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Quivers with potentials and their representations I: Mutations

Harm Derksen, Jerzy Weyman and Andrei Zelevinsky

Abstract. We study quivers with relations given by noncommutative analogs of Jacobian ideals in the complete path algebra. This framework allows us to give a representation-theoretic interpretation of quiver mutations at arbitrary vertices. This gives a far-reaching generalization of Bernstein–Gelfand–Ponomarev reflection functors. The motivations for this work come from several sources: superpotentials in physics, Calabi–Yau algebras, cluster algebras.

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Contents

1. Introduction

The main objects of study in this paper are *quivers with potentials* (QPs for short). Roughly speaking, a QP is a quiver Q together with an element S of the path algebra of Q such that S is a linear combination of cyclic paths. We associate to S the two-sided ideal $J(S)$ in the path algebra generated by the (noncommutative) partial derivatives of S with respect to the arrows of Q. We refer to $J(S)$ as the *Jacobian ideal*, and to the quotient of the path algebra modulo $J(S)$ as the *Jacobian algebra*. They appeared in physicists' work on *superpotentials* in the context of the Seiberg duality in mirror symmetry (see, e.g., [17, 2, 6]). Since in some of their work the superpotentials are required to satisfy some form of Serre duality, we prefer not to use this terminology, and just refer to S as a *potential*; another reason for this is that we are working with the completed path algebra, so our potentials are possibly infinite linear combinations of cyclic paths. The Jacobian algebras also play an important role in the recent work on Calabi–Yau algebras [4, 25, 26, 27].

In this paper we introduce and study *mutations* for QPs and their (decorated) representations. In the context of Calabi–Yau algebras, the mutations were discussed in [26] but our approach is much more elementary and down-to-earth. Namely, we develop the setup that directly extends to QPs the Bernstein–Gelfand–Ponomarev reflection functors [3] and their "decorated" version [28].

The original motivation for our study comes from the theory of cluster algebras introduced and studied in a series of papers [18, 19, 1, 20]. In this paper, we deal only with the underlying combinatorics of this theory embodied in skew-symmetrizable integer matrices and their mutations. Furthermore, we restrict our attention to *skew-symmetric* integer matrices. Such matrices can be encoded by quivers without loops and oriented 2-cycles. Namely, a skew-symmetric integer $n\times n$ matrix $B = (b_{i,j})$ corresponds to a quiver $Q(B)$ with vertices $1,\dots,n$, and $b_{i,j}$ arrows from j to i whenever $b_{i,j} > 0$. For every vertex k, the *mutation* at k transforms B into another skew-symmetric integer $n \times n$ matrix $\mu_k(B) = \overline{B} = (\overline{b}_{i,j})$. The formula for $\overline{b}_{i,j}$ is given below in (7.4). It is well-known (see Proposition 7.1 below) that the quiver $Q(\overline{B})$ can be obtained from $Q(B)$ by the following three-step procedure:

Step 1. For every incoming arrow $a : j \to k$ and every outgoing arrow $b : k \to i$, create a "composite" arrow $[ba] : j \to i$; thus, whenever $b_{i,k}, b_{k,j} > 0$, we create $b_{i,k}b_{k,j}$ new arrows from j to i.

Step 2. Reverse all arrows at k; that is, replace each arrow $a : j \to k$ with $a^\star : k \to j$, and $b : k \to i$ with $b^\star : i \to k$.

Step 3. Remove any maximal disjoint collection of oriented 2-cycles (that can appear as a result of creating new arrows in Step 1).

In the case where k is a source or a sink of $Q(B)$, the first and last steps of the above procedure are not applicable, so $Q(\overline{B})$ is obtained from $Q(B)$ by just reversing all the arrows at k. In this situation, J. Bernstein, I. Gelfand, and V. Ponomarev [3] introduced the *reflection functor* at k sending representations of a quiver $Q(B)$

(without relations) into representations of $Q(\overline{B})$. A modification of these functors acting on *decorated representations* was introduced in [28] to establish a link between cluster algebras and quiver representations (the definition of decorated representations for general QPs is given below in Section 10).

The elementary approach of [28] has not been further pursued until now, giving way to a more sophisticated approach via cluster categories and cluster-tilted algebras developed in [7–10, 12–15] and many other publications. Most of the results in these papers are for the quivers obtained by mutations from hereditary algebras (i.e., quivers without oriented cycles and without relations). In this paper we return to the more elementary point of view of [28] and propose an alternative approach (which is in fact more general, since we do not impose any restrictions on quivers in question). In this approach, the mutations at arbitrary vertices (not just sources or sinks) are defined for QPs and their decorated representations. The construction for QPs is carried out in Section 5, and for their representations in Section 10. It turns out to be rather delicate and requires a lot of technical preparation. The first two steps of the above mutation procedure extend to QPs in a relatively straightforward way, but Step 3 presents a real challenge: we need to accompany the removal of oriented 2-cycles from a quiver with a suitable modification of the potential, leaving the corresponding Jacobian algebra unchanged. Our main device in dealing with this difficulty is Theorem 4.6, which is the crucial technical result of the paper. Roughly speaking, Theorem 4.6 asserts that every potential S can be transformed by an automorphism of the path algebra into the sum of two potentials S_{triv} and S_{red} on the disjoint sets of arrows, where the *trivial* part S_{triv} is a linear combination of cyclic 2-paths, while the *reduced* part S_{red} involves only cyclic d-paths with $d \geq 3$. Furthermore, the Jacobian algebra of S_{red} is isomorphic to that of S.

Several comments on this result are in order. First, our arguments heavily depend on the setup using completed path algebras, thus allowing potentials to involve infinite sums of cyclic paths. Second, the reduction $S \mapsto S_{\text{red}}$ is not given by a canonical procedure. As a consequence, our construction of mutations for QPs and their representations is not functorial in any obvious sense. On the positive side, we prove that every mutation is a well-defined transformation on the right-equivalence classes of QPs (and their representations), where, roughly speaking, two QPs are *right-equivalent* if they can be obtained from each other by an automorphism of the path algebra (for more precise definitions see Definitions 4.2 and 10.2).

Finally, it is important to keep in mind that, even with the help of Theorem 4.6, in order to get rid of all oriented 2-cycles in the mutated QP, one needs to impose some "genericity" conditions on the initial potential S. These conditions are studied in Section 7. They are not very explicit in general, but we introduce an important class or *rigid* QPs (see Definitions 6.7 and 6.10) for which the absence of oriented 2-cycles after any sequence of QP mutations is guaranteed.

We now describe the contents of the paper in more detail. In Section 2 we introduce an algebraic setup for dealing with quivers and their path algebras. We fix a base field K, and encode a quiver with the vertex set Q_0 and the arrow set

Q_1 by its *vertex span* $R = K^{Q_0}$ and *arrow span* $A = K^{Q_1}$. Thus, R is a finite-dimensional commutative K-algebra, and A is a finite-dimensional R-bimodule. We then introduce the *path algebra*

$$R\langle A\rangle = \bigoplus_{d=0}^{\infty} A^d,$$

and, more importantly for our purposes, the *complete path algebra*

$$R\langle\langle A\rangle\rangle = \prod_{d=0}^{\infty} A^d;$$

here A^d stands for the d-fold tensor power of A as an R-bimodule. We view $R\langle\langle A\rangle\rangle$ as a topological algebra via the $\mathfrak{m}$-adic topology, where $\mathfrak{m}$ is the two-sided ideal generated by A.

In Section 3 we introduce some of our main objects of study: potentials and their Jacobian ideals. It is natural to view potentials as elements of the *trace space* $R\langle\langle A\rangle\rangle/\{R\langle\langle A\rangle\rangle, R\langle\langle A\rangle\rangle\}$, where $\{R\langle\langle A\rangle\rangle, R\langle\langle A\rangle\rangle\}$ is the closure of the vector subspace in $R\langle\langle A\rangle\rangle$ spanned by all commutators. It is more convenient for us to define a potential S as an element of the cyclic part of $R\langle\langle A\rangle\rangle$; for all practical purposes, S can be replaced by a *cyclically equivalent* potential, that is, the one with the same image in the trace space. To define the Jacobian ideal $J(S)$ and derive its basic properties, we develop the formalism of *cyclic derivatives*, in particular, establishing "cyclic" versions of the Leibniz rule and the chain rule. The main result of Section 3 is Proposition 3.7 that asserts that any isomorphism φ of path algebras sends $J(S)$ to $J(\varphi(S))$. Note that cyclic derivatives for general noncommutative algebras were introduced in [29], and the results we present can be easily deduced from those given there. For the convenience of the reader, we present complete independent proofs. Victor Ginzburg informed us that in the context of path algebras of quivers, cyclic derivatives were introduced and studied in [5, 24], and that Proposition 3.7 is a consequence of the geometric interpretation of $J(S)$ given in [25, Definition 5.1.1, Lemma 5.1.3].

In Section 4 we introduce quivers with potentials (QPs) and define the right-equivalence relation on them, which plays an important role in the paper. We then state and prove the key technical result of the paper: Splitting Theorem 4.6, already discussed above. The proof is elementary but pretty involved; it uses in an essential way the topology in a complete path algebra. In order not to interrupt the argument, we move to the Appendix our treatment of the topological properties needed for the proof of one of the technical lemmas.

In Section 5 we finally introduce the mutations of QPs. Using Theorem 4.6, we prove that the mutation at an arbitrary vertex is a well-defined involution on the set of right-equivalence classes of reduced QPs (Theorem 5.7).

In Section 6, we study some mutation invariants of QPs. In particular, we show that mutations preserve the class of QPs with finite-dimensional Jacobian algebras (Corollary 6.6). Another important property of QPs preserved by mutations is *rigidity* (Corollary 6.11), which was already mentioned above. For the precise definition of rigid QPs see Definitions 6.7 and 6.10 below; intuitively, a QP is rigid if its right-equivalence class is invariant under infinitesimal deformations.

In Section 7, we introduce and study *nondegenerate* QPs, that is, those to which one can apply an arbitrary sequence of mutations without creating oriented 2-cycles. In Corollary 7.4 we show that nondegeneracy is guaranteed by nonvanishing of countably many nonzero polynomial functions on the space of potentials. In particular, if the base field K is uncountable, a nondegenerate QP exists for every underlying quiver.

Section 8 contains some examples of rigid and nonrigid potentials and some further results illustrating the importance of rigidity. A simple but important Proposition 8.1 asserts that rigid QPs have no oriented 2-cycles. Combining this with the fact that rigidity is preserved by mutations, we conclude that every rigid QP is nondegenerate. Using a result by Keller–Reiten [27], we show in Example 8.7 that the class of rigid QPs (as well as the class of QPs with finite-dimensional Jacobian algebras) is strictly greater than the class of QPs mutation-equivalent to acyclic ones. On the other hand, Example 8.6 exhibits an underlying quiver without oriented 2-cycles that does not admit a rigid QP; thus, the class of nondegenerate QPs is strictly greater than the class of rigid ones.

In Section 9, we consider quivers that are mutation-equivalent to a Dynkin quiver. For every such underlying quiver, we compute explicitly the corresponding rigid QP (Proposition 9.1). Comparing this result with the description of cluster-tilted algebras obtained in [13, 9], we conclude in Corollary 9.3 that in the case in question, every cluster-tilted algebra can be identified with the Jacobian algebra of the corresponding rigid QP. Thus, Jacobian algebras can be viewed as generalizations of cluster-tilted algebras.

In Section 10 we introduce decorated representations of QPs (QP-representations, for short) and their right-equivalence (Definitions 10.1 and 10.2). We then present a representation-theoretic extension of Splitting Theorem 4.6 by defining the reduced part of a QP-representation $\mathcal{M}$ (Definition 10.4) and proving that, up to right-equivalence, it is determined by the right-equivalence class of $\mathcal{M}$ (Proposition 10.5). We use this result to introduce mutations of QP-representations and to prove a representation-theoretic extension of Theorem 5.7: the mutation at every vertex is an involution on the set of right-equivalence classes of reduced QP-representations (Theorem 10.13).

Some examples of QP-representations and their mutations are given in Section 11. All these examples treat quivers with three vertices. In particular, we describe the effect of mutations on a special family of *band representations* coming from the theory of string algebras [11, 21].

The concluding Section 12 contains some open problems about QPs and their representations that we find essential for better understanding of the theory.

In the forthcoming continuation of this paper, we plan to discuss applications of QP-representations and their mutations to the structure of the corresponding cluster algebras.

2. Quivers and path algebras

A *quiver* $Q = (Q_0, Q_1, h, t)$ consists of a pair of finite sets Q_0 (*vertices*) and Q_1 (*arrows*) supplied with two maps $h : Q_1 \to Q_0$ (*head*) and $t : Q_1 \to Q_0$ (*tail*). It is represented as a directed graph with the set of vertices Q_0 and directed edges $a : ta \to ha$ for $a \in Q_1$. Note that this definition allows the underlying graph to have multiple edges and (multiple) loops.

We fix a field K, and associate to a quiver Q two vector spaces $R = K^{Q_0}$ and $A = K^{Q_1}$ consisting of K-valued functions on Q_0 and Q_1, respectively. We will sometimes refer to R as the *vertex span* of Q, and to A as the *arrow span* of Q. The space R is a commutative algebra under the pointwise multiplication of functions. The space A is an R-bimodule, with the bimodule structure defined as follows: if $e \in R$ and $f \in A$ then $(e \cdot f)(a) = e(ha)f(a)$ and $(f \cdot e)(a) = f(a)e(ta)$ for all $a \in Q_1$.

We denote by $Q^\star$ the *dual* or *opposite* quiver $Q^\star$ obtained by reversing the arrows in Q (i.e., replacing $Q = (Q_0, Q_1, h, t)$ with $Q^\star = (Q_0, Q_1, t, h)$). The corresponding arrow span is naturally identified with the dual bimodule $A^\star$ (the dual vector space of A with the standard R-bimodule structure).

For a given vertex set Q_0 with the vertex span R, every finite-dimensional R-bimodule B is the arrow span of some quiver on Q_0. To see this, consider the elements $e_i \in R$ for $i \in Q_0$ given by $e_i(j) = \delta_{i,j}$ (the Kronecker delta symbol). They form a basis of idempotents of R, hence every R-bimodule B has a direct sum decomposition

$$B = \bigoplus_{i,j \in Q_0} B_{i,j},$$

where $B_{i,j} = e_i B e_j \subseteq B$ for every $i, j \in Q_0$. If B is finite-dimensional, we can identify the (finite) set of arrows Q_1 with a K-basis in B which is the union of bases in all components $B_{i,j}$; under this identification, every $a \in Q_1 \cap B_{i,j}$ has $h(a) = i$ and $t(a) = j$.

