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Nonlinearizable holomorphic group actions

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1. Introduction

The aim of this paper is to give a negative solution to the following long standing question.

Holomorphic linearization problem *Let* $G \hookrightarrow \mathrm{Aut}_{\mathrm{hol}}(\mathbb{C}^n)$ *be a complex reductive subgroup of the group* $\mathrm{Aut}_{\mathrm{hol}}(\mathbb{C}^n)$ *of holomorphic automorphisms of* $\mathbb{C}^n$*. Can one conjugate this subgroup by a single automorphism into the general linear group* $\mathrm{GL}_n(\mathbb{C}) \subset \mathrm{Aut}_{\mathrm{hol}}(\mathbb{C}^n)$*, i.e., is every action of a complex reductive group on* $\mathbb{C}^n$ *linearizable?*

The complex algebraic analogue has been studied quite well. For an overview on this subject see the article by KRAFT ([12]). The first non-linearizable complex algebraic actions were found by SCHWARZ in 1989 (see [16]), using non-trivial algebraic G-vector bundles over representations. The problem is still open for abelian groups. For the case $\mathbb{C}^\star$, see the survey by KRAFT ([11]). Recently ASANUMA ([2]) has constructed non-linearizable multiplicative group actions in the real algebraic category and also in positive characteristic.

Concerning holomorphic linearization several positive results are known: First SUZUKI proved that holomorphic $\mathbb{C}^*$-actions on $\mathbb{C}^2$ are linearizable (see [18]). His result was generalized by JIANG (see [10]) who proved using the methods of KRAFT and SCHWARZ (see [14]) that actions of reductive groups on $\mathbb{C}^n$ with one-dimensional quotient are linearizable. Surprisingly the method of constructing counterexamples to the algebraic linearization problem does not work in the holomorphic setting. It follows from the equivariant Oka-principle proved by

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HEINZNER and the second author (see [9]) that holomorphic actions arising from equivariant vector bundles over representation spaces are linearizable. Recently AHERN and RUDIN proved linearization for finite cyclic groups acting on $\mathbb{C}^2$ by automorphisms lying in a dense subgroup of $\mathrm{Aut}_{\mathrm{hol}}(\mathbb{C}^2)$ the group of overshears (see [1]). This result was generalized by KRAFT and the second author to arbitrary complex reductive groups (see [13]).

In this paper we construct non-linearizable holomorphic actions on $\mathbb{C}^n$ for all reductive groups. Our main result is (see Theorem 4.14)

Theorem *For every complex reductive Lie group G (except the trivial group) there exists a natural number N_G such that for all $l \geq N_G$ there exists an effective non-linearizable holomorphic action of G on $\mathbb{C}^l$.*

Our construction is an adaptation of a method of ASANUMA to the holomorphic setting. In [2] he uses exotic algebraic embeddings of $\mathbb{R}$ in $\mathbb{R}^3$ to construct a non-linearizable algebraic $\mathbb{R}^\star$-action on $\mathbb{R}^5$. For a field K of positive characteristic, he constructs a non-linearizable $K^\star$ action on K^4 using an exotic embedding $K \hookrightarrow K^2$. Our construction depends on the existance of non-rectifiable holomorphic embeddings of $\mathbb{C}$ into $\mathbb{C}^n$ for $n \geq 2$. We collected the facts we need about these embeddings for our construction in the second paragraph. In the third paragraph we define the Rees spaces, a holomorphic analogue of the Rees algebra, and prove some of their properties. The main idea of the construction is explained in the first part of paragraph 4, where we give counterexamples for actions of $\mathbb{C}^*$. In the remaining part of the paper we give similar constructions and use some representation theory to obtain counterexamples for all reductive groups.

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2. Non-rectifiable embeddings

In this section we explain how one constructs holomorphic embeddings of $\mathbb{C}^k$ into $\mathbb{C}^n$ $(0 < k < n)$ which are not equivalent to the standard embedding. Such embeddings were found by FORSTNERIC, GLOBEVNIK and ROSAY in [7]. Since we need some special version of their result (see Corollary 2.4) we recall the main ideas.

A holomorphic map $F : \mathbb{C}^n \to \mathbb{C}^n$ is called nondegenerate if $JF(p) \neq 0$ for some $p \in \mathbb{C}^n$, where JF is the determinant of the Jacobi matrix.

Theorem 2.1 (Rudin, Rosay) *If $n > 1$ then there exists a discrete subset $E \subset \mathbb{C}^n$ such that all nondegenerate $F : \mathbb{C}^n \to \mathbb{C}^n$ satisfy $F(\mathbb{C}^n) \cap E \neq \emptyset$ (cf. [15]).*

Such discrete subsets $E \subset \mathbb{C}^n$ are called *unavoidable*.

Theorem 2.2 (Forstneric, Globevnik, Rosay) *For any discrete subset $E \subset \mathbb{C}^2$ there exists a proper holomorphic embedding $\varphi : \mathbb{C} \to \mathbb{C}^2$ such that $\varphi(\mathbb{C}) \supset E$ (cf. [7]).*

Definition. *A proper holomorphic embedding $\varphi : \mathbb{C}^k \to \mathbb{C}^n$ is called rectifiable if there exists a $\beta \in \mathrm{Aut}_{\mathrm{hol}}(\mathbb{C}^n)$ such that*

$$\beta \circ \varphi(z_1, \dots, z_k) = (z_1, \dots, z_k, 0, \dots, 0).$$

Remark 2.3. Every automorphism of $\mathbb{C}^k \times \{0\} \subset \mathbb{C}^n$ extends to an automorphism of $\mathbb{C}^n$. It follows that a proper holomorphic embedding $\varphi : \mathbb{C}^k \to \mathbb{C}^n$ is rectifiable if and only if there exists a $\beta \in \mathrm{Aut}_{\mathrm{hol}}(\mathbb{C}^n)$ such that $\beta(\varphi(\mathbb{C}^k)) = \mathbb{C}^k \times \{0\} \subset \mathbb{C}^n$.

