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The kernel of a derivation

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Abstract

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Let K be a field of characteristic 0. Nagata and Nowicki have shown that the kernel of a derivation on $K[X_1, \ldots, X_n]$ is of finite type over K if $n \leq 3$. We construct a derivation of a polynomial ring in 32 variables which kernel is not of finite type over K. Furthermore we show that for every field extension L over K of finite transcendence degree, every intermediate field which is algebraically closed in L is the kernel of a K-derivation of L.

1. The counterexample

In this section we construct a derivation on a polynomial ring in 32 variables, based on Nagata's counterexample to the fourteenth problem of Hilbert. For the reader's convenience we recall the fourteenth problem of Hilbert and Nagata's counterexample (cf. [1–3]).

Hilbert's fourteenth problem. Let L be a subfield of $\mathbb{C}(X_1, \ldots, X_n)$. Is the ring $L \cap \mathbb{C}[X_1, \ldots, X_n]$ of finite type over $\mathbb{C}$?

Nagata's counterexample. Let $R = \mathbb{C}[X_1, \ldots, X_r, Y_1, \ldots, Y_r]$, $t = Y_1 Y_2 \cdots Y_r$ and $v_i = tX_i/Y_i$ for $i = 1, 2, \ldots, r$. Choose for $j = 1,2,3$ and $i = 1,2, \ldots, r$ elements $a_{j,i} \in \mathbb{C}$ algebraically independent over $\mathbb{Q}$. Define $w_j = \sum_{i=1}^{r} a_{j,i} v_i$ for $j = 1,2,3$. Define $L = \mathbb{C}(w_1, w_2, w_3, t)$. If r is a square ≥ 16 then $R \cap L$ is not of finite type.

Using this example, we will construct a derivation of R with kernel equal to

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$L \cap R$. Therefore the kernel of a derivation on $\mathbb{C}[X_1, \ldots, X_n]$ does not have to be of finite type.

Let D be the derivation

$$\frac{\partial}{\partial X_1} + X_2 \frac{\partial}{\partial X_2} + X_2 X_3 \frac{\partial}{\partial X_3} + \cdots + X_2 X_3 \cdots X_n \frac{\partial}{\partial X_n}$$

on $K(X_1, \ldots, X_n)$.

Theorem 1. *The kernel of D is equal to K.*

Proof. First we will assume that $K = \mathbb{C}$. Define complex differentiable functions $H_1, H_2, \ldots, H_n$ by $H_1(z) = z$ and $H_i(z) = \exp(H_{i-1}(z))$ for $i = 2, 3, \ldots, n$. $H_1, H_2, \ldots, H_n$ are algebraically independent over $\mathbb{C}$. Let $\mathcal{M}$ be the set of meromorphic functions on $\mathbb{C}$. Define a field-inclusion $\phi : \mathbb{C}(X_1, \ldots, X_n) \to \mathcal{M}$ by $\phi(X_i) = H_i$ for $i = 1, 2, \ldots, n$. It is easy to verify that $\phi(Df)(z) = \frac{d}{dz}\phi(f)(z)$ for all $f \in \mathbb{C}(X_1, \ldots, X_n)$ (it is sufficient to verify it for $f = X_1, \ldots, X_n$). If $Df = 0$ then $\phi(Df) = \frac{d}{dz}\phi(f) = 0$. So $\phi(f) \in \mathbb{C}$ and therefore $f \in \mathbb{C}$.

Now we will treat the general case. Suppose $f \in K(X_1, \ldots, X_n)$ with $Df = 0$. So f is a fraction of polynomials. Each polynomial is a finite K-linear combination of monomials. Therefore, one can choose $\alpha_1, \ldots, \alpha_k \in K$ such that $f \in \mathbb{Q}(\alpha_1, \ldots, \alpha_k)(X_1, \ldots, X_k)$. Let $L = \mathbb{Q}(\alpha_1, \ldots, \alpha_k)$. Then L can be embedded in $\mathbb{C}$ and so $L(X_1, \ldots, X_n)$ can be embedded in $\mathbb{C}(X_1, \ldots, X_n)$. Because $Df = 0$ we have that $f \in \mathbb{C} \cap L(X_1, \ldots, X_n) = L$. So $f \in K$. □

Corollary 2. *If K is a field with characteristic 0, then there exists a derivation D of $K(X_1, \ldots, X_n)$ with kernel K.* □

Look again at the Nagata-counterexample. Choose also $a_{j,i}$ for $j = 4, 5, \ldots, r$ and $i = 1, 2, \ldots, r$ such that all $a_{j,i}$ with $i, j \in \{1, 2, \ldots, r\}$ are algebraically independent over $\mathbb{Q}$. Define $w_j = \sum_{i=1}^{r} a_{j,i} v_i$ for $j = 1, 2, \ldots, r$, $w_j = Y_{j-r}$ for $j = r+1, r+2, \ldots, 2r-1$ and $w_{2r} = t$.