It is convenient to represent an R-bimodule B by a matrix of vector spaces $(B_{i,j})$ whose rows and columns are labeled by vertices. In this model, the left (resp. right) action of an element $c = \sum_i c_i e_i \in R$ is given by the left (resp. right) multiplication by the diagonal matrix with diagonal entries c_i. And the tensor product over R is given by the usual matrix multiplication: if $B = \bigoplus_{i,j} B_{i,j}$ and $C = \bigoplus_{i,j} C_{i,j}$, then

$$(B \otimes_R C)_{i,j} = \bigoplus_k (B_{i,k} \otimes C_{k,j}).$$

Returning to a quiver Q with the arrow span A, for each nonnegative integer d, let A^d denote the R-bimodule

$$A^d = \underbrace{A \otimes_R \cdots \otimes_R A}_{d},$$

with the convention $A^0 = R$.

Definition 2.1. The *path algebra* of Q is defined as the (graded) tensor algebra

$$R\langle A\rangle = \bigoplus_{d=0}^{\infty} A^d.$$

For each $i, j \in Q_0$, the component $R\langle A\rangle_{i,j} = e_i R\langle A\rangle e_j$ is called the *space of paths* from j to i.

As above, we identify the set of arrows Q_1 with some basis of A consisting of homogeneous elements, that is, each $a \in Q_1$ belongs to some component $A_{i,j}$. Then for every $d \geq 1$, the products $a_1 \cdots a_d$ such that all a_k belong to Q_1, and $t(a_k) = h(a_{k+1})$ for $1 \leq k < d$, form a K-basis of A^d. We call this basis the *path basis* of A^d associated to Q_1. For $d = 0$, we call $\{e_i \mid i \in Q_0\}$ the path basis of $A^0 = R$. We refer to the union of path bases for all d as the path basis of $R\langle A\rangle$. The elements of the path basis will be sometimes referred to simply as *paths.* We depict $a_1 \cdots a_d$ as a path of length d starting in the vertex $t(a_d)$ and ending in $h(a_1)$. Note that the product $(a_1 \cdots a_d)(a_{d+1} \cdots a_{d+k})$ of two paths is 0 unless $t(a_d) = h(a_{d+1})$, in which case the product is given by concatenation of paths. This description implies the following:

$$\text{If } 0 \neq p \in A^k e_i \text{ and } 0 \neq q \in e_i A^l \text{ for some vertex } i \text{ then } pq \neq 0. \tag{2.1}$$

Definition 2.2. The *complete path algebra* of Q is defined as

$$R\langle\langle A\rangle\rangle = \prod_{d=0}^{\infty} A^d.$$

Thus, the elements of $R\langle\langle A\rangle\rangle$ are (possibly infinite) K-linear combinations of the elements of a path basis in $R\langle A\rangle$; and the multiplication in $R\langle\langle A\rangle\rangle$ naturally extends the multiplication in $R\langle A\rangle$.

Note that, if the quiver Q is *acyclic* (that is, has no oriented cycles), then $A^d = \{0\}$ for $d \gg 0$, hence in this case $R\langle\langle A\rangle\rangle = R\langle A\rangle$, and this algebra is finite-dimensional.

Example 2.3. Consider the quiver $Q = (Q_0, Q_1)$ with $Q_0 = \{1\}$ and $Q_1 = \{a\}$ with $a : 1 \to 1$. This is the loop quiver:

$$a \circlearrowleft 1 \, .$$

In this case $R = K^{Q_0} = K$, and $A = K^{Q_1} = Ka$. We have $R\langle A\rangle = K[a]$, and $R\langle\langle A\rangle\rangle = K[[a]]$, the algebra of formal power series.

Let $\mathfrak{m} = \mathfrak{m}(A)$ denote the (two-sided) ideal of $R\langle\langle A\rangle\rangle$ given by

$$\mathfrak{m} = \mathfrak{m}(A) = \prod_{d=1}^{\infty} A^d. \tag{2.2}$$

Thus the powers of $\mathfrak{m}$ are given by

$$\mathfrak{m}^n = \prod_{d=n}^{\infty} A^d.$$

We view $R\langle\langle A\rangle\rangle$ as a topological K-algebra via the $\mathfrak{m}$-*adic topology* having the powers of $\mathfrak{m}$ as a basic system of open neighborhoods of 0. Thus, the closure of any subset $U \subseteq R\langle\langle A\rangle\rangle$ is given by

$$\overline{U} = \bigcap_{n=0}^{\infty} (U + \mathfrak{m}^n). \tag{2.3}$$

It is clear that $R\langle A\rangle$ is a dense subalgebra of $R\langle\langle A\rangle\rangle$.

In dealing with $R\langle\langle A\rangle\rangle$, the following fact is quite useful: every (noncommutative) formal power series over R in a finite number of variables can be evaluated at arbitrary elements of $\mathfrak{m}$ to obtain a well-defined element of $R\langle\langle A\rangle\rangle$. To illustrate, let us show that $\mathfrak{m}$ is the unique maximal two-sided ideal of $R\langle\langle A\rangle\rangle$ having zero intersection with $R = A^0$. Indeed, it is enough to show that any element $x \in R\langle\langle A\rangle\rangle - \mathfrak{m}$ generates an ideal having nonzero intersection with R. Let $x = c + y$ with c a nonzero element of R, and $y \in \mathfrak{m}$. Multiplying x on both sides by suitable elements of R, we can assume that $c = e_i$ for some $i \in Q_0$, and $e_i y = y e_i = y$. But then $z = e_i - y + y^2 - y^3 + \cdots$ is a well-defined element of $R\langle\langle A\rangle\rangle$, and we have $xz = e_i$, proving our claim.

This characterization of $\mathfrak{m}$ implies that it is invariant under any algebra automorphism φ of $R\langle\langle A\rangle\rangle$ such that $\varphi|_R$ is the identity. Thus, φ is continuous, i.e., is an automorphism of $R\langle\langle A\rangle\rangle$ as a topological algebra.

The same argument shows that, more generally, if A and A' are finite-dimensional R-bimodules then any algebra homomorphism $\varphi : R\langle\langle A\rangle\rangle \to R\langle\langle A'\rangle\rangle$ with $\varphi|_R = \mathrm{id}$ sends $\mathfrak{m}(A)$ into $\mathfrak{m}(A')$, hence is continuous. Thus, φ is uniquely determined by its restriction to $A^1 = A$, which is an R-bimodule homomorphism $A \to \mathfrak{m}(A') = A' \oplus \mathfrak{m}(A')^2$. We write $\varphi|_A = (\varphi^{(1)}, \varphi^{(2)})$, where $\varphi^{(1)} : A \to A'$ and $\varphi^{(2)} : A \to \mathfrak{m}(A')^2$ are R-bimodule homomorphisms.

Proposition 2.4. *Any pair $(\varphi^{(1)}, \varphi^{(2)})$ of R-bimodule homomorphisms $\varphi^{(1)} : A \to A'$ and $\varphi^{(2)} : A \to \mathfrak{m}(A')^2$ gives rise to a unique homomorphism of topological algebras $\varphi : R\langle\langle A\rangle\rangle \to R\langle\langle A'\rangle\rangle$ such that $\varphi|_R = \mathrm{id}$, and $\varphi|_A = (\varphi^{(1)}, \varphi^{(2)})$. Furthermore, φ is an isomorphism if and only if $\varphi^{(1)}$ is an R-bimodule isomorphism $A \to A'$.*

Proof. The uniqueness of φ is clear. To show the existence, let $Q_1 = \{a_1, \dots, a_N\} \subset A = A^1$ be the set of arrows in A. Writing an element $x \in R\langle\langle A\rangle\rangle$ as an infinite K-linear combination of elements of the corresponding path basis in $R\langle A\rangle$, we express x as a (noncommutative) formal power series $F(a_1, \dots, a_N)$. Therefore, a desired algebra homomorphism can be obtained by setting

$$\varphi(x) = F(\varphi^{(1)}(a_1) + \varphi^{(2)}(a_1), \dots, \varphi^{(1)}(a_N) + \varphi^{(2)}(a_N)).$$

If φ is an isomorphism then $\varphi^{(1)} : A \to A'$ is clearly an isomorphism of R-bimodules. To show the converse implication, we can identify A and A' with the help of $\varphi^{(1)}$, and so assume that $\varphi^{(1)}$ is the identity automorphism of A. Then the (infinite) matrix that expresses φ as a K-linear map in the path basis of $R\langle\langle A\rangle\rangle$ is lower-triangular with all the diagonal entries equal to 1 (we order the basis elements so that their degrees weakly increase). Clearly, such a matrix is invertible, completing the proof of Proposition 2.4. □

Definition 2.5. Let φ be the automorphism of $R\langle\langle A\rangle\rangle$ corresponding to a pair $(\varphi^{(1)}, \varphi^{(2)})$ as in Proposition 2.4. If $\varphi^{(2)} = 0$, then we call φ a *change of arrows*. If $\varphi^{(1)}$ is the identity automorphism of A, we say that φ is a *unitriangular* automorphism; furthermore, we say that φ is of *depth* $d \geq 1$ if $\varphi^{(2)}(A) \subset \mathfrak{m}^{d+1}$.

The following property of unitriangular automorphisms is immediate from the definitions:

$$\text{If } \varphi \text{ is an unitriangular automorphism of } R\langle\langle A\rangle\rangle \text{ of depth } d, \text{ then } \varphi(u) - u \in \mathfrak{m}^{n+d} \text{ for } u \in \mathfrak{m}^n. \tag{2.4}$$

3. Potentials and their Jacobian ideals

In this section we introduce some of our main objects of study: potentials and their Jacobian ideals in the complete path algebra $R\langle\langle A\rangle\rangle$ given by Definition 2.2. We fix a path basis in $R\langle A\rangle$; recall that it consists of the elements $e_i \in R = A^0$ together with the products $a_1 \cdots a_d$ (paths) such that all a_k belong to Q_1, and $t(a_k) = h(a_{k+1})$ for $1 \leq k < d$. Then each space A^d has a direct R-bimodule decomposition $A^d = \bigoplus_{i,j\in Q_0} A^d_{i,j}$, where the component $A^d_{i,j}$ is spanned by the paths $a_1 \cdots a_d$ with $h(a_1) = i$ and $t(a_d) = j$.

Definition 3.1.

- For each $d \geq 1$, we define the *cyclic part* of A^d as the sub-R-bimodule $A^d_{\text{cyc}} = \bigoplus_{i\in Q_0} A^d_{i,i}$. Thus, A^d_{cyc} is the span of all paths $a_1 \cdots a_d$ with $h(a_1) = t(a_d)$; we call such paths *cyclic*.
- We define a closed vector subspace $R\langle\langle A\rangle\rangle_{\text{cyc}} \subseteq R\langle\langle A\rangle\rangle$ by setting
$$R\langle\langle A\rangle\rangle_{\text{cyc}} = \prod_{d=1}^{\infty} A^d_{\text{cyc}},$$
and call the elements of $R\langle\langle A\rangle\rangle_{\text{cyc}}$ *potentials*.
- For every $\xi \in A^\star$, we define the *cyclic derivative* ∂_ξ as the continuous K-linear map $R\langle\langle A\rangle\rangle_{\text{cyc}} \to R\langle\langle A\rangle\rangle$ acting on paths by
$$\partial_\xi(a_1 \cdots a_d) = \sum_{k=1}^{d} \xi(a_k) a_{k+1} \cdots a_d a_1 \cdots a_{k-1}. \tag{3.1}$$
- For every potential S, we define its *Jacobian ideal* $J(S)$ as the closure of the (two-sided) ideal in $R\langle\langle A\rangle\rangle$ generated by the elements $\partial_\xi(S)$ for all $\xi \in A^\star$ (see (2.3)); clearly, $J(S)$ is a two-sided ideal in $R\langle\langle A\rangle\rangle$.
- We call the quotient $R\langle\langle A\rangle\rangle/J(S)$ the *Jacobian algebra* of S, and denote it by $\mathcal{P}(Q, S)$ or $\mathcal{P}(A, S)$.

An easy check shows that a cyclic derivative $\partial_\xi : R\langle\langle A\rangle\rangle_{\text{cyc}} \to R\langle\langle A\rangle\rangle$ does not depend on the choice of the path basis. Furthermore, cyclic derivatives do not distinguish between potentials that are equivalent in the following sense.

Definition 3.2. Two potentials S and S' are *cyclically equivalent* if $S - S'$ lies in the closure of the span of all elements of the form $a_1 \cdots a_d - a_2 \cdots a_d a_1$, where $a_1 \cdots a_d$ is a cyclic path.

The following proposition is immediate from (3.1).

Proposition 3.3. *If two potentials S and S' are cyclically equivalent, then $\partial_\xi(S) = \partial_\xi(S')$ for all $\xi \in A^\star$, hence $J(S) = J(S')$ and $\mathcal{P}(A,S) = \mathcal{P}(A,S')$.*

It is easy to see that the definition of cyclic equivalence does not depend on the choice of the path basis. In fact, it can be given in more invariant terms as follows.

Definition 3.4. For any topological K-algebra U, its *trace space* $\mathrm{Tr}(U)$ is defined as $\mathrm{Tr}(U) = U/\{U,U\}$, where $\{U,U\}$ is the closure of the vector subspace in U spanned by all commutators. We denote by $\pi = \pi_U : U \to \mathrm{Tr}(U)$ the canonical projection.

The following proposition is a direct consequence of the definitions.

Proposition 3.5. *Two potentials S and S' are cyclically equivalent if and only if $\pi_{R\langle\langle A\rangle\rangle}(S) = \pi_{R\langle\langle A\rangle\rangle}(S')$. Thus, the Jacobian ideal and the Jacobian algebra of a potential S depend only on the image of S in* $\mathrm{Tr}(R\langle\langle A\rangle\rangle)$.

Recall that we identify the set of arrows Q_1 with a K-basis in $A = A^1$. For $a \in Q_1$, we will use the notation ∂_a for the cyclic derivative $\partial_{a^\star}$, where $Q_1^\star = \{a^\star \mid a \in Q_1\}$ is the dual basis of Q_1 in $A^\star$.