Take an unavoidable subset $E \subset \mathbb{C}^2$ and let $\varphi : \mathbb{C} \to \mathbb{C}^2$ be a proper holomorphic embedding such that $\varphi(\mathbb{C}) \supset E$. Define $\varphi_k : \mathbb{C} \times \mathbb{C}^k \to \mathbb{C}^2 \times \mathbb{C}^k$ by $\varphi_k(z, y) = (\varphi(z), y)$.

Corollary 2.4. *The embedding φ_k is not rectifiable.*

Proof. Suppose that φ_k is rectifiable. Then $\mathbb{C}^{2+k} \setminus \varphi_k(\mathbb{C}^{1+k}) \cong (\mathbb{C}^2 \setminus \varphi(\mathbb{C})) \times \mathbb{C}^k$ is biholomorphic to $\mathbb{C}^{k+1} \times \mathbb{C}^\star$. So there exists a non-degenerate map π from $\mathbb{C}^{2+k}$ to $\mathbb{C}^{2+k} \setminus \varphi_k(\mathbb{C}^{1+k})$, for instance the universal covering map. Let

$$p : (\mathbb{C}^2 \setminus \varphi(\mathbb{C})) \times \mathbb{C}^k \to \mathbb{C}^2 \setminus \varphi(\mathbb{C})$$

be the projection onto the first factor. The composition $p \circ \pi : \mathbb{C}^{k+2} \to \mathbb{C}^2 \setminus \varphi(\mathbb{C})$ has generically maximal rank 2. The restriction of $p \circ \pi$ to a generic two-dimensional affine subspace is nondegenerate. This contradicts the unavoidability of $E \subset \varphi(\mathbb{C})$. □

The following lemma shows that every embedding of $\mathbb{C}^k$ into $\mathbb{C}^n$ whose image lies in a linear subspace of codimension at least k is rectifiable, so adding some additional dimensions to $\mathbb{C}^n$ one can rectify every embedding inside the higher-dimensional space. We need this fact at the end of paragraph 3 to show that the product of some Rees spaces with $\mathbb{C}^l$ ($l \gg 0$) are affine spaces.

Lemma 2.5. *Let X be a Stein space. If $\varphi_1 : X \to \mathbb{C}^n$ and $\varphi_2 : X \to \mathbb{C}^m$ are two proper holomorphic embeddings, then there exists a $\beta \in \mathrm{Aut}_{\mathrm{hol}}(\mathbb{C}^n \times \mathbb{C}^m)$ such that $\beta \circ (\varphi_1 \times 0) = 0 \times \varphi_2$. (compare with Lemma 3.2 in [2]).*

Proof. By a Corollary of CARTAN's Theorem B the holomorphic map $\varphi_1 : X \to \mathbb{C}^n$ can be extended to a holomorphic map μ_1 from $\mathbb{C}^m \supseteq \varphi_2(X)$ to $\mathbb{C}^n$ (so $\mu_1 \circ \varphi_2 = \varphi_1$). Likewise there exists a holomorphic map $\mu_2 : \mathbb{C}^n \to \mathbb{C}^m$ such that $\mu_2 \circ \varphi_1 = \varphi_2$. Define $\beta_1, \beta_2 \in \mathrm{Aut}_{\mathrm{hol}}(\mathbb{C}^n \times \mathbb{C}^m)$ by $\beta_1(x, y) = (x, y + \mu_2(x))$ and $\beta_2(x, y) = (x + \mu_1(y), y)$. Now we have

$$\beta_1 \circ (\varphi_1 \times 0) = \varphi_1 \times \varphi_2 = \beta_2 \circ (0 \times \varphi_2).$$

So take $\beta = \beta_2^{-1} \circ \beta_1$. □

3. Rees spaces

In this section we introduce the Rees spaces. Important for the next paragraph are especially Example 3.5 and Corollary 3.7.

Let X, Y and Z be Stein spaces such that $X \subset Y$ and let $f \in \mathscr{O}(Z)$ be a holomorphic function on Z. Suppose that there are finitely many holomorphic functions $h_1, h_2, \ldots, h_r$ on Y generating the ideal $I_X(Y)$ of X. This is for instance true if X and Y are manifolds (see [6], Satz 5.5). Consider the map:

$$\psi : Y \times (Z \setminus \{f = 0\}) \to Y \times Z \times \mathbb{C}^r$$
$$\psi(y, z) = \left(y, z, \frac{h_1(y)}{f(z)}, \frac{h_2(y)}{f(z)}, \ldots, \frac{h_r(y)}{f(z)} \right).$$

Definition. *We define the Rees space $\mathscr{R}(X, Y, Z, f)$ as $\overline{\mathrm{Im}(\psi)}$, the holomorphic Zariski closure of the image of ψ within $Y \times Z \times \mathbb{C}^r$.*

Remark 3.1. The holomorphic Zariski closure of $\mathrm{Im}(\psi)$ within $Y \times Z \times \mathbb{C}^r$ equals the topological closure.