Lemma 3. $\mathbb{C}(w_1, \ldots, w_{2r}) = \mathbb{C}(X_1, \ldots, X_r, Y_1, \ldots, Y_r)$.

Proof. It is sufficient to prove that $X_1, \ldots, X_r, Y_1, \ldots, Y_r \in \mathbb{C}(w_1, \ldots, w_{2r})$. Let $A = (a_{j,i})_{j,i=1}^{r}$. Then $\det(A) \neq 0$ for otherwise there would we an algebraic relation between the $a_{j,i}$. By definition

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_r \end{pmatrix} = A \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_r \end{pmatrix}, \quad \text{so} \quad \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_r \end{pmatrix} = A^{-1} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_r \end{pmatrix}.$$

Therefore, $v_1, v_2, \ldots, v_r \in \mathbb{C}(w_1, \ldots, w_{2r})$. Furthermore, $Y_1, \ldots, Y_{r-1}, t \in \mathbb{C}(w_1, \ldots, w_{2r})$ and also $Y_r = t/Y_1Y_2 \cdots Y_{r-1} \in \mathbb{C}(w_1, \ldots, w_{2r})$. Finally, $X_i = Y_iv_i/t \in \mathbb{C}(w_1, \ldots, w_{2r})$ for all i. □

Using Lemma 3 we see that the transcendence degree of $\mathbb{C}(w_1, \ldots, w_{2r})$ over $\mathbb{C}$ equals $2r$. So $w_4, \ldots, w_{2r-1}$ are algebraically independent over $L = \mathbb{C}(w_1, w_2, w_3, w_{2r})$. Corollary 2 says that there is a derivation D of $L(w_4, \ldots, w_{2r-1})$ with kernel L. Choose some $h \in R\backslash\{0\}$ such that $hD(X_1), \ldots, hD(Y_r) \in R$. Then $E := hD$ is a derivation of R with kernel $L \cap R$.

Theorem 4. *Notations as above. Then the kernel of E on $\mathbb{C}[X_1, \ldots, X_r, Y_1, \ldots, Y_r]$ is not of finite type over $\mathbb{C}$.* □

2. The kernel of a derivation

In this section we will generalize Corollary 2 in the following way: If L is a field extension of K of finite transcendence degree such that K is algebraically closed in L then K appears as kernel of a derivation of L. In particular, every subfield of $K(X_1, \ldots, X_n)$ which is algebraically closed in $K(X_1, \ldots, X_n)$ and contains K, is the kernel of a derivation of $K(X_1, \ldots, X_n)$ over K.

Theorem 5. *Let L be a field extension of K of transcendence degree n such that K is algebraically closed in L. Then there exists a derivation of L with kernel K.*

Proof. Choose a transcendence basis $f_1, \ldots, f_n$ of L over K. Corollary 2 gives us a derivation D of $K(f_1, \ldots, f_n)$ with kernel K. Because L is an algebraic extension of $K(f_1, \ldots, f_n)$ there exists a unique extension $\tilde{D}$ of D which is a derivation on L. Suppose $\alpha \in \ker(\tilde{D})$. Then α is algebraic over $K(f_1, \ldots, f_n)$. Let $h \in K(f_1, \ldots, f_n)[X]$ be the minimum polynomial of α over $K(f_1, \ldots, f_n)$. Write

$$h(X) = X^k + h_{k-1}X^{k-1} + \cdots + h_1X + h_0$$

with $h_0, \ldots, h_{k-1} \in K(f_1, \ldots, f_n)$. Then

$$\begin{aligned} 0 &= \tilde{D}(h(\alpha)) \\ &= D(h_{k-1})\alpha^{k-1} + D(h_{k-2})\alpha^{k-2} + \cdots + D(h_1)\alpha + D(h_0) . \end{aligned}$$

So $D(h_0) = D(h_1) = \cdots = D(h_{k-1}) = 0$ because h was already the minimum polynomial of α. Then $h_0, h_1, \ldots, h_{k-1} \in \ker(D) = K$. So α is algebraic over K and therefore $\alpha \in K$. This proves that the kernel of $\tilde{D}$ is equal to K. □

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