Example 3.6. Consider the quiver $Q = (Q_0, Q_1)$ with $Q_0 = \{1,2\}$ and $Q_1 = \{a,b\}$, where $a : 1 \to 2$ and $b : 2 \to 1$:

$$1 \underset{b}{\overset{a}{\rightleftarrows}} 2 \,.$$

The vertex and arrow spans of Q are given by $R = K^{Q_0} = Ke_1 \oplus Ke_2$, and $A = K^{Q_1} = Ka \oplus Kb$. The paths in $R\langle\langle A\rangle\rangle$ are e_1, e_2 and all products of the generators a and b in which the factors a and b alternate. The potentials are (possibly infinite) linear combinations of the elements $(ab)^n$ and $(ba)^n$ for all $n \geq 1$. Using (3.1), we obtain

$$\partial_a((ab)^n) = \partial_a((ba)^n) = nb(ab)^{n-1}, \quad \partial_b((ab)^n) = \partial_b((ba)^n) = na(ba)^{n-1} \quad (n \geq 1).$$

Up to cyclic equivalence, every potential can be written in the form

$$\sum_{n=1}^{\infty} \alpha_n (ab)^n \quad (\alpha_n \in K).$$

Returning to the general theory, it is clear that every algebra homomorphism $\varphi : R\langle\langle A\rangle\rangle \to R\langle\langle A'\rangle\rangle$ with $\varphi|_R = \mathrm{id}$ sends potentials to potentials.

Proposition 3.7. *Every algebra isomorphism $\varphi : R\langle\langle A\rangle\rangle \to R\langle\langle A'\rangle\rangle$ with $\varphi|_R =$ id sends $J(S)$ onto $J(\varphi(S))$, inducing an isomorphism of the Jacobian algebras $\mathcal{P}(A,S) \to \mathcal{P}(A',\varphi(S))$.*

Proof. We start by developing some "differential calculus" for cyclic derivatives. We need a few pieces of notation. We set

$$R\langle\langle A\rangle\rangle \mathbin{\widehat{\otimes}} R\langle\langle A\rangle\rangle = \prod_{d,e\geq 0}(A^d \otimes A^e)$$

(the tensor product on the right is over the base field K), and view this space as a topological vector space with a basic system of open neighborhoods of 0 formed by the sets $\prod_{d+e\geq n}(A^d\otimes A^e)$ for all $n\geq 0$; thus, $R\langle A\rangle \otimes R\langle A\rangle$ is dense in $R\langle\langle A\rangle\rangle \mathbin{\widehat{\otimes}} R\langle\langle A\rangle\rangle$. Now, for every $\xi \in A^\star$, we define a continuous K-linear map

$$\Delta_\xi : R\langle\langle A\rangle\rangle \to R\langle\langle A\rangle\rangle \mathbin{\widehat{\otimes}} R\langle\langle A\rangle\rangle$$

by setting $\Delta_\xi(e) = 0$ for $e \in R = A^0$, and

$$\Delta_\xi(a_1\cdots a_d) = \sum_{k=1}^{d} \xi(a_k)a_1\cdots a_{k-1}\otimes a_{k+1}\cdots a_d \tag{3.2}$$

for any path $a_1\cdots a_d$ of length $d\geq 1$. Note that Δ_ξ does not depend on the choice of the path basis. We will use the same convention as for cyclic derivatives: for $a\in Q_1$, we write Δ_a instead of $\Delta_{a^\star}$. For instance, in the situation of Example 3.6, we have

$$\Delta_a((ab)^n) = \sum_{k=1}^{n}(ab)^{k-1}\otimes b(ab)^{n-k}, \quad \Delta_b((ab)^n) = \sum_{k=1}^{n}(ab)^{k-1}a\otimes (ab)^{n-k}.$$

Next, we denote by $(f,g)\mapsto f\,\square\, g$ a continuous K-bilinear map

$$(R\langle\langle A\rangle\rangle \mathbin{\widehat{\otimes}} R\langle\langle A\rangle\rangle)\times R\langle\langle A\rangle\rangle \to R\langle\langle A\rangle\rangle$$

given by

$$(u\otimes v)\,\square\, g = vgu \tag{3.3}$$

for $u,v\in R\langle A\rangle$. We are now ready to state the Leibniz rule.

Lemma 3.8 (Cyclic Leibniz rule). *Let $f\in R\langle\langle A\rangle\rangle_{i,j}$ and $g\in R\langle\langle A\rangle\rangle_{j,i}$ for some vertices i and j. Then for every $\xi\in A^\star$, we have*

$$\partial_\xi(fg) = \Delta_\xi(f)\,\square\, g + \Delta_\xi(g)\,\square\, f. \tag{3.4}$$

More generally, for any finite sequence of vertices $i_1,\ldots,i_d,i_{d+1}=i_1$ and for any $f_1,\ldots,f_d$ such that $f_k\in R\langle\langle A\rangle\rangle_{i_k,i_{k+1}}$, we have

$$\partial_\xi(f_1\cdots f_d) = \sum_{k=1}^{d}\Delta_\xi(f_k)\,\square\,(f_{k+1}\cdots f_d f_1\cdots f_{k-1}). \tag{3.5}$$

Proof. It is enough to check (3.4) in the case where $f = a_1\cdots a_d$ and $g = a_{d+1}\cdots a_{d+s}$ are two paths such that $t(a_d) = h(a_{d+1})$ and $t(a_{d+s}) = h(a_1)$. Using (3.1), we obtain

$$\partial_\xi(fg) = \sum_{k=1}^{d+s}\xi(a_k)a_{k+1}\cdots a_{d+s}a_1\cdots a_{k-1}.$$

Comparing this expression with (3.2) and (3.3), we see that the part of the last sum where k runs from 1 to d (resp. from $d+1$ to $d+s$) is equal to $\Delta_\xi(f) \square g$ (resp. to $\Delta_\xi(g) \square f$), proving (3.4). The identity (3.5) follows from (3.4) by induction on d. □

Lemma 3.9 (Cyclic chain rule). *Suppose that* $\varphi : R\langle\langle A\rangle\rangle \to R\langle\langle A'\rangle\rangle$ *is an algebra homomorphism as in Proposition* 2.4. *Then, for every potential* $S \in R\langle\langle A\rangle\rangle_{\mathrm{cyc}}$ *and* $\xi \in A'^\star$, *we have*

$$\partial_\xi(\varphi(S)) = \sum_{a \in Q_1} \Delta_\xi(\varphi(a)) \square \varphi(\partial_a(S)). \tag{3.6}$$

Proof. It suffices to treat the case where $S = a_1 \cdots a_d$ is a cyclic path. Applying (3.5) and (3.1), we obtain

$$\begin{aligned}
\partial_\xi(\varphi(S)) &= \sum_{k=1}^{d} \Delta_\xi(\varphi(a_k)) \square (\varphi(a_{k+1} \cdots a_d a_1 \cdots a_{k-1})) \\
&= \sum_{a \in Q_1} \Delta_\xi(\varphi(a)) \square \varphi\Big(\sum_{k:\, a_k = a} a_{k+1} \cdots a_d a_1 \cdots a_{k-1}\Big) \\
&= \sum_{a \in Q_1} \Delta_\xi(\varphi(a)) \square \varphi(\partial_a(S)),
\end{aligned}$$

as desired. □

Now we are ready to prove Proposition 3.7. By Lemma 3.9, for every $\xi \in A'^\star$, the element $\partial_\xi(\varphi(S))$ lies in the ideal generated by the elements $\varphi(\partial_a(S))$ for $a \in Q_1$, hence, it lies in $\varphi(J(S))$. Thus, we have the inclusion

$$J(\varphi(S)) \subseteq \varphi(J(S)).$$

We can also apply this to the inverse isomorphism φ^{-1} and the potential $\varphi(S)$:

$$J(S) = J(\varphi^{-1}(\varphi(S))) \subseteq \varphi^{-1}(J(\varphi(S)).$$

Applying φ to both sides yields $\varphi(J(S)) \subseteq J(\varphi(S))$, completing the proof. □

4. Quivers with potentials

We now introduce our main objects of study.

Definition 4.1. Suppose Q is a quiver with the arrow span A, and $S \in R\langle\langle A\rangle\rangle_{\mathrm{cyc}}$ is a potential. We say that a pair (Q, S) (or (A, S)) is a *quiver with potential* (QP for short) if it satisfies the following two conditions:

The quiver Q has no loops, i.e., $A_{i,i} = 0$ for all $i \in Q_0$. (4.1)

No two cyclically equivalent cyclic paths appear in the decomposition of S. (4.2)

In view of (4.1), every potential S belongs to $\mathfrak{m}(A)^2$; and condition (4.2) excludes, for instance, any nonzero potential S cyclically equivalent to 0.

Definition 4.2. Let (A, S) and (A', S') be QPs on the same vertex set Q_0. By a *right-equivalence* between (A, S) and (A', S') we mean an algebra isomorphism $\varphi : R\langle\langle A\rangle\rangle \to R\langle\langle A'\rangle\rangle$ with $\varphi|_R = \mathrm{id}$ and $\varphi(S)$ is cyclically equivalent to S' (see Definition 3.2).

In view of Proposition 3.5, any algebra homomorphism $R\langle\langle A\rangle\rangle \to R\langle\langle A'\rangle\rangle$ with $\varphi|_R = \mathrm{id}$ sends cyclically equivalent potentials to cyclically equivalent ones. It follows that right-equivalences of QPs have the expected properties: the composition of two right-equivalences, as well as the inverse of a right-equivalence, is again a right-equivalence. Note also that an isomorphism $\varphi : R\langle\langle A\rangle\rangle \to R\langle\langle A'\rangle\rangle$ induces an isomorphism of R-bimodules A and A' (cf. Proposition 2.4), so in dealing with right-equivalent QPs we can assume without loss of generality that $A = A'$.

In view of Propositions 3.3 and 3.7, any right-equivalence of QPs $(A, S) \cong (A', S')$ induces an isomorphism of the Jacobian ideals $J(S) \cong J(S')$ and of the Jacobian algebras $\mathcal{P}(A, S) \cong \mathcal{P}(A', S')$.

For any two QPs (A, S) and (A', S') (on the same set of vertices Q_0), we can form their *direct sum* $(A, S) \oplus (A', S') = (A \oplus A', S + S')$; it is well-defined since both complete path algebras $R\langle\langle A\rangle\rangle$ and $R\langle\langle A'\rangle\rangle$ have canonical embeddings into $R\langle\langle A \oplus A'\rangle\rangle$ as closed R-subalgebras.

We start our analysis of QPs with the case $S \in A^2$. In this case, $J(S)$ is the closure of the ideal generated by the subspace

$$\partial S = \{\partial_\xi(S) \mid \xi \in A^\star\} \subseteq A. \tag{4.3}$$

Definition 4.3. We say that a QP (A, S) is *trivial* if $S \in A^2$ and $\partial S = A$, or equivalently $\mathcal{P}(A, S) = R$.

The following description of trivial QPs is seen by standard linear algebra.

Proposition 4.4. *A QP (A, S) with $S \in A^2$ is trivial if and only if the set of arrows Q_1 consists of $2N$ distinct arrows $a_1, b_1, \ldots, a_N, b_N$ such that each $a_k b_k$ is a cyclic 2-path, and there is a change of arrows φ (see Definition 2.5) such that $\varphi(S)$ is cyclically equivalent to $a_1 b_1 + \cdots + a_N b_N$.*

Returning to general QPs, we now show that taking direct sums with trivial ones does not affect the Jacobian algebra.

Proposition 4.5. *If (A, S) is an arbitrary QP, and (C, T) is a trivial one, then the canonical embedding $R\langle\langle A\rangle\rangle \to R\langle\langle A \oplus C\rangle\rangle$ induces an isomorphism of Jacobian algebras $\mathcal{P}(A, S) \to \mathcal{P}(A \oplus C, S + T)$.*

Proof. Let L denote the closure of the two-sided ideal in $R\langle\langle A \oplus C\rangle\rangle$ generated by C; thus, L is the set of all (possibly infinite) linear combinations of paths, each of which contains at least one arrow from C. The definitions readily imply that

$$R\langle\langle A \oplus C\rangle\rangle = R\langle\langle A\rangle\rangle \oplus L \quad \text{and} \quad J(S + T) = J(S) \oplus L$$

(in the last equality, $J(S)$ is understood as the Jacobian ideal of S in $R\langle\langle A\rangle\rangle$). Therefore,

$$\begin{aligned}\mathcal{P}(A \oplus C, S+T) &= R\langle\langle A \oplus C\rangle\rangle / J(S+T) = (R\langle\langle A\rangle\rangle \oplus L)/(J(S) \oplus L)\\ &\cong R\langle\langle A\rangle\rangle / J(S) = \mathcal{P}(A,S),\end{aligned}$$

as desired. □

For an arbitrary QP (A, S), we denote by $S^{(2)} \in A^2$ the degree 2 homogeneous component of S. We call (A, S) *reduced* if $S^{(2)} = 0$, i.e., $S \in \mathfrak{m}(A)^3$. We define the *trivial* and *reduced* arrow spans of (A, S) as the finite-dimensional R-bimodules given by

$$A_{\mathrm{triv}} = A_{\mathrm{triv}}(S) = \partial S^{(2)}, \qquad A_{\mathrm{red}} = A_{\mathrm{red}}(S) = A/\partial S^{(2)} \tag{4.4}$$

(see (4.3)).

The following statement will play a crucial role in later sections.

Theorem 4.6 (Splitting Theorem). *For every QP (A, S) with the trivial arrow span A_{triv} and the reduced arrow span A_{red}, there exist a trivial QP $(A_{\mathrm{triv}}, S_{\mathrm{triv}})$ and a reduced QP $(A_{\mathrm{red}}, S_{\mathrm{red}})$ such that (A, S) is right-equivalent to the direct sum $(A_{\mathrm{triv}}, S_{\mathrm{triv}}) \oplus (A_{\mathrm{red}}, S_{\mathrm{red}})$. Furthermore, the right-equivalence class of each of the QPs $(A_{\mathrm{triv}}, S_{\mathrm{triv}})$ and $(A_{\mathrm{red}}, S_{\mathrm{red}})$ is determined by the right-equivalence class of (A, S).*

Let us first prove the existence of a desired right-equivalence

$$(A, S) \cong (A_{\mathrm{triv}}, S_{\mathrm{triv}}) \oplus (A_{\mathrm{red}}, S_{\mathrm{red}}). \tag{4.5}$$

There is nothing to prove if (A, S) is reduced, so let us assume that $S^{(2)} \neq 0$. Using Proposition 4.4 and replacing S if necessary by a cyclically equivalent potential, we can assume that S is of the form

$$S = \sum_{k=1}^{N} (a_k b_k + a_k u_k + v_k b_k) + S', \tag{4.6}$$

where each $a_k b_k$ is a cyclic 2-path, the arrows $a_1, b_1, \ldots, a_N, b_N$ form a basis of A_{triv}, the elements u_k and v_k belong to $\mathfrak{m}^2$, and the potential $S' \in \mathfrak{m}^3$ is a linear combination of cyclic paths containing none of the arrows a_k or b_k. The existence of a right-equivalence (4.5) becomes a consequence of the following lemma.