Remark 3.2. The definition of the Rees space doesn't depend on the choice of generators $h_1, h_2, \ldots, h_r$. Suppose that $g_1, g_2, \ldots, g_s$ also generate $I_X(Y)$, and let ψ' be the corresponding map $Y \times (Z \setminus \{f = 0\}) \to Y \times Z \times \mathbb{C}^s$. It is easy to see that there exist holomorphic maps $\varphi : Y \times Z \times \mathbb{C}^r \to Y \times Z \times \mathbb{C}^s$ and $\varphi' : Y \times Z \times \mathbb{C}^s \to Y \times Z \times \mathbb{C}^r$ such that $\psi' = \varphi \circ \psi$ and $\psi = \varphi' \circ \psi'$. Now $\varphi' \circ \varphi = \mathrm{Id}$ on $\mathrm{Im}(\psi)$, therefore $\varphi' \circ \varphi = \mathrm{Id}$ on $\overline{\mathrm{Im}(\psi)}$ and likewise $\varphi \circ \varphi' = \mathrm{Id}$ on $\overline{\mathrm{Im}(\psi')}$. So the restriction of φ defines a biholomorphic map $\overline{\mathrm{Im}(\psi)} \xrightarrow{\cong} \overline{\mathrm{Im}(\psi')}$.

Remark 3.3. If we take $Z := \mathbb{C}$ and $f := z$, then the Rees space is just the holomorphic analogue of the Rees algebra.

We will use the symbol $\cong$ to denote biholomorphic.

Lemma 3.4. *Rees spaces have the following properties:*

1. $\mathscr{R}(X, Y, Z, f) \cong \mathscr{R}(\varphi(X), Y, Z, f)$ *for* $\varphi \in \mathrm{Aut}_{\mathrm{hol}}(Y)$,
2. $\mathscr{R}(X \times W, Y \times W, Z, f) \cong \mathscr{R}(X, Y, Z, f) \times W$ *for any Stein space* W,
3. $\mathscr{R}(X \times \{0\}, Y \times \mathbb{C}^m, Z, f) \cong \mathscr{R}(X, Y, Z, f) \times \mathbb{C}^m$.

Proof. The first two properties are trivial. We show the third property. Suppose $I_X(Y)$ is generated by $h_1, h_2, \ldots, h_r \in \mathscr{O}(Y)$. Then $I_{X \times \{0\}}(Y \times \mathbb{C}^m)$ is generated by $h_1, \ldots, h_r, u_1, \ldots, u_m$ where the u_i are the coordinate functions of $\mathbb{C}^m$. Consider the map

$$\psi : Y \times \mathbb{C}^m \times (Z \setminus \{f = 0\}) \to Y \times \mathbb{C}^m \times Z \times \mathbb{C}^r \times \mathbb{C}^m$$
$$\psi(y, u, z) = (y, u, z, f(z)^{-1}.h, f(z)^{-1}.u),$$

where $h = (h_1, \ldots, h_r)$ and $u = (u_1, \ldots, u_m)$. Define
$\varphi \in \mathrm{Aut}_{\mathrm{hol}}(Y \times \mathbb{C}^m \times Z \times \mathbb{C}^r \times \mathbb{C}^m)$ by $\varphi(y, u, z, v, w) = (y, u - f(z).w, z, v, w)$.
Now we have

$$\varphi \circ \psi(y, u, z) = (y, 0, z, f(z)^{-1}.h, f(z)^{-1}.u),$$

and we get

$$\mathscr{R}(X \times \{0\}, Y \times \mathbb{C}^m, Z, f) = \overline{\operatorname{Im}(\psi)} \cong \overline{\operatorname{Im}(\varphi \circ \psi)} \cong \mathscr{R}(X, Y, Z, f) \times \mathbb{C}^m.$$

□

Example 3.5. Suppose $X \subset Y$ is given by $h = 0$ with $h \in \mathscr{O}(Y)$ ($I_X(Y) = \mathscr{O}(Y)h$). Let $\psi : Y \times (Z \setminus \{f = 0\}) \to Y \times Z \times \mathbb{C}$ given by $\psi(y, z) = (y, z, h(y)/f(z))$. Then we get

$$\mathscr{R}(X, Y, Z, f) = \overline{\operatorname{Im}(\psi)} = \{(y, z, w) \in Y \times Z \times \mathbb{C} \mid f(z)w = h(y)\}.$$

Lemma 3.6. *If $\varphi_1 : X \to \mathbb{C}^n$ and $\varphi_2 : X \to \mathbb{C}^m$ are proper holomorphic embeddings, then*

$$\mathscr{R}(\varphi_1(X), \mathbb{C}^n, Z, f) \times \mathbb{C}^m \cong \mathscr{R}(\varphi_2(X), \mathbb{C}^m, Z, f) \times \mathbb{C}^n$$

(compare with [2], Corollary 4.2).

