j'}\}$. Notice, then, that $f_{\mu_i} = \bar{l}$ for every $j' \in \hat{J}(f)$. The following claim shows that it is common knowledge among firms in $\hat{J}$, at $w$, that some firm in $\hat{J}$ is matched to a worker with the highest type.

Claim 3: In every $w' \in \mathcal{M}_{\hat{J}(f)}^\mu (w)$, there exists $\hat{j} \in \hat{J}(f)$ such that $w_{\mu_{\hat{j}}} = k_{w'}$.

Proof. Suppose, contrary to hypothesis, that there exists $w' \in \mathcal{M}_{\hat{J}(f)}^\mu (w)$ at which no firm $j'$ in $\hat{J}(f)$ is matched to a worker of type $k_{w'}$. By Claim 1 we have $\mu_{j'} \neq \emptyset$ for every $j' \in \hat{J}(f)$. Thus, $w_{\mu_{j'}} < k_{w'}$ for every $j' \in \hat{J}(f)$. By construction, there exists a worker $i'$ of type $k_{w'}$ such that, regardless of whether $\mu_{j'} \neq \emptyset$ or not, $f_{\mu_{j'}} < \bar{l}$. Yet this contradicts that $(\mu, w', f) \in N$. \hfill \Box

I show the desired result by induction, using Claims 2 and 3. If there exists some $w' \in \mathcal{M}_{\hat{J}(f)}^\mu (w)$ such that $k_{w'} = 1$, then $k_{w'} = k_{w'}$. But then, the very definitions of $k_{w'}$ and $k_{w'}$ imply that $w_{\hat{i}} = k_{w'}$ for every $\hat{i}$. Hence, $(\mu, w', f) \in P$, as desired.

Suppose, as induction hypothesis, that the desired result is true if $k_{w'} \leq t$ in any $w' \in \mathcal{M}_{\hat{J}(f)}^\mu (w)$. Take, then, any $w' \in \mathcal{M}_{\hat{J}(f)}^\mu (w)$ and assume that $k_{w'} = t + 1$. If $k_{w'} = t + 1$, then we are again done. Thus, assume that $k_{w'} < t + 1$, and take any $\hat{j} \in \hat{J}(f)$ such that $w_{\mu_{\hat{j}}} = k_{w'}$. Since $(\mu, w', f) \in \sum$, Assumption 1 entails that there must be some $w'' \in \Pi_{\hat{j}}^\mu (w') \subseteq \mathcal{M}_{\hat{J}(f)}^\mu (w') = \mathcal{M}_{\hat{J}(f)}^\mu (w)$ such that $w''_{\hat{i}} \leq w_{\mu_{\hat{j}}} = k_{w'} < t + 1$. But then, $w''_{\hat{i}} \leq t$. Hence, the induction hypothesis delivers the desired result. \hfill \Box
References