Lemma 4.7. *For every potential S of the form (4.6), there exists a unitriangular automorphism φ of $R\langle\langle A\rangle\rangle$ such that $\varphi(S)$ is cyclically equivalent to a potential of the form (4.6) with $u_k = v_k = 0$ for all k.*

We say that a potential S is *d-split* if it is of the form (4.6) with $u_k, v_k \in \mathfrak{m}^{d+1}$ for all k. To prove Lemma 4.7, we first show the following.

Lemma 4.8. *Suppose a potential S is d-split for some $d \geq 1$. There exists a unitriangular automorphism φ of $R\langle\langle A\rangle\rangle$ having depth d and such that $\varphi(S)$ is cyclically equivalent to a $2d$-split potential $\tilde{S}$ with $\varphi(S) - \tilde{S} \in \mathfrak{m}^{2d+2}$.*

Proof. Let us write S in the form (4.6) with $u_k, v_k \in \mathfrak{m}^{d+1}$. Let φ be the unitriangular automorphism of $R\langle\langle A\rangle\rangle$ acting on arrows as follows:

$$\varphi(a_k) = a_k - v_k,\ \varphi(b_k) = b_k - u_k,\ \varphi(c) = c \quad (c \in Q_1 - \{a_1, b_1, \dots, a_N, b_N\}).$$

Then φ is of depth d, so by (2.4), for each k, we have

$$\varphi(u_k) = u_k + u'_k,\ \varphi(v_k) = v_k + v'_k \quad (u'_k, v'_k \in \mathfrak{m}^{2d+1}).$$

Therefore, we obtain

$$\begin{aligned}\varphi(S) &= \sum_k ((a_k - v_k)(b_k - u_k) + (a_k - v_k)(u_k + u'_k) + (v_k + v'_k)(b_k - u_k)) + S' \\ &= \sum_k (a_k b_k + a_k u'_k + v'_k b_k) + S_1 + S',\end{aligned}$$

where

$$S_1 = -\sum_k (v_k u_k + v_k u'_k + v'_k u_k) \in \mathfrak{m}^{2d+2}.$$

In view of Definition 3.2, S_1 is cyclically equivalent to a potential of the form $\sum_k (a_k u''_k + v''_k b_k) + S''$, where $u''_k, v''_k \in \mathfrak{m}^{2d+1}$, and S'' is a linear combination of cyclic paths containing none of the a_k or b_k. Furthermore, we have

$$S_1 - S'' - \sum_k (a_k u''_k + v''_k b_k) \in \mathfrak{m}^{2d+2}.$$

We see that the desired potential $\tilde{S}$ can be chosen as

$$\tilde{S} = \sum_k (a_k b_k + a_k(u'_k + u''_k) + (v'_k + v''_k) b_k) + S' + S'',$$

completing the proof of Lemma 4.8. □

Proof of Lemma 4.7. Starting with a potential S of the form (4.6) and using repeatedly Lemma 4.8, we construct a sequence of potentials $S_1, S_2, \dots,$ and a sequence of unitriangular automorphisms $\varphi_1, \varphi_2, \dots,$ with the following properties:

(1) $S_1 = S$.
(2) S_d is 2^{d-1}-split.
(3) φ_d is of depth 2^{d-1}.
(4) $\varphi_d(S_d)$ is cyclically equivalent to S_{d+1}, and $\varphi_d(S_d) - S_{d+1} \in \mathfrak{m}^{2^d+2}$.

By property (3), setting

$$\varphi = \lim_{n\to\infty} \varphi_n \varphi_{n-1} \cdots \varphi_1, \tag{4.7}$$

we obtain a well-defined unitriangular automorphism φ of $R\langle\langle A\rangle\rangle$; indeed, in view of (2.4), for any $u \in R\langle\langle A\rangle\rangle$, if we write $\varphi_n \varphi_{n-1} \cdots \varphi_1(u)$ as $\sum_{d=0}^{\infty} u_n^{(d)}$ with $u_n^{(d)} \in A^d$, then each homogeneous component $u_n^{(d)}$ stabilizes as $n \to \infty$.

We claim that this automorphism φ satisfies the required properties in Lemma 4.7. To see this, for $d \geq 1$, define $C_d = \varphi_d(S_d) - S_{d+1}$. By (4), we have $C_d \in \{R\langle\langle A\rangle\rangle, R\langle\langle A\rangle\rangle\} \cap \mathfrak{m}^{2^d+2}$ (recall from Definition 3.4 that $\{R\langle\langle A\rangle\rangle, R\langle\langle A\rangle\rangle\}$

denotes the closure of the vector subspace in $R\langle\langle A\rangle\rangle$ spanned by all commutators). Using (1), it is easy to see that

$$\varphi_n\varphi_{n-1}\cdots\varphi_1(S) = S_{n+1} + \sum_{d=1}^{n}\varphi_n\varphi_{n-1}\cdots\varphi_{d+1}(C_d)$$

for every $n \geq 1$; passing to the limit as $n \to \infty$ yields

$$\varphi(S) = \lim_{n\to\infty} S_n + \varphi\Big(\sum_{d=1}^{\infty}(\varphi_d\cdots\varphi_1)^{-1}(C_d)\Big)$$

(the convergence of the series on the right is clear since any automorphism of $R\langle\langle A\rangle\rangle$ preserves the powers of $\mathfrak{m}$). We conclude that $\varphi(S)$ is cyclically equivalent to $\lim_{n\to\infty} S_n$. In view of (2), the latter element is of the form (4.6) with $u_k = v_k = 0$ for all k. This completes the proofs of Lemma 4.7 and of the existence of a right-equivalence (4.5). □

The above argument makes it clear that the right-equivalence class of $(A_{\mathrm{triv}}, S_{\mathrm{triv}})$ is determined by the right-equivalence class of (A, S) . To prove Theorem 4.6, it remains to show that the same is true for $(A_{\mathrm{red}}, S_{\mathrm{red}})$. Changing notation a little bit, we need to prove the following.

Proposition 4.9. *Let (A, S) and (A, S') be reduced QPs, and (C, T) a trivial QP. If $(A \oplus C, S + T)$ is right-equivalent to $(A \oplus C, S' + T)$ then (A, S) is right-equivalent to (A, S').*

We deduce Proposition 4.9 from the following result of independent interest.

Proposition 4.10. *Let (A, S) and (A, S') be reduced QPs such that $S' - S \in J(S)^2$. Then:*

(1) *$J(S') = J(S)$.*
(2) *(A, S) is right-equivalent to (A, S'). More precisely, there exists an algebra automorphism φ of $R\langle\langle A\rangle\rangle$ such that $\varphi(S)$ is cyclically equivalent to S', and $\varphi(u) - u \in J(S)$ for all $u \in R\langle\langle A\rangle\rangle$.*

Proof. (1) Since (A, S) is reduced, we have $J(S) \subseteq \mathfrak{m}^2$. As an easy consequence of the cyclic Leibniz rule (3.4), we see that

$$\partial_\xi(J(S)^2)_{\mathrm{cyc}} \subseteq \mathfrak{m}J(S) + J(S)\mathfrak{m}$$

for any $\xi \in A^\star$. It follows that

$$\partial_\xi S' - \partial_\xi S \in \mathfrak{m}J(S) + J(S)\mathfrak{m}, \tag{4.8}$$

implying that $J(S') \subseteq J(S)$.

To show the reverse inclusion, note that (4.8) also implies that

$$J(S) \subseteq J(S') + (\mathfrak{m}J(S) + J(S)\mathfrak{m}).$$

Applying the same inclusion to each of the terms $J(S)$ on the right, we obtain

$$J(S) \subseteq J(S') + (\mathfrak{m}^2J(S) + \mathfrak{m}J(S)\mathfrak{m} + J(S)\mathfrak{m}^2).$$

Continuing in the same way, we get

$$J(S) \subseteq J(S') + \sum_{k=0}^{n} \mathfrak{m}^k J(S)\mathfrak{m}^{n-k} \subseteq J(S') + \mathfrak{m}^{n+2}$$

for any $n \geq 1$. Remembering the definition of topology in $R\langle\langle A\rangle\rangle$ (see (2.3)) and the fact that $J(S')$ is closed, we conclude that $J(S) \subseteq J(S')$, finishing the proof of part (1) of Proposition 4.10.

(2) Let $Q_1 = \{a_1, \dots, a_N\}$ be the set of arrows (that is, a basis of A). Then a unitriangular automorphism φ of $R\langle\langle A\rangle\rangle$ is specified by an N-tuple of elements $b_1, \dots, b_N \in \mathfrak{m}^2$ such that

$$\varphi(a_k) = a_k + b_k \quad (k = 1, \dots, N). \tag{4.9}$$

Lemma 4.11. *Let (A, S) be a reduced QP, and let φ be a unitriangular automorphism of $R\langle\langle A\rangle\rangle$ given by (4.9). Then the potential $\varphi(S) - S - \sum_{k=1}^{N} b_k \partial_{a_k} S$ is cyclically equivalent to an element of $\mathfrak{m}I^2$, where I is the closure of the ideal in $R\langle\langle A\rangle\rangle$ generated by $b_1, \dots, b_N$.*

Proof. First consider the case where $S = a_{k_1} \cdots a_{k_d}$ is a cyclic path of length $d \geq 3$. Then $\varphi(S) = (a_{k_1} + b_{k_1}) \cdots (a_{k_d} + b_{k_d})$. Expanding this product, we see that the term that contains no factors b_{k_j} is equal to S, while the sum of the terms that contain exactly one factor b_{k_j} is easily seen to be cyclically equivalent to $\sum_{k=1}^{N} b_k \partial_{a_k} S$ (cf. (3.1)), and the rest of the terms are cyclically equivalent to elements of $\sum_{k=1}^{N} \mathfrak{m}(\mathfrak{m}^{d-1} \cap I) b_k$.

Writing a general potential $S \in \mathfrak{m}^3$ as a linear combination of cyclic paths, we see that $\varphi(S) - S - \sum_{k=1}^{N} b_k \partial_{a_k} S$ is cyclically equivalent to $\sum_{k=1}^{N} c_k b_k$, where each c_k is of the form

$$c_k = \sum_{l=1}^{N} a_l \sum_{d=3}^{\infty} c_{kl}^{(d)}$$

with $c_{kl}^{(d)} \in \mathfrak{m}^{d-1} \cap I$. Since I is closed, each c_k is a well-defined element of $\mathfrak{m}I$, implying the assertion of Lemma 4.11. □

We will also need one more lemma whose proof will be given in Section 13.

Lemma 4.12. *Let I be a closed ideal of $R\langle\langle A\rangle\rangle$, and J be the closure of an ideal generated by finitely many elements $f_1, \dots, f_N$, which are bi-homogeneous with respect to the vertex bigrading. Then every potential belonging to the ideal IJ is cyclically equivalent to an element of the form $\sum_{k=1}^{N} b_k f_k$, where all b_k belong to I.*

To prove part (2) of Proposition 4.10, we construct a sequence of N-tuples

$$(b_{1n}, \dots, b_{Nn}) \quad (n \geq 1)$$

of elements of $\mathfrak{m}^2$ and the corresponding unitriangular automorphisms φ_n of $R\langle\langle A\rangle\rangle$ (so that $\varphi_n(a_k) = a_k + b_{kn}$ for $k = 1, \dots, N$) such that, for all $n \geq 1$, we have

(a) $b_{kn} \in \mathfrak{m}^{n+1} \cap J(S)$ for $k = 1, \ldots, N$.

(b) S' is cyclically equivalent to $\varphi_0\varphi_1 \cdots \varphi_{n-1}(S + \sum_{k=1}^N b_{kn}\partial_{a_k} S)$ (with the convention that φ_0 is the identity automorphism).

We proceed by induction on n. In the basic case $n = 1$, the existence of an N-tuple $(b_{11}, \ldots, b_{N1})$ with the desired properties follows from Lemma 4.12 applied to $I = J = J(S)$ and $f_k = \partial_{a_k} S$ (note that $J(S) \subseteq \mathfrak{m}^2$, since (A, S) is assumed to be reduced).

Now assume that, for some $n \geq 1$, we have already defined the elements b_{kl} for $k = 1, \ldots, N$ and $l = 1, \ldots, n$, satisfying (a) and (b). Applying Lemma 4.11 to $b_k = b_{kn}$ (so that $\varphi = \varphi_n$), we find that $\varphi_n(S) - (S + \sum_{k=1}^N b_{kn}\partial_{a_k} S)$ is cyclically equivalent to an element of $\mathfrak{m}(\mathfrak{m}^{n+1} \cap J(S))^2$. We have

$$\mathfrak{m}(\mathfrak{m}^{n+1} \cap J(S))^2 \subseteq (\mathfrak{m}^{n+2} \cap J(S))J(S).$$

This implies in particular that $\varphi_n(S) - S$ is cyclically equivalent to an element of $J(S)^2$. Combining Proposition 3.7 with the already proved part (1) of Proposition 4.10, we conclude that $\varphi_n(J(S)) = J(\varphi_n(S)) = J(S)$. It follows that the difference $\varphi_n(S) - (S + \sum_{k=1}^N b_{kn}\partial_{a_k} S)$ is cyclically equivalent to an element of $\varphi_n((\mathfrak{m}^{n+2} \cap J(S))J(S))$.

Applying Lemma 4.12 to $I = \mathfrak{m}^{n+2} \cap J(S)$, $J = J(S)$ and $f_k = \partial_{a_k} S$, we see that every potential in $(\mathfrak{m}^{n+2} \cap J(S))J(S)$ is cyclically equivalent to a potential of the form

$$\sum_{k=1}^N b_{k,n+1}\partial_{a_k} S$$

for some $b_{k,n+1} \in \mathfrak{m}^{n+2} \cap J(S)$. It follows that $S + \sum_{k=1}^N b_{kn}\partial_{a_k} S$ is cyclically equivalent to $\varphi_n(S + \sum_{k=1}^N b_{k,n+1}\partial_{a_k} S)$. Thus, conditions (a) and (b) get satisfied with n replaced by $n + 1$, completing our inductive step.