Proof. From Lemma 2.5 and Lemma 3.4 it follows that

$$\begin{aligned}\mathscr{R}(\varphi_1(X), \mathbb{C}^n, Z, f) \times \mathbb{C}^m &\cong \mathscr{R}(\varphi_1(X) \times \{0\}, \mathbb{C}^n \times \mathbb{C}^m, Z, f) \cong \\ \cong \mathscr{R}(\{0\} \times \varphi_2(X), \mathbb{C}^n \times \mathbb{C}^m, Z, f) &\cong \mathscr{R}(\varphi_2(X), \mathbb{C}^m, Z, f) \times \mathbb{C}^n.\end{aligned}$$

□

Corollary 3.7. *For an embedding $\varphi : \mathbb{C}^k \to \mathbb{C}^n$ and $f \in \mathscr{O}(\mathbb{C}^l)$ we have*

$$\mathscr{R}(\varphi(\mathbb{C}^k), \mathbb{C}^n, \mathbb{C}^l, f) \times \mathbb{C}^k \cong \mathscr{R}(\mathbb{C}^k, \mathbb{C}^k, \mathbb{C}^l, f) \times \mathbb{C}^n \cong \mathbb{C}^{k+l+n}.$$

4. Non-linearizable group actions

The goal of this section is to find for every non-trivial reductive group a non-linearizable action on $\mathbb{C}^n$ for some n. As usually we denote by X^G the fixed point set of an action of G on X. Several times we will use the notion of categorical quotient $\pi_X : X \to X/\!/G$ for a holomorphic action of a reductive group G on a Stein space X. Recall that this is the quotient of X with respect to the equivalence relation which is given by the invariant holomorphic functions and that $X/\!/G$ carries a complex structure such that π_X is a holomorphic map. Since G-invariant holomorphic functions on a closed G-invariant subspace Y of a holomorphic Stein G-space X extend to G-invariant holomorphic functions on X, the categorical quotient $Y \to Y/\!/G$ is just given by restricting π_X to Y. Beside some easy examples of quotients for linear actions this is the only fact we use. So for further details on categorical quotients we refer to the papers of SNOW ([17]) and HEINZNER ([8]).

For the remaining part of the paper let $\varphi = \varphi_0 : \mathbb{C} \to \mathbb{C}^2$ be a non-rectifiable proper holomorphic embedding like in Corollary 2.4. Let $h : \mathbb{C}^2 \to \mathbb{C}$ be a holomorphic function whose zero set is $\varphi(\mathbb{C})$ and who vanishes there with multiplicity 1 (i.e., the gradient of h is nonvanishing on $\varphi(\mathbb{C})$). Such a function h always exists, because of the solution of the second Cousin problem.

Non-linearizable $\mathbb{C}^\star$-actions

Define

$$X = \{(x, y, z, w) \in \mathbb{C}^4 \mid h(x, y) = zw\}.$$

As in Example 3.5 X is equal to the Rees space $\mathscr{R}(\varphi(\mathbb{C}), \mathbb{C}^2, \mathbb{C}, z)$. By Corollary 3.7 we have

$$X \times \mathbb{C} \cong \mathbb{C}^4.$$

Of course we have for each $k \geq 1$ a biholomorphism $\alpha_k : X \times \mathbb{C}^k \xrightarrow{\cong} \mathbb{C}^{k+3}$.

Remark 4.1. If h is any holomorphic function on $\mathbb{C}^2$, whose zero set is biholomorphic to $\mathbb{C}$, and with a non-vanishing gradient on its zero set, then from the above discussion follows that

$$\{(x, y, z, w) \in \mathbb{C}^4 \mid h(x, y) = zw\} \times \mathbb{C} \cong \mathbb{C}^4$$

but it is not clear whether the manifold $\{(x, y, z, w) \in \mathbb{C}^4 \mid h(x, y) = zw\}$ itself is biholomorphic to $\mathbb{C}^3$. If it is not biholomorphic to $\mathbb{C}^3$ then this would be a counterexample to the following open problem:

Problem 4.2 (Holomorphic Zariski Cancellation Problem) *Let Z be a complex manifold such that $Z \times \mathbb{C}$ is biholomorphic to $\mathbb{C}^{n+1}$ ($n \geq 2$). Does it follow that $Z \cong \mathbb{C}^n$?*

Now $X \times \mathbb{C}^k \subset \mathbb{C}^4 \times \mathbb{C}^k$ is stable under the linear $\mathbb{C}^\star$-action on $\mathbb{C}^4 \times \mathbb{C}^k$ given by

$$\mathbb{C}^\star \times \mathbb{C}^4 \times \mathbb{C}^k \to \mathbb{C}^4 \times \mathbb{C}^k$$

$$\lambda \cdot (x, y, z, w, t_1, \ldots, t_k) = (x, y, \lambda z, \lambda^{-1} w, t_1, \ldots, t_k).$$

The restriction of this action to $X \times \mathbb{C}^k$ induces via α_k a holomorphic $\mathbb{C}^\star$-action $\sigma_k : \mathbb{C}^\star \times \mathbb{C}^{k+3} \to \mathbb{C}^{k+3}$.

Proposition 4.3. *The action σ_k is not linearizable ($k \geq 1$). So for all $l \geq 4$ there exists a non-linearizable $\mathbb{C}^\star$ action on $\mathbb{C}^l$.*

Proof. Suppose $\alpha_k : X \times \mathbb{C}^k \xrightarrow{\cong} \mathbb{C}^{3+k}$ is a biholomorphic $\mathbb{C}^\star$-equivariant map, where $\mathbb{C}^\star$ acts linearly on $\mathbb{C}^{3+k}$. This representation of $\mathbb{C}^\star$ on $\mathbb{C}^{k+3}$ must be isomorphic to the representation of $\mathbb{C}^\star$ on the tangent space of some fixed point of $X \times \mathbb{C}^k$. With respect to some coordinates, this action is given by

$$\lambda \cdot (z, w, u_1, \dots, u_{k+1}) = (\lambda z, \lambda^{-1} w, u_1, \dots, u_{k+1}).$$

The categorical quotient of $X \times \mathbb{C}^k$ is given by $\pi_1 : X \times \mathbb{C}^k \to \mathbb{C}^{2+k}$,

$$\pi_1(x, y, z, w, t_1, \dots, t_k) = (x, y, t_1, \dots, t_k)$$

and the categorical quotient of $\mathbb{C}^{3+k}$ is given by $\pi_2 : \mathbb{C}^{3+k} \to \mathbb{C}^{2+k}$,

$$\pi_2(z, w, u_1, \dots, u_{k+1}) = (zw, u_1, \dots, u_{k+1}).$$

The fixed point set $(X \times \mathbb{C}^k)^{\mathbb{C}^*}$ is

$$\{(x, y, z, w, t_1, \dots, t_k) \in X \times \mathbb{C}^k \mid f(x, y) = z = w = 0\}.$$

Its image under π_1 is $\varphi(\mathbb{C}) \times \mathbb{C}^k \subset \mathbb{C}^2 \times \mathbb{C}^k$. On the other hand $(\mathbb{C}^{3+k})^{\mathbb{C}^*}$ is

$$\{(z, w, u_1, \dots, u_{k+1}) \in \mathbb{C}^{3+k} \mid z = w = 0\}$$

and its image under π_2 is $\{0\} \times \mathbb{C}^{1+k} \subset \mathbb{C}^{2+k}$. Now $\alpha_k : X \times \mathbb{C}^k \xrightarrow{\simeq} \mathbb{C}^{3+k}$ induces a biholomorphism $\gamma : \mathbb{C}^{2+k} \xrightarrow{\simeq} \mathbb{C}^{2+k}$ of the categorical quotients such that $\gamma(\varphi(\mathbb{C}) \times \mathbb{C}^k) = \{0\} \times \mathbb{C}^{k+1}$. This contradicts Corollary 2.4. $\square$

Remark 4.4. If X would be biholomorphic to $\mathbb{C}^3$ then we would even have a non-linearizable $\mathbb{C}^\star$-action on $X \cong \mathbb{C}^3$. If X is not biholomorphic to $\mathbb{C}^3$, then the cancellation problem has a negative answer (see Remark 4.1).

Non-linearizable actions for finite cyclic groups

Suppose $G \cong \mathbb{Z}/n\mathbb{Z}$ ($n \geq 2$). Define a holomorphic function $f \in \mathscr{O}(\mathbb{C}^{n-1})$ by $f(z_1, z_2, \dots, z_{n-1}) = z_1 z_2 \cdots z_{n-1}$. Consider the Rees space $X := \mathscr{R}(\varphi(\mathbb{C}), \mathbb{C}^2, \mathbb{C}^{n-1}, f)$ which is given by (see Example 3.5)

$$X = \{(x, y, z_1, z_2, \dots, z_n) \in \mathbb{C}^{2+n} \mid h(x, y) = z_1 z_2 \cdots z_n\}.$$

From Corollary 3.7 we get $X \times \mathbb{C}^k \cong \mathbb{C}^{1+n+k}$ for $k \geq 1$. Let G act on $\mathbb{C}^2 \times \mathbb{C}^n$ by

$$g \cdot (x, y, z_1, z_2, \dots, z_n) = (x, y, z_2, z_3, \dots, z_n, z_1)$$

where g is a generator of G. This induces by restriction an action of G on X. If we let G act trivially on $\mathbb{C}^k$ then we get an action of G on $\mathbb{C}^{1+n+k} \cong X \times \mathbb{C}^k$ ($k \geq 1$) which we denote by σ_k. The fixed point set of G on $X \times \mathbb{C}^k$ is biholomorphic to $Y_n \times \mathbb{C}^k$ where

$$Y_n = \{(x, y, z) \mid h(x, y) = z^n\}.$$

We define a G action on Y_n by

$$g \cdot (x, y, z) = (x, y, \zeta z)$$

where ζ is a primitive n-th root of unity. If we let G act trivially on $\mathbb{C}^k$ then we get an action of G on $Y_n \times \mathbb{C}^k$ which we will denote by τ_k.

Proposition 4.5. *For $k \geq 1$ at least one of the following statements is true:*

1. *The action σ_k on $\mathbb{C}^{n+k+1}$ is not linearizable.*
2. *$Y_n \times \mathbb{C}^l \cong \mathbb{C}^{2+l}$ and the action τ_l of G on $Y_n \times \mathbb{C}^l$ is not linearizable for all $l \geq k$.*

Proof. If the action σ_k is linearizable, then its fixed point set $Y_n \times \mathbb{C}^k$ must be a vector space, so $Y_n \times \mathbb{C}^l \cong \mathbb{C}^{2+l}$ for all $l \geq k$. The fixed point set of G on $Y_n \times \mathbb{C}^l$ is isomorphic to $W \times \mathbb{C}^l$ where W is given by

$$W = \{(x, y, z) \in \mathbb{C}^3 \mid h(x, y) = z = 0\}$$

If τ_l is linearizable, then $Y_n \times \mathbb{C}^l \setminus W \times \mathbb{C}^l = (Y_n \setminus W) \times \mathbb{C}^l$ is biholomorphic to $\mathbb{C}^\star \times \mathbb{C}^{l+1}$. But this contradicts Lemma 4.6 below. □