In view of (a), $\lim_{n\to\infty} \varphi_1 \cdots \varphi_n$ is a well-defined automorphism φ of $R\langle\langle A\rangle\rangle$ such that $\varphi(u) - u \in J(S)$ for all $u \in R\langle\langle A\rangle\rangle$. Letting $n \to \infty$ in (2), we conclude that S' is cyclically equivalent to $\varphi(S)$, completing the proof of part (2) of Proposition 4.10. □

Proof of Proposition 4.9. We abbreviate $J = J(S)$ and $J' = J(S')$ (understood as the Jacobian ideals of S and S' in $R\langle\langle A\rangle\rangle$). As in Proposition 4.5, let L denote the closure of the two-sided ideal in $R\langle\langle A \oplus C\rangle\rangle$ generated by C. Then we have

$$R\langle\langle A \oplus C\rangle\rangle = R\langle\langle A\rangle\rangle \oplus L, \quad J(S + T) = J \oplus L, \quad J(S' + T) = J' \oplus L. \tag{4.10}$$

Let φ be an automorphism of $R\langle\langle A \oplus C\rangle\rangle$ such that $\varphi(S + T)$ is cyclically equivalent to $S' + T$. In view of (4.10) and Proposition 3.7, we have

$$\varphi(J \oplus L) = J' \oplus L. \tag{4.11}$$

Let $\psi : R\langle\langle A\rangle\rangle \to R\langle\langle A\rangle\rangle$ denote the restriction to $R\langle\langle A\rangle\rangle$ of the composition $p\varphi$, where p is the projection of $R\langle\langle A \oplus C\rangle\rangle$ onto $R\langle\langle A\rangle\rangle$ along L. In view of

Proposition 4.10, it suffices to show the following:

$$\begin{array}{l}\psi \text{ is an automorphism of } R\langle\langle A\rangle\rangle \text{ such that}\\ S' - \psi(S) \text{ is cyclically equivalent to an element of } \psi(J^2)\end{array} \tag{4.12}$$

(indeed, assuming (4.12) and using Proposition 3.7, we see that $\psi(J^2) = J(\psi(S))^2$, hence one can apply Proposition 4.10 to potentials S' and $\psi(S)$).

Clearly, ψ is an algebra homomorphism, so can be represented by a pair $(\psi^{(1)}, \psi^{(2)})$ as in Proposition 2.4. To show that ψ is an automorphism of $R\langle\langle A\rangle\rangle$, it suffices to show that $\psi^{(1)}$ is an R-bimodule automorphism of A. By the definition, if we write the R-bimodule automorphism $\varphi^{(1)}$ of $A \oplus C$ as a matrix

$$\begin{pmatrix} \varphi_{AA} & \varphi_{AC} \\ \varphi_{CA} & \varphi_{CC} \end{pmatrix},$$

then $\psi^{(1)} = \varphi_{AA}$. Since

$$\varphi(C) \subset \varphi(J \oplus L) = J' \oplus L \subseteq \mathfrak{m}(A)^2 \oplus L,$$

it follows that $\varphi_{AC} = 0$, implying that $\psi^{(1)} = \varphi_{AA}$ is an R-bimodule automorphism of A, and that ψ is an automorphism of $R\langle\langle A\rangle\rangle$.

Since $S' + T$ is cyclically equivalent to $\varphi(S + T)$, the same is true for the potentials obtained from them by applying the projection p; it follows that $S' - \psi(S)$ is cyclically equivalent to $p\varphi(T)$. Since $T \in C^2$, the claim that $S' - \psi(S)$ is cyclically equivalent to an element of $\psi(J^2)$ follows from the fact that $p\varphi(L) \subseteq \psi(J)$, or equivalently, that $\varphi(L) \subseteq \varphi(J) + L$. Applying the inverse automorphism φ^{-1} to both sides, it suffices to show that $L \subseteq J + \varphi^{-1}(L)$. Using the obvious symmetry between J and J', it is enough to show the inclusion $L \subseteq J' + \varphi(L)$.

Let us abbreviate $M = \mathfrak{m}(A \oplus C)$ and $I = J' + \varphi(L)$. Since $\varphi(J) \subseteq J' \oplus L$ and $J \subseteq \mathfrak{m}(A)^2$, it follows that $\varphi(J) \subseteq J' \oplus (L \cap M^2) = J' + ML + LM$. Therefore,

$$L \subseteq J' + L = \varphi(J) + \varphi(L) \subseteq I + ML + LM.$$

Substituting this upper bound for L into the right hand side, we deduce

$$L \subseteq I + M^2L + MLM + LM^2.$$

Continuing in the same way, for every $n > 0$, we have the inclusion

$$L \subseteq I + \sum_{k=0}^{n} M^k L M^{n-k} \subseteq I + M^{n+1}.$$

In view of (2.3), it follows that L is contained in $\overline{I}$, the closure of I in $R\langle\langle A \oplus C\rangle\rangle$. However, it is easy to see that $I = J' + \varphi(L)$ is closed in $R\langle\langle A \oplus C\rangle\rangle$ (indeed, the closedness of I is equivalent to that of $\varphi^{-1}(I) = \varphi^{-1}(J') + L$, and so, by symmetry, it is enough to show that $\varphi(J) + L$ is closed; but this is clear since $\varphi(J) + L = p^{-1}(\psi(J))$ is the inverse image of the closed ideal $\psi(J)$ of $R\langle\langle A\rangle\rangle$). This completes the proofs of Proposition 4.9 and Theorem 4.6. $\square$

Definition 4.13. We call the component $(A_{\mathrm{red}}, S_{\mathrm{red}})$ in the decomposition (4.5) the *reduced part* of a QP (A,S) (by Theorem 4.6, it is determined by (A,S) up to right-equivalence).

Definition 4.14. We call a quiver Q (as well as its arrow span A) 2-*acyclic* if it has no oriented 2-cycles, i.e., satisfies the following condition:

$$\text{For every pair of vertices } i \neq j, \text{ either } A_{i,j} = \{0\} \text{ or } A_{j,i} = \{0\}. \tag{4.13}$$

In the rest of this section we study conditions on a QP (A,S) guaranteeing that its reduced part is 2-acyclic. We need some preparation.

For a quiver Q with the arrow span A, let $\mathcal{C} = \mathcal{C}(A)$ denote the set of cyclic paths on A up to cyclic equivalence. Thus, $\mathcal{C}$ is either empty (if Q has no oriented cycles at all), or countable. The space of potentials up to cyclic equivalence is naturally identified with $K^{\mathcal{C}}$. We say that a K-valued function on $K^{\mathcal{C}}$ is *polynomial* if it depends on finitely many components of a potential S and can be expressed as a polynomial in these components. For a nonzero polynomial function F, we denote by $U(F) \subset K^{\mathcal{C}}$ the set of all potentials S such that $F(S) \neq 0$. By a *regular function* on $U(F)$ we mean a ratio of two polynomial functions on $K^{\mathcal{C}}$ such that the denominator vanishes nowhere on $U(F)$; in particular, any function of the form G/F^n, where G is a polynomial, is regular on $U(F)$. If A' is the arrow span of another quiver Q', we say that a map $K^{\mathcal{C}(A)} \to K^{\mathcal{C}(A')}$ is *polynomial* if each of its components is a polynomial function; similarly, a map $U(F) \to K^{\mathcal{C}(A')}$ is *regular* if each of its components is a regular function on $U(F)$.

Now suppose that the arrow span A satisfies (4.1), and let $\{a_1, b_1, \ldots, a_N, b_N\}$ be any maximal collection of distinct arrows in Q such that $b_k a_k$ is a cyclic 2-path for $k = 1, \ldots, N$. Then the quiver obtained from Q by removing this collection of arrows is clearly 2-acyclic. To such a collection we associate a nonzero polynomial function on $K^{\mathcal{C}(A)}$ given by

$$D^{b_1,\ldots,b_N}_{a_1,\ldots,a_N}(S) = \det(x_{b_q a_p})_{p,q=1,\ldots,N}, \tag{4.14}$$

where $x_{b_q a_p}$ is the sum of the coefficients of $b_q a_p$ and $a_p b_q$ in a potential S, with the convention that $x_{b_q a_p} = 0$ unless $b_q a_p$ is a cyclic 2-path.

Proposition 4.15. *The reduced part $(A_{\mathrm{red}}, S_{\mathrm{red}})$ of a QP (A,S) is 2-acyclic if and only if $D^{b_1,\ldots,b_N}_{a_1,\ldots,a_N}(S) \neq 0$ for some collection of arrows as above. Furthermore, if A' is the arrow span of the quiver obtained from Q by removing all arrows $a_1, b_1, \ldots, a_N, b_N$, then there exists a regular map $H : U(D^{b_1,\ldots,b_N}_{a_1,\ldots,a_N}) \to K^{\mathcal{C}(A')}$ such that, for any QP (A,S) with $S \in U(D^{b_1,\ldots,b_N}_{a_1,\ldots,a_N})$, the reduced part $(A_{\mathrm{red}}, S_{\mathrm{red}})$ is right-equivalent to $(A', H(S))$.*

The proof of Proposition 4.15 follows by tracing the construction of $(A_{\mathrm{red}}, S_{\mathrm{red}})$ given in the proof of Lemma 4.7. Note that we use the following convention. If A is 2-acyclic from the start then the only collection $\{a_1, b_1, \ldots, a_N, b_N\}$ as above is the empty set; in this case, the function $D^{b_1,\ldots,b_N}_{a_1,\ldots,a_N}$ is understood to be equal to 1, and H is just the identity mapping.

5. Mutations of quivers with potentials

Let (A, S) be a QP. Suppose that a vertex $k \in Q_0$ does not belong to an oriented 2-cycle. In other words, k satisfies the following condition:

$$\text{For every vertex } i, \text{ either } A_{i,k} \text{ or } A_{k,i} \text{ is zero.} \tag{5.1}$$

Replacing S if necessary with a cyclically equivalent potential, we can also assume that

$$\text{No cyclic path occurring in the expansion of } S \text{ starts (and ends) at } k. \tag{5.2}$$

Under these conditions, we associate to (A, S) a QP $\widetilde{\mu}_k(A, S) = (\widetilde{A}, \widetilde{S})$ on the same set of vertices Q_0. We define the homogeneous components $\widetilde{A}_{i,j}$ as follows:

$$\widetilde{A}_{i,j} = \begin{cases} (A_{j,i})^\star & \text{if } i = k \text{ or } j = k; \\ A_{i,j} \oplus A_{i,k}A_{k,j} & \text{otherwise;} \end{cases} \tag{5.3}$$

here the product $A_{i,k}A_{k,j}$ is understood as a subspace of $A^2 \subseteq R\langle\langle A \rangle\rangle$. Thus, the R-bimodule $\widetilde{A}$ is given by

$$\widetilde{A} = \overline{e}_k A \overline{e}_k \oplus A e_k A \oplus (e_k A)^\star \oplus (A e_k)^\star, \tag{5.4}$$

where we use the notation

$$\overline{e}_k = 1 - e_k = \sum_{i \in Q_0 - \{k\}} e_i. \tag{5.5}$$

We associate to Q_1 the set of arrows $\widetilde{Q}_1$ in the following way:

- Take all the arrows $c \in Q_1$ not incident to k.
- For each incoming arrow a and outgoing arrow b at k, create a "composite" arrow $[ba]$ corresponding to the product $ba \in A e_k A$.
- Replace each incoming arrow a (resp. each outgoing arrow b) at k by the corresponding arrow $a^\star$ (resp. $b^\star$) oriented in the opposite way.

More formally, for $i = k$ or $j = k$, we set

$$\widetilde{Q}_1 \cap \widetilde{A}_{i,j} = \{a^\star \mid a \in Q_1 \cap A_{j,i}\} \tag{5.6}$$

(the dual basis); and for i and j different from k, we define

$$\widetilde{Q}_1 \cap \widetilde{A}_{i,j} = (Q_1 \cap A_{i,j}) \sqcup \{[ba] \mid b \in Q_1 \cap A_{i,k},\ a \in Q_1 \cap A_{k,j}\}, \tag{5.7}$$

where $[ba] \in \widetilde{Q}_1 \cap A_{i,k}A_{k,j}$ denotes the arrow in $\widetilde{Q}_1$ associated with the product ba.

We now associate to S the potential $\widetilde{\mu}_k(S) = \widetilde{S} \in R\langle\langle \widetilde{A} \rangle\rangle$ given by

$$\widetilde{S} = [S] + \Delta_k, \tag{5.8}$$

where

$$\Delta_k = \Delta_k(A) = \sum_{a,b \in Q_1 \colon h(a) = t(b) = k} [ba] a^\star b^\star, \tag{5.9}$$

and $[S]$ is obtained by substituting $[a_p a_{p+1}]$ for each factor $a_p a_{p+1}$ with $t(a_p) = h(a_{p+1}) = k$ of any cyclic path $a_1 \cdots a_d$ occurring in the expansion of S (recall that none of these cyclic paths starts at k). It is easy to see that neither $[S]$ nor Δ_k depends on the choice of the basis Q_1 of A.

The following proposition is immediate from the definitions.

Proposition 5.1. *Suppose a QP (A,S) satisfies* (5.1) *and* (5.2)*, and a QP (A',S') is such that $e_kA' = A'e_k = \{0\}$. Then*

$$\widetilde{\mu}_k(A\oplus A', S+S') = \widetilde{\mu}_k(A,S)\oplus(A',S'). \tag{5.10}$$

Theorem 5.2. *The right-equivalence class of the QP $(\widetilde{A},\widetilde{S}) = \widetilde{\mu}_k(A,S)$ is determined by the right-equivalence class of (A,S).*

Proof. Let $\widehat{A}$ be the finite-dimensional R-bimodule given by

$$\widehat{A} = A\oplus(e_kA)^\star\oplus(Ae_k)^\star. \tag{5.11}$$

The natural embedding $A\to\widehat{A}$ identifies $R\langle\langle A\rangle\rangle$ with a closed subalgebra in $R\langle\langle\widehat{A}\rangle\rangle$. We also have a natural embedding $\widetilde{A}\to R\langle\langle\widehat{A}\rangle\rangle$ (sending each arrow $[ba]$ to the product ba). This allows us to identify $R\langle\langle\widetilde{A}\rangle\rangle$ with another closed subalgebra in $R\langle\langle\widehat{A}\rangle\rangle$, namely, with the closure of the linear span of the paths $\widehat{a}_1\cdots\widehat{a}_d$ such that $\widehat{a}_1\notin e_kA$ and $\widehat{a}_d\notin Ae_k$. Under this identification, the potential $\widetilde{S}$ given by (5.8) and viewed as an element of $R\langle\langle\widehat{A}\rangle\rangle$ is cyclically equivalent to the potential

$$S+\Big(\sum_{b\in Q_1\cap Ae_k} b^\star b\Big)\Big(\sum_{a\in Q_1\cap e_kA} aa^\star\Big).$$

Taking this into account, we see that Theorem 5.2 becomes a consequence of the following lemma.

Lemma 5.3. *Every automorphism φ of $R\langle\langle A\rangle\rangle$ can be extended to an automorphism $\widehat{\varphi}$ of $R\langle\langle\widehat{A}\rangle\rangle$ satisfying*

$$\widehat{\varphi}(R\langle\langle\widetilde{A}\rangle\rangle) = R\langle\langle\widetilde{A}\rangle\rangle, \tag{5.12}$$

and

$$\widehat{\varphi}\Big(\sum_{a\in Q_1\cap e_kA} aa^\star\Big) = \sum_{a\in Q_1\cap e_kA} aa^\star,\quad \widehat{\varphi}\Big(\sum_{b\in Q_1\cap Ae_k} b^\star b\Big) = \sum_{b\in Q_1\cap Ae_k} b^\star b. \tag{5.13}$$

In order to extend φ to an automorphism $\widehat{\varphi}$ of $R\langle\langle\widehat{A}\rangle\rangle$, we need only define the elements $\widehat{\varphi}(a^\star)$ and $\widehat{\varphi}(b^\star)$ for all arrows $a\in Q_1\cap e_kA$ and $b\in Q_1\cap Ae_k$.

We first deal with $\widehat{\varphi}(a^\star)$. Let $Q_1\cap e_kA = \{a_1,\dots,a_s\}$. In view of Proposition 2.4, the action of φ on these arrows is given by

$$(\varphi(a_1)\ \ \cdots\ \ \varphi(a_s)) = (a_1\ \ \cdots\ \ a_s)(C_0+C_1), \tag{5.14}$$

where:

- C_0 is an invertible $s\times s$ matrix with entries in K such that its (p,q)-entry is 0 unless $t(a_p) = t(a_q)$;
- C_1 is an $s\times s$ matrix whose (p,q)-entry belongs to $\mathfrak{m}(A)_{t(a_p),t(a_q)}$.

Note that $C_0 + C_1$ is invertible, and its inverse is of the same form: indeed,

$$(C_0 + C_1)^{-1} = (I + C_0^{-1}C_1)^{-1}C_0^{-1} = \Big(I + \sum_{n=1}^{\infty}(-1)^n(C_0^{-1}C_1)^n\Big)C_0^{-1}.$$

Now we define the elements $\widehat{\varphi}(a_p^\star)$ by setting

$$\begin{pmatrix}\widehat{\varphi}(a_1^\star)\\ \vdots \\ \widehat{\varphi}(a_s^\star)\end{pmatrix} = (C_0 + C_1)^{-1}\begin{pmatrix}a_1^\star\\ \vdots \\ a_s^\star\end{pmatrix}.$$

It follows that

$$\begin{aligned}\widehat{\varphi}\Big(\sum_p a_p a_p^\star\Big) &= (\widehat{\varphi}(a_1) \ \cdots \ \widehat{\varphi}(a_s))\begin{pmatrix}\widehat{\varphi}(a_1^\star)\\ \vdots \\ \widehat{\varphi}(a_s^\star)\end{pmatrix}\\ &= (a_1 \ \cdots \ a_s)\begin{pmatrix}a_1^\star\\ \vdots \\ a_s^\star\end{pmatrix} = \sum_p a_p a_p^\star.\end{aligned}$$

For $b \in Q_1 \cap Ae_k$, we define $\widehat{\varphi}(b^\star)$ in a similar way. Namely, let $Q_1 \cap Ae_k = \{b_1, \dots, b_t\}$. As above, the action of φ on these arrows is given by

$$\begin{pmatrix}\varphi(b_1)\\ \vdots \\ \varphi(b_t)\end{pmatrix} = (D_0 + D_1)\begin{pmatrix}b_1\\ \vdots \\ b_t\end{pmatrix}, \tag{5.15}$$

where:

- D_0 is an invertible $t \times t$ matrix with entries in K such that its (p, q)-entry is 0 unless $h(b_p) = h(b_q)$;
- D_1 is a $t \times t$ matrix whose (p, q)-entry belongs to $\mathfrak{m}(A)_{h(b_p),h(b_q)}$.

As above, we see that $D_0 + D_1$ is invertible, and its inverse is of the same form. Now we define the elements $\widehat{\varphi}(b_q^\star)$ by setting

$$(\widehat{\varphi}(b_1^\star) \ \cdots \ \widehat{\varphi}(b_t^\star)) = (b_1^\star \ \cdots \ b_t^\star)(D_0 + D_1)^{-1}.$$

It follows that

$$\begin{aligned}\widehat{\varphi}\Big(\sum_q b_q^\star b_q\Big) &= (\widehat{\varphi}(b_1)^\star \ \cdots \ \widehat{\varphi}(b_t^\star))\begin{pmatrix}\widehat{\varphi}(b_1)\\ \vdots \\ \widehat{\varphi}(b_t)\end{pmatrix}\\ &= (b_1^\star \ \cdots \ b_t^\star)\begin{pmatrix}b_1\\ \vdots \\ b_t\end{pmatrix} = \sum_q b_q^\star b_q.\end{aligned}$$

The condition (5.13) is then clearly satisfied; the construction also makes it clear that the automorphism $\widehat{\varphi}$ of $R\langle\langle\widehat{A}\rangle\rangle$ preserves the subalgebra $R\langle\langle\widetilde{A}\rangle\rangle$.

As a consequence of Proposition 2.4, $\widehat{\varphi}$ restricts to an automorphism of $R\langle\langle \widetilde{A} \rangle\rangle$, satisfying (5.12) and completing the proofs of Lemma 5.3 and Theorem 5.2. $\square$

Note that even if a QP (A, S) is assumed to be reduced, the QP $\widetilde{\mu}_k(A, S) = (\widetilde{A}, \widetilde{S})$ is not necessarily reduced because the component $[S]^{(2)} \in \widetilde{A}^2$ may be nonzero. Combining Theorems 4.6 and 5.2, we obtain the following corollary.

Corollary 5.4. *Suppose a QP* (A, S) *satisfies* (5.1) *and* (5.2)*, and let* $\widetilde{\mu}_k(A, S) = (\widetilde{A}, \widetilde{S})$*. Let* $(\overline{A}, \overline{S})$ *be a reduced QP such that*

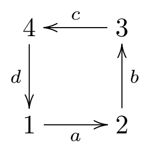
$$(\widetilde{A}, \widetilde{S}) \cong (\widetilde{A}_{\mathrm{triv}}, \widetilde{S}^{(2)}) \oplus (\overline{A}, \overline{S}) \tag{5.16}$$

(see (4.5)*). Then the right-equivalence class of* $(\overline{A}, \overline{S})$ *is determined by the right-equivalence class of* (A, S)*.*

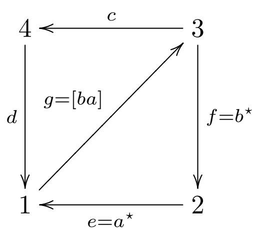
Definition 5.5. In the situation of Corollary 5.4, we use the notation $\mu_k(A, S) = (\overline{A}, \overline{S})$ and call the correspondence $(A, S) \mapsto \mu_k(A, S)$ the *mutation at vertex* k.

Note that if a QP (A, S) satisfies (5.1) then the same is true for $\widetilde{\mu}_k(A, S)$ and for $\mu_k(A, S)$. Thus, the mutation μ_k is a well-defined transformation on the set of right-equivalence classes of reduced QPs satisfying (5.1). (With some abuse of notation, we sometimes denote a right-equivalence class by the same symbol as any of its representatives.)

Example 5.6. Consider the quiver Q with vertices $\{1, 2, 3, 4\}$ and arrows $a : 1 \to 2$, $b : 2 \to 3$, $c : 3 \to 4$ and $d : 4 \to 1$:



Let $S = dcba$. Let us perform the mutation at vertex 2. The arrow a is replaced by $e := a^\star : 2 \to 1$, and b is replaced by $f := b^\star : 3 \to 2$. We also have a new arrow $g := [ba] : 1 \to 3$. So $\widetilde{\mu}_2(A)$ corresponds to the quiver with vertices $\{1, 2, 3, 4\}$ and arrows c, d, e, f, g:

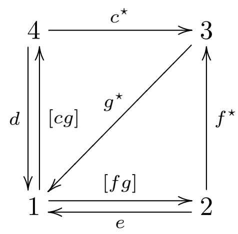


The potential $\widetilde{\mu}_2(S) = \widetilde{S}$ is given by

$$\widetilde{S} = dcg + gef;$$

thus, $\widetilde{\mu}_2(A, S)$ is reduced, and we have $\widetilde{\mu}_2(A, S) = \mu_2(A, S)$.

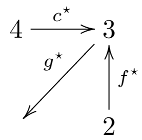
Note that $\widetilde{S}$ does not satisfy condition (5.2) with respect to vertex $k = 3$ since the path gef starts and ends at 3. But we can fix this condition by replacing $\widetilde{S}$ with a cyclically equivalent potential, say $S' = dcg + efg$. Now let us mutate $(\widetilde{A}, S')$ at vertex 3. The arrows c, f, g are replaced by $c^\star : 4 \to 3$, $f^\star : 2 \to 3$ and $g^\star : 3 \to 1$, respectively. We also add new arrows $[cg] : 1 \to 4$ and $[fg] : 1 \to 2$. Thus, $\widetilde{\mu}_3(\widetilde{A}, S')$ has arrows $\{d, e, c^\star, f^\star, g^\star, [cg], [fg]\}$:



The potential $\widetilde{\mu}_3(S')$ is given by

$$\mu_3(S') = d[cg] + e[fg] + [fg]g^\star f^\star + [cg]g^\star c^\star.$$

It is not reduced, so to obtain the reduced QP $\mu_3(\widetilde{A}, S')$, we need to remove the trivial part of $\widetilde{\mu}_3(\widetilde{A}, S')$. The resulting quiver is as follows:



Since it is acyclic (that is, has no oriented cycles), the corresponding potential is 0.

Our next result is that every mutation is an involution.

Theorem 5.7. *The correspondence $\mu_k : (A, S) \to (\overline{A}, \overline{S})$ acts as an involution on the set of right-equivalence classes of reduced QPs satisfying* (5.1)*, that is, $\mu_k^2(A, S)$ is right-equivalent to (A, S).*

Proof. Let (A, S) be a reduced QP satisfying (5.1) and (5.2). Let $\widetilde{\mu}_k(A, S) = (\widetilde{A}, \widetilde{S})$ and $\widetilde{\mu}_k^2(A, S) = \widetilde{\mu}_k(\widetilde{A}, \widetilde{S}) = (\widetilde{\widetilde{A}}, \widetilde{\widetilde{S}})$. In view of Theorem 4.6 and Proposition 5.1, it is enough to show that

$$(\widetilde{\widetilde{A}}, \widetilde{\widetilde{S}}) \text{ is right-equivalent to } (A, S) \oplus (C, T), \text{ where } (C, T) \text{ is a trivial QP.} \quad (5.17)$$

Using (5.4) twice, and identifying $(e_k A)^\star$ with $A^\star e_k$, and $(Ae_k)^\star$ with $e_k A^\star$, where $A^\star$ is the dual R-bimodule of A, we conclude that

$$\widetilde{\widetilde{A}} = A \oplus Ae_k A \oplus A^\star e_k A^\star. \quad (5.18)$$

Furthermore, the basis of arrows in $\widetilde{\widetilde{A}}$ consists of the original set of arrows Q_1 in A together with the arrows $[ba] \in Ae_k A$ and $[a^\star b^\star] \in A^\star e_k A^\star$ for $a \in Q_1 \cap e_k A$ and $b \in Q_1 \cap Ae_k$.

Remembering (5.8) and (5.9), we see that the potential $\widetilde{\widetilde{S}}$ is given by

$$\widetilde{\widetilde{S}} = [[S]] + [\Delta_k(A)] + \Delta_k(\widetilde{A}) = [S] + \sum_{a,b \in Q_1 :\, h(a)=t(b)=k} ([ba][a^\star b^\star] + [a^\star b^\star]ba), \quad (5.19)$$

hence it is cyclically equivalent to

$$S_1 = [S] + \sum_{a,b \in Q_1 :\, h(a)=t(b)=k} ([ba] + ba)[a^\star b^\star] \quad (5.20)$$

(recall that $[S]$ is obtained by substituting $[a_p a_{p+1}]$ for each factor $a_p a_{p+1}$ with $t(a_p) = h(a_{p+1}) = k$ of any cyclic path $a_1 \cdots a_d$ occurring in the path expansion of S). Let us abbreviate

$$(C, T) = \Big(Ae_k A \oplus A^\star e_k A^\star, \sum_{a,b \in Q_1 :\, h(a)=t(b)=k} [ba][a^\star b^\star] \Big).$$

This is a trivial QP (cf. Proposition 4.4); therefore to prove Theorem 5.7 it suffices to show that the QP $(\widetilde{\widetilde{A}}, S_1)$ given by (5.18) and (5.20) is right-equivalent to $(A, S) \oplus (C, T)$. We proceed in several steps.

Step 1: Let φ_1 be the change of arrows automorphism of $R\langle\langle \widetilde{\widetilde{A}} \rangle\rangle$ (see Definition 2.5) multiplying each arrow $b \in Q_1 \cap Ae_k$ by -1, and fixing the rest of the arrows in $\widetilde{\widetilde{A}}$. Then the potential $S_2 = \varphi_1(S_1)$ is given by

$$S_2 = [S] + \sum_{a,b \in Q_1 :\, h(a)=t(b)=k} ([ba] - ba)[a^\star b^\star].$$

Step 2: Let φ_2 be the unitriangular automorphism of $R\langle\langle \widetilde{\widetilde{A}} \rangle\rangle$ (see Definition 2.5) sending each arrow $[ba] \in Ae_k A$ to $[ba] + ba$, and fixing the rest of the arrows in $\widetilde{\widetilde{A}}$. Remembering the definition of $[S]$, it is easy to see that the potential $\varphi_2(S_2)$ is cyclically equivalent to a potential of the form

$$S_3 = S + \sum_{a,b \in Q_1 :\, h(a)=t(b)=k} [ba]([a^\star b^\star] + f(a, b))$$

for some elements $f(a, b) \in \mathfrak{m}(A \oplus Ae_k A)^2$.