Lemma 4.6. *For $k \geq 0$ the space $(Y_n \setminus W) \times \mathbb{C}^k$ is not biholomorphic to $\mathbb{C}^\star \times \mathbb{C}^{1+k}$.*

Proof. If $(Y_n \setminus W) \times \mathbb{C}^k$ is biholomorphic to $\mathbb{C}^\star \times \mathbb{C}^{1+k}$, then there exists a nondegenerate map π from $\mathbb{C}^{2+k}$ to $(Y_n \setminus W) \times \mathbb{C}^k$. The map ψ given by

$$(x, y, z, w_1, w_2, \dots, w_k) \in (Y_n \setminus W) \times \mathbb{C}^k \mapsto (x, y, w_1, w_2, \dots, w_k) \in (\mathbb{C}^2 \setminus \varphi(\mathbb{C})) \times \mathbb{C}^k$$

is an n-sheeted unramified covering. The composition $\pi' = \psi \circ \pi$ is a nondegenerate map $\mathbb{C}^{2+k} \to (\mathbb{C}^2 \setminus \varphi(\mathbb{C})) \times \mathbb{C}^k$. We now get a contradiction like in the proof of Corollary 2.4.

□

Remark 4.7. We do not know which of the two statements in Proposition 4.5 is true (possibly both statements are true), because we cannot decide whether $Y_n \times \mathbb{C}^l \cong \mathbb{C}^{2+l}$ for $l \geq 0$. For example if $Y_n := \{(x, y, z) \in \mathbb{C}^3 \mid h(x, y) = z^n\}$ is biholomorphic to $\mathbb{C}^2$, then the automorphism of $Y_n \cong \mathbb{C}^2$ given by $(x, y, z) \mapsto (x, y, \zeta z)$ (ζ an n-th primitive root of unity) would be a nonlinearizable automorphism of order n.

Corollary 4.8. *For $l \geq n + 2$ there exists a non-linearizable action of $\mathbb{Z}/n\mathbb{Z}$ on $\mathbb{C}^l$.*

Nonlinearizable actions for simple non-abelian groups

Define

$$X_n := \{(x, y, z_1, \dots, z_n, w_1, \dots, w_n) \in \mathbb{C}^{2n+2} \mid h(x, y) = z_1 w_1 + \dots + z_n w_n\}.$$

Lemma 4.9. *The space $X_n \times \mathbb{C}^{3.2^n - 2n - 3}$ is biholomorphic to $\mathbb{C}^{3.2^n - 2}$.*

Proof. We prove the lemma by induction. For $n = 0$ this is trivial. Suppose that $X_n \times \mathbb{C}^{3.2^n-2n-3} \cong \mathbb{C}^{3.2^n-2}$. Observe that $X_{n+1} \cong \mathscr{R}(X_n, \mathbb{C}^{2n+2}, \mathbb{C}, z)$ (see Example 3.5). By Corollary 3.7 we have

$$X_{n+1} \times \mathbb{C}^{3.2^{n+1}-2(n+1)-3} \cong \mathscr{R}(X_n, \mathbb{C}^{2n+2}, \mathbb{C}, z) \times \mathbb{C}^{3.2^n-2n-3} \times \mathbb{C}^{3.2^n-2} \cong$$
$$\mathscr{R}(X_n \times \mathbb{C}^{3.2^n-2n-3}, \mathbb{C}^{2n+2} \times \mathbb{C}^{3.2^n-2n-3}, \mathbb{C}, z) \times \mathbb{C}^{3.2^n-2} \cong \mathbb{C}^{3.2^{n+1}-2}.$$

□

Suppose that there exists a subgroup H of G, an n-dimensional representation V of G and a homogeneous G-invariant polynomial a on V of degree 2 such that:

- $V^G = \{0\}$,
- $\dim V^H = 1$ and **(*)**
- a is not identical to 0 on V^H.

Then we define

$$\begin{aligned} U_1 &:= \{(x,y,v) \in \mathbb{C}^2 \times V \mid h(x,y) = a(v)\} \\ U_2 &:= \{(x,y,v,w) \in \mathbb{C}^2 \times V \times V \mid h(x,y) = a(v) + a(w)\}. \end{aligned}$$

The spaces U_1 and U_2 carry G-actions defined by

$$\begin{aligned} g \cdot (x,y,v) &= (x,y,g \cdot v) \qquad (x,y,v) \in U_1 \subset \mathbb{C}^2 \times V, g \in G \\ g \cdot (x,y,v,w) &= (x,y,v,g \cdot w) \qquad (x,y,v,w) \in U_2 \subset \mathbb{C}^2 \times V \times V, g \in G. \end{aligned}$$

Proposition 4.10. *For $k \geq 3.2^n - 2n - 3$ we have $U_2 \times \mathbb{C}^k \cong \mathbb{C}^{k+2n+1}$ and at least one of the following statements is true:*

1. *The action of G on $U_2 \times \mathbb{C}^k$ is not linearizable.*
2. *$U_1 \times \mathbb{C}^l \cong \mathbb{C}^{l+n+1}$ for $l \geq k$ and the action of G on $U_1 \times \mathbb{C}^l$ is not linearizable.*