Step 3: Let φ_3 be the unitriangular automorphism of $R\langle\langle \widetilde{\widetilde{A}} \rangle\rangle$ sending each arrow $[a^\star b^\star] \in A^\star e_k A^\star$ to $[a^\star b^\star] - f(a, b)$, and fixing the rest of the arrows in $\widetilde{\widetilde{A}}$. Then $\varphi_3(S_3) = S + T$.

Combining these three steps, we conclude that the QP $(\widetilde{\widetilde{A}}, S_1)$ is right-equivalent to $(\widetilde{\widetilde{A}}, S + T) = (A, S) \oplus (C, T)$, finishing the proof of Theorem 5.7. $\square$

6. Some mutation invariants

In this section we fix a vertex k and study the effect of the mutation μ_k on the Jacobian algebra $\mathcal{P}(A,S)$. We will use the following notation: for an R-bimodule B, define

$$B_{\hat{k},\hat{k}} = \overline{e}_k B \overline{e}_k = \bigoplus_{i,j\neq k} B_{i,j} \tag{6.1}$$

(see (5.5)). Note that if B is a (topological) algebra then $B_{\hat{k},\hat{k}}$ is a (closed) subalgebra of B.

Proposition 6.1. *Suppose a QP (A,S) satisfies* (5.1) *and* (5.2)*, and let $(\widetilde{A},\widetilde{S}) = \widetilde{\mu}_k(A,S)$ be given by* (5.4) *and* (5.8)*. Then the algebras $\mathcal{P}(A,S)_{\hat{k},\hat{k}}$ and $\mathcal{P}(\widetilde{A},\widetilde{S})_{\hat{k},\hat{k}}$ are isomorphic to each other.*

Proof. In view of (5.4), we have

$$\widetilde{A}_{\hat{k},\hat{k}} = A_{\hat{k},\hat{k}} \oplus Ae_kA. \tag{6.2}$$

Thus, the algebra $R\langle\langle \widetilde{A}_{\hat{k},\hat{k}}\rangle\rangle$ is generated by the arrows $c \in Q_1 \cap A_{\hat{k},\hat{k}}$ and $[ba]$ for $a \in Q_1 \cap e_kA$ and $b \in Q_1 \cap Ae_k$. The following fact is immediate from the definitions.

Lemma 6.2. *The correspondence sending each $c \in Q_1 \cap A_{\hat{k},\hat{k}}$ to itself, and each generator $[ba]$ to ba, extends to an algebra isomorphism*

$$R\langle\langle \widetilde{A}_{\hat{k},\hat{k}}\rangle\rangle \to R\langle\langle A\rangle\rangle_{\hat{k},\hat{k}}.$$

Let $u \mapsto [u]$ denote the isomorphism $R\langle\langle A\rangle\rangle_{\hat{k},\hat{k}} \to R\langle\langle \widetilde{A}_{\hat{k},\hat{k}}\rangle\rangle$ inverse to that in Lemma 6.2. It acts in the same way as the correspondence $S \mapsto [S]$ in (5.8): $[u]$ is obtained by substituting $[a_pa_{p+1}]$ for each factor a_pa_{p+1} with $t(a_p) = h(a_{p+1}) = k$ of any path $a_1 \cdots a_d$ occurring in the path expansion of u.

Lemma 6.3. *The correspondence $u \mapsto [u]$ induces an algebra epimorphism*

$$\mathcal{P}(A,S)_{\hat{k},\hat{k}} \to \mathcal{P}(\widetilde{A},\widetilde{S})_{\hat{k},\hat{k}}.$$

Proof. It is enough to prove the following two facts:

$$R\langle\langle \widetilde{A}\rangle\rangle_{\hat{k},\hat{k}} = R\langle\langle \widetilde{A}_{\hat{k},\hat{k}}\rangle\rangle + J(\widetilde{S})_{\hat{k},\hat{k}}; \tag{6.3}$$

$$[J(S)_{\hat{k},\hat{k}}] \subseteq R\langle\langle \widetilde{A}_{\hat{k},\hat{k}}\rangle\rangle \cap J(\widetilde{S})_{\hat{k},\hat{k}}. \tag{6.4}$$

To show (6.3), we note that if a path $\widetilde{a}_1 \cdots \widetilde{a}_d \in R\langle\langle \widetilde{A}\rangle\rangle_{\hat{k},\hat{k}}$ does not belong to $R\langle\langle \widetilde{A}_{\hat{k},\hat{k}}\rangle\rangle$ then it must contain one or more factors of the form $a^\star b^\star$ with $h(a) = t(b) = k$. In view of (5.8) and (5.9), we have

$$a^\star b^\star = \partial_{[ba]}\widetilde{S} - \partial_{[ba]}[S]. \tag{6.5}$$

Substituting this expression for each factor $a^\star b^\star$, we see that $\widetilde{a}_1 \cdots \widetilde{a}_d \in R\langle\langle \widetilde{A}_{\hat{k},\hat{k}}\rangle\rangle + J(\widetilde{S})_{\hat{k},\hat{k}}$, as desired.

To show (6.4), we note that $J(S)_{\hat{k},\hat{k}}$ is easily seen to be the closure of the ideal in $R\langle\langle A\rangle\rangle_{\hat{k},\hat{k}}$ generated by the elements $\partial_c S$ for all arrows $c \in Q_1$ with $t(c) \neq k$ and $h(c) \neq k$, together with the elements $(\partial_a S)a'$ for $a, a' \in Q_1 \cap e_k A$, and $b'(\partial_b S)$ for $b, b' \in Q_1 \cap Ae_k$. Let us apply the map $u \mapsto [u]$ to these generators. First, we have

$$[\partial_c S] = \partial_c \widetilde{S}. \tag{6.6}$$

With a little bit more work (using (6.5)), we obtain

$$\begin{aligned}[(\partial_a S)a'] &= \sum_{t(b)=k} (\partial_{[ba]}[S])[ba'] = \sum_{t(b)=k} (\partial_{[ba]}\widetilde{S} - a^\star b^\star)[ba'] \\ &= \sum_{t(b)=k} (\partial_{[ba]}\widetilde{S})[ba'] - a^\star \partial_{a'^\star}\widetilde{S},\end{aligned} \tag{6.7}$$

and

$$\begin{aligned}[b'(\partial_b S)] &= \sum_{h(a)=k} [b'a](\partial_{[ba]}[S]) = \sum_{h(a)=k} [b'a](\partial_{[ba]}\widetilde{S} - a^\star b^\star) \\ &= \sum_{h(a)=k} [b'a](\partial_{[ba]}\widetilde{S}) - (\partial_{b'^\star}\widetilde{S})b^\star.\end{aligned} \tag{6.8}$$

This implies the desired inclusion in (6.4). □

To finish the proof of Proposition 6.1, it is enough to show that the epimorphism in Lemma 6.3 (let us denote it by α) is in fact an isomorphism. To do this, we construct the left inverse algebra homomorphism $\beta : \mathcal{P}(\widetilde{A}, \widetilde{S})_{\hat{k},\hat{k}} \to \mathcal{P}(A, S)_{\hat{k},\hat{k}}$ (so that $\beta\alpha$ is the identity map on $\mathcal{P}(A, S)_{\hat{k},\hat{k}}$). We define β as the composition of three maps. First, we apply the epimorphism $\mathcal{P}(\widetilde{A}, \widetilde{S})_{\hat{k},\hat{k}} \to \mathcal{P}(\widetilde{\widetilde{A}}, \widetilde{\widetilde{S}})_{\hat{k},\hat{k}}$ defined in the same way as α. Remembering the proof of Theorem 5.7 and using the notation introduced there, we then apply the isomorphism $\mathcal{P}(\widetilde{\widetilde{A}}, \widetilde{\widetilde{S}})_{\hat{k},\hat{k}} \to \mathcal{P}(A \oplus C, S + T)_{\hat{k},\hat{k}}$ induced by the automorphism $\varphi_3\varphi_2\varphi_1$ of $R\langle\langle A \oplus C\rangle\rangle$. Finally, we apply the isomorphism $\mathcal{P}(A \oplus C, S + T)_{\hat{k},\hat{k}} \to \mathcal{P}(A, S)_{\hat{k},\hat{k}}$ given in Proposition 4.5.

Since all the maps involved are algebra homomorphisms, it is enough to check that $\beta\alpha$ fixes the generators $p(c)$ and $p(ba)$ of $\mathcal{P}(A, S)_{\hat{k},\hat{k}}$, where p is the projection $R\langle\langle A\rangle\rangle \to \mathcal{P}(A, S)$, and a, b, c have the same meaning as above. This is done by direct tracing of the definitions. □

Proposition 6.4. *In the situation of Proposition* 6.1*, if the Jacobian algebra* $\mathcal{P}(A, S)$ *is finite-dimensional then so is* $\mathcal{P}(\widetilde{A}, \widetilde{S})$.

Proof. We start by showing that finite dimensionality of $\mathcal{P}(A, S)$ follows from a seemingly weaker condition.

Lemma 6.5. *Let* $J \subseteq \mathfrak{m}(A)$ *be a closed ideal in* $R\langle\langle A\rangle\rangle$. *Then the quotient algebra* $R\langle\langle A\rangle\rangle/J$ *is finite-dimensional provided the subalgebra* $R\langle\langle A\rangle\rangle_{\hat{k},\hat{k}}/J_{\hat{k},\hat{k}}$ *is. In particular, the Jacobian algebra* $\mathcal{P}(A, S)$ *is finite-dimensional if and only if so is the subalgebra* $\mathcal{P}(A, S)_{\hat{k},\hat{k}}$.

Proof. Similarly to (6.1), for an R-bimodule B, we define

$$B_{k,\hat{k}} = e_k B \overline{e}_k = \bigoplus_{j \neq k} B_{k,j}, \quad B_{\hat{k},k} = \overline{e}_k B e_k = \bigoplus_{i \neq k} B_{i,k}.$$

We need to show that if $R\langle\langle A\rangle\rangle_{\hat{k},\hat{k}}/J_{\hat{k},\hat{k}}$ is finite-dimensional then so is each of the spaces $R\langle\langle A\rangle\rangle_{k,\hat{k}}/J_{k,\hat{k}}$, $R\langle\langle A\rangle\rangle_{\hat{k},k}/J_{\hat{k},k}$ and $R\langle\langle A\rangle\rangle_{k,k}/J_{k,k}$. Let us treat $R\langle\langle A\rangle\rangle_{k,k}/J_{k,k}$; the other two cases are similar (and a little simpler).

Let

$$Q_1 \cap A_{k,\hat{k}} = \{a_1, \dots, a_s\}, \quad Q_1 \cap A_{\hat{k},k} = \{b_1, \dots, b_t\}.$$

We have

$$R\langle\langle A\rangle\rangle_{k,k} = Ke_k \oplus \bigoplus_{l,m} a_l R\langle\langle A\rangle\rangle_{\hat{k},\hat{k}} b_m.$$

It follows that there is a surjective map $\alpha : K \times \mathrm{Mat}_{s\times t}(R\langle\langle A\rangle\rangle_{\hat{k},\hat{k}}) \to R\langle\langle A\rangle\rangle_{k,k}/J_{k,k}$ given by

$$\alpha(c, C) = p\left(ce_k + (a_1 \ \cdots \ a_s) C \begin{pmatrix} b_1 \\ \vdots \\ b_t \end{pmatrix}\right),$$

where $\mathrm{Mat}_{s\times t}(B)$ stands for the space of $s \times t$ matrices with entries in B, and p is the projection $R\langle\langle A\rangle\rangle \to R\langle\langle A\rangle\rangle/J$. The kernel of α contains $\mathrm{Mat}_{s\times t}(J_{\hat{k},\hat{k}})$, hence $R\langle\langle A\rangle\rangle_{k,k}/J_{k,k}$ is isomorphic to a quotient of the finite-dimensional space $K \times \mathrm{Mat}_{s\times t}(R\langle\langle A\rangle\rangle_{\hat{k},\hat{k}}/J_{\hat{k},\hat{k}})$. Thus, $R\langle\langle A\rangle\rangle_{k,k}/J_{k,k}$ is finite-dimensional, as desired. □

To finish the proof of Proposition 6.4, suppose that $\mathcal{P}(A, S)$ is finite-dimensional. Then so is $\mathcal{P}(\widetilde{A}, \widetilde{S})_{\hat{k},\hat{k}}$ by Proposition 6.1. Applying Lemma 6.5 to the QP $(\widetilde{A}, \widetilde{S})$, we conclude that $\mathcal{P}(\widetilde{A}, \widetilde{S})$ is finite-dimensional, as desired. □

Remembering (5.16) and using Proposition 4.5, we see that Propositions 6.1 and 6.4 have the following corollary.

Corollary 6.6. *Suppose* (A, S) *is a reduced QP satisfying* (5.1)*, and let* $(\overline{A}, \overline{S}) = \mu_k(A, S)$ *be a reduced QP obtained from* (A, S) *by the mutation at* k*. Then the algebras* $\mathcal{P}(A, S)_{\hat{k},\hat{k}}$ *and* $\mathcal{P}(\overline{A}, \overline{S})_{\hat{k},\hat{k}}$ *are isomorphic to each other, and* $\mathcal{P}(A, S)$ *is finite-dimensional if and only if so is* $\mathcal{P}(\overline{A}, \overline{S})$*.*

We see that the class of QPs with finite-dimensional Jacobian algebras is invariant under mutations. Let us now present another such class.

Definition 6.7. For every QP (A, S), we define its *deformation space* $\mathrm{Def}(A, S)$ by

$$\mathrm{Def}(A, S) = \mathrm{Tr}(\mathcal{P}(A, S))/R \tag{6.9}$$

(see Definitions 3.1 and 3.4).

Remark 6.8. Definition 6.7 can be motivated as follows (we keep the following arguments informal although with some work they can be made rigorous). Let $G = \mathrm{Aut}(R\langle\langle A\rangle\rangle)$ be the group of algebra automorphisms of $R\langle\langle A\rangle\rangle$ (acting as the identity on R). Using Proposition 2.4, we can think of G as an infinite-dimensional algebraic group. In view of Definition 3.4, G acts naturally on the trace space $\mathrm{Tr}(R\langle\langle A\rangle\rangle)$. Remembering Definition 4.2, it is natural to think of the deformation space of a QP (A, S) as the normal space at $\pi(S)$ of the orbit $G \cdot \pi(S)$ in the ambient space $\pi(\mathfrak{m}(A)^2)$ (recall that π stands for the natural projection $R\langle\langle A\rangle\rangle \to \mathrm{Tr}(R\langle\langle A\rangle\rangle)$). Arguing as in Lemma 4.11, we conclude that the infinitesimal action of (the Lie algebra of) G on $\pi(\mathfrak{m}(A)^2)$ is by the transformations

$$\pi(u) \mapsto \pi\Big(\sum_{k=1}^{N} b_k \partial_{a_k} u\Big),$$

where $Q_1 = \{a_1, \ldots, a_N\}$ is the set of arrows, and $b_k \in \mathfrak{m}(A)_{h(a_k),t(a_k)}$ (this is well defined in view of Proposition 3.3). This makes it natural to identify the tangent space at $\pi(S)$ of $G \cdot \pi(S)$ with $\pi(J(S))$, hence to identify the corresponding normal space with $\pi(\mathfrak{m}(A))/\pi(J(S))$, or equivalently, with the space $\mathrm{Def}(S)$ given by (6.9).