Proof. After a change of coordinates we have

$$a(z) = z_1^2 + z_2^2 + \ldots + z_r^2$$

for $z = (z_1, z_2, \ldots, z_n)$ with respect to some coordinates on V. So

$$a(z) + a(w) = z_1^2 + z_2^2 + \ldots + z_r^2 + w_1^2 + w_2^2 + \ldots + w_r^2$$

where $w = (w_1, w_2, \ldots, w_n)$. Again after a linear change of coordinates in $V \times V$ we get

$$a(z) + a(w) = z_1' w_1' + z_2' w_2' + \ldots + z_r' w_r'.$$

So $U_2 \cong X_r \times \mathbb{C}^{2n-2r}$ and it follows from Corollary 3.7 that $U_2 \times \mathbb{C}^k \cong \mathbb{C}^{k+2n+1}$ for all $k \geq 3.2^n - 2n - 3$. The fixed point set of G on $U_2 \times \mathbb{C}^k$ is $U_1 \times \mathbb{C}^k$. So if the action of G on $U_2 \times \mathbb{C}^k$ is linearizable then $U_1 \times \mathbb{C}^l \cong \mathbb{C}^{l+n+1}$ for all $l \geq k$. Take $p \in V$ such that $V^H = \mathbb{C}.p$. Then we have

$$(U_1 \times \mathbb{C}^l)^H = \{(x, y, z) \in \mathbb{C}^2 \times \mathbb{C}.p \mid h(x, y) = a(z)\} \times \mathbb{C}^l$$

Because $a(p) \neq 0$, this is biholomorphic to

$$\{(x, y, z) \in \mathbb{C}^3 \mid h(x, y) = z^2\} \times \mathbb{C}^l = Y_2 \times \mathbb{C}^l.$$

On the other hand $(U_1 \times \mathbb{C}^l)^G$ is the subset given by

$$\{(x, y, z) \in \mathbb{C}^3 \mid h(x, y) = z = 0\} \times \mathbb{C}^l = W \times \mathbb{C}^l.$$

The space $(U_1 \times \mathbb{C}^l)^H \setminus (U_1 \times \mathbb{C}^l)^G = (Y_2 \setminus W) \times \mathbb{C}^l$ is not biholomorphic to $\mathbb{C}^\star \times \mathbb{C}^{l+1}$ by Lemma 4.6. Therefore the G-action on $U_1 \times \mathbb{C}^l$ cannot be linearizable. $\square$

In the remaining part of this paragraph we show in a case by case study that for all simple non-abelian groups the setting **(*)** can be achieved thus by Proposition 4.10 constructing non-linearizable actions.

First suppose G is a positive dimensional simple Lie group. Let $V := \mathfrak{g}$ be the Lie algebra of G. Consider the adjoint action of G on $\mathfrak{g}$. Fix a maximal torus $T \subset G$. The fixed point set of T is $\mathfrak{t}$, the Lie algebra of T. Now the Weyl group $W = N_G(T)/T$ acts on $\mathfrak{t}$. Suppose that $s_1, s_2, \ldots, s_r$ are the simple generating reflections of W corresponding with the simple roots $\alpha_1, \ldots, \alpha_r$. The Killing form $\langle \cdot, \cdot \rangle$ given by $\langle a, b \rangle := \mathrm{Trace}(\mathrm{ad}(a)\mathrm{ad}(b))$ is a G-invariant bilinear form on $\mathfrak{g}$. The restriction to $\mathfrak{t}$ is nondegenerate, so we can identify $\mathfrak{t}$ with its dual. Let W' be the subgroup of W generated by $s_1, s_2, \ldots, s_{r-1}$. The fixed point set of W' is one dimensional and spanned by the fundamental weight λ_r. Let H be the group containing T such that $H/T = W'$. If we define $a(v) := \langle v, v \rangle$ then the restriction of a to V^H is not identically 0 because $a(\lambda_r) = \langle \lambda_r, \lambda_r \rangle > 0$. So by Proposition 4.10 we obtain:

Corollary 4.11. *If G is a positive dimensional simple Lie group then for all $l \geq 3.2^{\dim G} - 2$ there exist a non-linearizable action of G on $\mathbb{C}^l$.*

Second we consider the case where G is a non-abelian finite simple group. The proof of the following lemma was shown to us by Jürgen MÜLLER:

Lemma 4.12. *For any non-abelian finite simple group G, there exist a subgroup H and a real representation $V_\mathbb{R}$ such that $V_\mathbb{R}^G = \{0\}$ and $V_\mathbb{R}^H$ is one-dimensional.*

Proof. We use the classification of finite simple groups.

Alternating groups: For $G = A_n$ $(n \geq 5)$, take

$$V_\mathbb{R} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1 + x_2 + \ldots + x_n = 0\}$$

and let G act on it by permuting the coordinates. Take $H = A_{n-1}$, the alternating subgroup which permutes only the first $n - 1$ coordinates. It is clear that $V_\mathbb{R}^H$ is one-dimensional, spanned by the vector $(1, 1, \ldots, 1, 1 - n)$.

For the groups of Lie type and the sporadic ones we will look for a non-trivial irreducible real character χ such that $\langle \chi, (1_H)^G \rangle = 1$, where $(1_H)^G$ is the character of G induced by the trivial character 1_H of H. Now $(1_H)^G$ is the character of the real representation $\mathbb{R}[G/H]$. Let $V_{\mathbb{R}}$ be the isotypic component of χ in $\mathbb{R}[G/H]$. The multiplicity of χ in $V_{\mathbb{R}}$ is $\langle \chi, (1_H)^G \rangle = 1$. So $V_{\mathbb{R}}$ is a real representation with character χ. The dimension of $V_{\mathbb{R}}^H$ is $\langle \chi_H, 1_H \rangle$ which is equal to $\langle \chi, (1_H)^G \rangle = 1$ by Frobenius reciprocity.

Finite simple groups of Lie type: Take $H := B$, the maximal Borel subgroup. Let I be the set of simple reflections. For every subset $J \subseteq I$ we have a parabolic subgroup P_J. Let $(1_{P_J})^G$ be the character of G induced by the trivial character 1_{P_J} of P_J. Define a function St on G by

$$\mathrm{St} = \sum_{J \subseteq I} (-1)^{|J|} (1_{P_J})^G$$

This turns out to be an irreducible character and it has the following properties (see Chapter 6 of [3]):

- $\langle \mathrm{St}, (1_B)^G \rangle = 1$,
- $\mathrm{St}(1) = |U|$ where U is the maximal unipotent subgroup.

Now St is clearly a real character, and it belongs to a real representation $V_{\mathbb{R}}$ of dimension $|U|$ satisfying $\dim V_{\mathbb{R}}^H = 1$.

Sporadic groups: If $(1_H)^G = \sum_{i=1}^r a_i \chi_i$ where $a_1, a_2, \ldots, a_r \in \mathbb{Z}$ and $\chi_1, \chi_2, \ldots, \chi_r$ different irreducible characters, then $a_i = \langle \chi_i, (1_H)^G \rangle$. For most of the sporadic groups and some of its large subgroups the decomposition of $(1_H)^G$ can be found in the *Atlas of Finite Groups* (see [4]). The table below shows the possibilities for G, H and χ as they can be found in the Atlas. The notation of the Atlas is used. For example, if $G = J_3$ (the third Janko group) then 323a stands for the first 323-dimensional irreducible representation of J_3 as listed in the character table of J_3 in the Atlas.

group G	subgroup H	character
M_{11}	M_{10}	10a
M_{12}	M_{11}	11a
M_{22}	$L_3(4)$	21a
M_{23}	M_{22}	22a
M_{24}	M_{23}	23a
J_1	$L_2(11)$	56a
J_2	$U_3(3)$	36a
J_3	$L_2(16):2$	323a
Suz	$G_2(4)$	780a
HS	M_{22}	22a
McL	$U_4(3)$	22a
Co_3	$McL:2$	275a
Co_2	$U_6(2):2$	275a
He	$S_4(4):2$	680a
Fi_{22}	$2\cdot U_6(2)$	429a
Fi_{23}	$2\cdot Fi_{22}$	782a
Fi'_{24}	Fi_{23}	57477a
$O'N$	$L_3(7):2$	10944a
Ly	$G_2(5)$	45694a
Ru	${}^2F_4(2)$	783a

For the other sporadic groups the computer algebra system *GAP* can be used to find:

group G	subgroup H	character
J_4	$2^{11} : M_{24}$	889111a
Co_1	Co_2	299a
HN	A_{12}	133a
Th	${}^3D_4(2) : 3$	4123a
B	Th	96255a
M	$2 \cdot B$	196883a

For all sporadic groups we took for H the maximal subgroup of G, except in the cases $G = B$ and $G = M$ where the maximal subgroups aren't classified yet. $\square$

Now we realize the setting **(*)**. Take H and $V_{\mathbb{R}}$ as in Lemma 4.12. Put $V := V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Let $p \in V_{\mathbb{R}}^H \setminus \{0\}$. Choose a real linear function $b \in V_{\mathbb{R}}^\star$ such that $b(p) \neq 0$. Define the G-invariant polynomial $a = \sum_{g \in G}(g \cdot b)^2$ on V. Now $V^G = \{0\}$, $V^H = \mathbb{C} \cdot p$ and $a(p) > 0$. From Proposition 4.10 we obtain:

Corollary 4.13. *If G is a finite simple group, then there exists a non-linearizable action of G on $\mathbb{C}^l$ for all $l \geq 3.2^d - 2$ where*

- $d = n - 1$ *if* $G = A_n$,
- $d = \mathrm{St}(1) = |U|$ *if G is of Lie type,*
- d *is like in the tables of the proof of Lemma 4.12 if G is sporadic.*

Effective non-linearizable actions for all reductive groups

Theorem 4.14. *For every complex reductive Lie group G (except the trivial group) there exists a natural number N_G such that for all $l \geq N_G$ there exists an effective non-linearizable holomorphic action of G on $\mathbb{C}^l$.*

Proof. Let H be the maximal closed normal subgroup of G. For every $l \geq N_{G/H}$ there exists an effective non-linearizable action of G/H on $\mathbb{C}^l$ by Proposition 4.3, Corollary 4.8, Corollary 4.11 and Corollary 4.13. Take a G-representation V on which H acts effectively. Let $W = V/V^H$. Now H acts on W effectively and $W^H = \{0\}$. The action of G on $\mathbb{C}^l \times W$ is effective. If this action were linearizable, then the action of G on $(\mathbb{C}^l \times W)^H = \mathbb{C}^l \times \{0\}$ would be linearizable, but this is not the case. $\square$

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