Proposition 6.9. *In the situation of Proposition* 6.1*, the deformation spaces* $\mathrm{Def}(\widetilde{A}, \widetilde{S})$ *and* $\mathrm{Def}(A, S)$ *are isomorphic to each other.*

Proof. In view of Proposition 3.5, $\mathrm{Def}(A, S)$ is isomorphic to $\mathrm{Tr}(\mathcal{P}(A, S)_{\hat{k},\hat{k}})/R_{\hat{k},\hat{k}}$. Therefore, our assertion is immediate from Proposition 6.1. □

Definition 6.10. We call a QP (A, S) *rigid* if $\mathrm{Def}(A, S) = \{0\}$, i.e., $\mathrm{Tr}(\mathcal{P}(A, S)) = R$.

Combining Propositions 4.5 and 6.9, we obtain the following corollary.

Corollary 6.11. *If a reduced QP* (A, S) *satisfies* (5.1) *and is rigid, then the QP* $(\overline{A}, \overline{S}) = \mu_k(A, S)$ *is also rigid.*

Some examples of rigid and nonrigid QPs will be given in Section 8.

7. Nondegenerate QPs

If we wish to be able to apply to a reduced QP (A, S) the mutation at *every* vertex of Q_0, the R-bimodule A must satisfy (5.1) at all vertices. Thus, the arrow span A must be 2-acyclic (see Definition 4.14). Such an arrow span A can be encoded by a skew-symmetric integer matrix $B = B(A) = (b_{i,j})$ with rows and columns labeled by Q_0, by setting

$$b_{i,j} = \dim A_{i,j} - \dim A_{j,i}. \tag{7.1}$$

Indeed, the dimensions of the components $A_{i,j}$ are recovered from B by

$$\dim A_{i,j} = [b_{i,j}]_+, \tag{7.2}$$

where we use the notation

$$[x]_+ = \max(x, 0). \tag{7.3}$$

Proposition 7.1. *Let (A, S) be a 2-acyclic reduced QP, and suppose that the reduced QP $\mu_k(A, S) = (\overline{A}, \overline{S})$ obtained from (A, S) by the mutation at some vertex k (see Definition 5.5) is also 2-acyclic. Let $B(A) = (b_{i,j})$ and $B(\overline{A}) = (\overline{b}_{i,j})$ be the skew-symmetric integer matrices given by (7.1). Then*

$$\overline{b}_{i,j} = \begin{cases} -b_{i,j} & \text{if } i = k \text{ or } j = k; \\ b_{i,j} + [b_{i,k}]_+ \, [b_{k,j}]_+ - [-b_{i,k}]_+ \, [-b_{k,j}]_+ & \text{otherwise.} \end{cases} \tag{7.4}$$

Proof. First we note that by Proposition 4.4, if (C, T) is a trivial QP then $\dim C_{i,j} = \dim C_{j,i}$ for all i, j. In view of (5.16), this implies that

$$\overline{b}_{i,j} = \dim \overline{A}_{i,j} - \dim \overline{A}_{j,i} = \dim \widetilde{A}_{i,j} - \dim \widetilde{A}_{j,i}, \tag{7.5}$$

where $(\widetilde{A}, \widetilde{S}) = \widetilde{\mu}_k(A, S)$. Using (5.3), we obtain

$$\dim \widetilde{A}_{i,j} = \begin{cases} \dim A_{j,i} & \text{if } i = k \text{ or } j = k; \\ \dim A_{i,j} + \dim A_{i,k} \dim A_{k,j} & \text{otherwise.} \end{cases}$$

To obtain (7.4), it remains to substitute these expressions into (7.5) and use (7.2). □

An easy calculation using the obvious identity $x = [x]_+ - [-x]_+$ shows that the second case in (7.4) can be rewritten in several equivalent ways:

$$\begin{aligned} \overline{b}_{i,j} &= b_{i,j} + \operatorname{sgn}(b_{i,k})[b_{i,k}b_{k,j}]_+ = b_{i,j} + [-b_{i,k}]_+ \, b_{k,j} + b_{i,k}[b_{k,j}]_+ \\ &= b_{i,j} + \frac{|b_{i,k}|b_{k,j} + b_{i,k}|b_{k,j}|}{2}. \end{aligned}$$

It follows that the transformation $B \mapsto \overline{B}$ given by (7.4) coincides with the *matrix mutation* at k which plays a crucial part in the theory of cluster algebras (cf. [18, (4.3)], [20, (2.2), (2.5)]).

We see that the mutations of 2-acyclic QPs provide a natural framework for matrix mutations. With some abuse of notation, we denote by $\mu_k(A)$ the 2-acyclic R-bimodule such that the skew-symmetric matrix $B(\mu_k(A))$ is obtained from $B(A)$ by the mutation at k; thus, $\mu_k(A)$ is determined by A up to isomorphism.

Note that the matrix mutations at arbitrary vertices can be iterated indefinitely, while the 2-acyclicity condition (4.13) can be destroyed by a QP mutation. We will study QPs for which this does not happen.

Definition 7.2. Let $k_1, \dots, k_l \in Q_0$ be a finite sequence of vertices such that $k_p \neq k_{p+1}$ for $p = 1, \dots, l-1$. We say that a QP (A, S) is $(k_l, \dots, k_1)$-*nondegenerate* if all the QPs $(A, S), \mu_{k_1}(A, S), \mu_{k_2}\mu_{k_1}(A, S), \dots, \mu_{k_l} \cdots \mu_{k_1}(A, S)$ are 2-acyclic (hence well-defined). We say that (A, S) is *nondegenerate* if it is $(k_l, \dots, k_1)$-nondegenerate for every sequence of vertices as above.

To state our next result recall the terminology introduced before Proposition 4.15. In particular, for a given quiver with the arrow span A, the QPs on A are identified with the elements of $K^{\mathcal{C}(A)}$.

Proposition 7.3. *Suppose that the base field K is infinite, Q is a 2-acyclic quiver with the arrow span A, a sequence of vertices $k_1, \dots, k_l$ is as in Definition 7.2, and define $A' = \mu_{k_l} \cdots \mu_{k_1}(A)$. Then there exist a nonzero polynomial function $F : K^{\mathcal{C}(A)} \to K$ and a regular map $G : U(F) \to K^{\mathcal{C}(A')}$ such that every QP (A, S) with $S \in U(F)$ is $(k_l, \dots, k_1)$-nondegenerate, and, for any QP (A, S) with $S \in U(F)$, the QP $\mu_{k_l} \cdots \mu_{k_1}(A, S)$ is right-equivalent to $(A', G(S))$.*

Proof. We proceed by induction on l. First let us deal with the case $l = 1$, that is, with a single mutation μ_k. Recall that $\mu_k(A, S) = (\overline{A}, \overline{S})$ is the reduced part of the QP $\widetilde{\mu}_k(A, S) = (\widetilde{A}, \widetilde{S})$ given by (5.3) and (5.8). It is clear from the definition that $\widetilde{S} = \widetilde{G}(S)$ for a polynomial map $\widetilde{G} : K^{\mathcal{C}(A)} \to K^{\mathcal{C}(\widetilde{A})}$. Now let us apply Proposition 4.15 to the quiver with the arrow span $\widetilde{A}$. We see that there exists a polynomial function of the form $D^{d_1,\dots,d_N}_{c_1,\dots,c_N}$ on $K^{\mathcal{C}(\widetilde{A})}$ (see (4.14), where we have changed the notation for the arrows to avoid the notation conflict with Section 5) such that the reduced part $(\overline{A}, \overline{S})$ of a QP $(\widetilde{A}, \widetilde{S})$ is 2-acyclic whenever $\widetilde{S} \in U(D^{d_1,\dots,d_N}_{c_1,\dots,c_N})$. Furthermore, for $\widetilde{S} \in U(D^{d_1,\dots,d_N}_{c_1,\dots,c_N})$, the QP $(\overline{A}, \overline{S})$ is right-equivalent to $(A', H(\widetilde{S}))$ for some regular map $H : U(D^{d_1,\dots,d_N}_{c_1,\dots,c_N}) \to K^{\mathcal{C}(A')}$, where $A' = \mu_k(A)$. We now define a polynomial function $F : K^{\mathcal{C}(A)} \to K$ and a regular map $G : U(F) \to K^{\mathcal{C}(A')}$ by setting

$$F = D^{d_1,\dots,d_N}_{c_1,\dots,c_N} \circ \widetilde{G}, \qquad G = H \circ \widetilde{G}. \tag{7.6}$$

To finish the argument for $l = 1$, it remains to show that F is not identically equal to zero. But this is clear from the definitions (4.14) and (5.8), since the oriented 2-cycles in $\widetilde{A}$ (up to cyclic equivalence) are of the form $c[ba]$ and so are in a bijection with the oriented 3-cycles cba in A that pass through k.

Now assume that $l \geq 2$, and that our assertion holds if we replace l by $l - 1$. Let $A_1 = \mu_{k_1}(A)$, so $A' = \mu_{k_l} \cdots \mu_{k_2}(A_1)$. By the inductive assumption, there exist a nonzero polynomial function $F' : K^{\mathcal{C}(A_1)} \to K$ and a regular map $G' : U(F') \to K^{\mathcal{C}(A')}$ such that, for any QP (A_1, S_1) with $S_1 \in U(F')$, the QP $\mu_{k_l} \cdots \mu_{k_2}(A_1, S_1)$ is right-equivalent to $(A', G'(S_1))$. Also by the already established case $l = 1$, there exists a nonzero polynomial function $F'' : K^{\mathcal{C}(A_1)} \to K$ such that, for any QP (A_1, S_1) with $S_1 \in U(F'')$, the QP $\mu_{k_1}(A_1, S_1)$ is 2-acyclic, hence is right-equivalent to some QP on A. Since the base field K is assumed to be infinite, we have $U(F') \cap U(F'') \neq \emptyset$. Choose $S_1^{(0)} \in U(F') \cap U(F'')$, and let $(A, S_0) = \mu_k(A_1, S_1^{(0)})$. By Theorem 5.7, we have $\mu_k(A, S_0) = (A_1, S_1^{(0)})$. By the above argument for $l = 1$, there exist a nonzero polynomial function $F_1 : K^{\mathcal{C}(A)} \to K$ and a regular map $G_1 : U(F_1) \to K^{\mathcal{C}(A_1)}$ (of the type (7.6)) such that $\mu_k(A, S) = (A_1, G_1(S))$ for $S \in U(F_1)$. In particular, we have $G_1(S_0) = S_1^{(0)}$, implying that $F' \circ G_1$ is a nonzero polynomial function on $K^{\mathcal{C}(A)}$. It follows that the nonzero polynomial function $F(S) = F_1(S) F'(G_1(S))$ and the regular map $G = G' \circ G_1 : U(F) \to K^{\mathcal{C}(A')}$ are well-defined and satisfy all the required conditions. This completes the proof of Proposition 7.3. □

Corollary 7.4. *For every 2-acyclic arrow span A, there exists a countable family $\mathcal{F}$ of nonzero polynomial functions on $K^{\mathcal{C}(A)}$ such that the QP (A, S) is nondegenerate whenever $S \in \bigcap_{F \in \mathcal{F}} U(F)$. In particular, if the base field K is uncountable, then there exists a nondegenerate QP on A.*

Proof. By Proposition 7.3, for every sequence $k_1, \dots, k_l$ as in Definition 7.2, there exists a nonzero polynomial function $F_{k_1,\dots,k_l}$ on $K^{\mathcal{C}(A)}$ such that a QP (A, S) is $(k_l, \dots, k_1)$-nondegenerate for $S \in U(F_{k_1,\dots,k_l})$. These functions form a desired countable family $\mathcal{F}$.

It remains to prove that $\bigcap_{F \in \mathcal{F}} U(F) \neq \emptyset$ provided K is uncountable. If A is acyclic, i.e., has no oriented cycles, then $K^{\mathcal{C}(A)} = \{0\}$, and each of the functions in $\mathcal{F}$ is just a nonzero constant, so there is nothing to prove; no assumption on K is needed here. If A has at least one oriented cycle then the set $\mathcal{C}(A)$ is countable (recall that it consists of cyclic paths up to cyclic equivalence). Thus, we can realize $K^{\mathcal{C}(A)}$ as the polynomial ring $K[X_1, X_2, \dots]$ in countably many indeterminates. Since K is uncountable, we can choose x_1 so that $F(x_1) \neq 0$ for all $F \in \mathcal{F} \cap K[X_1]$. Then we choose x_2 so that $F(x_1, x_2) \neq 0$ for all $F \in \mathcal{F} \cap K[X_1, X_2]$. Continuing like this, we find a sequence $x_1, x_2, \dots$ such that $F(x_1, x_2, \dots) \neq 0$ for all $F \in \mathcal{F}$. □

8. Rigid QPs

Proposition 8.1. *Every rigid reduced QP (A, S) is 2-acyclic.*

Proof. First note that the definition of rigidity can be conveniently restated as follows:

$$\text{a QP } (A, S) \text{ is rigid if and only if every potential on } A \text{ is cyclically equivalent to an element of } J(S). \tag{8.1}$$

Now suppose for the sake of contradiction that for some $i \neq j$ both components $A_{i,j}$ and $A_{j,i}$ are nonzero. Choose nonzero elements $a \in A_{i,j}$ and $b \in A_{j,i}$. Remembering the definition of the Jacobian ideal (see Definition 3.1), it is easy to see that the cyclic part of $J(S)$ is contained in $\mathfrak{m}(A)^3$. It follows that ab is *not* cyclically equivalent to an element of $J(S)$, in contradiction with (8.1). □

Combining Proposition 8.1 with Corollary 6.11 yields the following result.

Corollary 8.2. *Any rigid QP is nondegenerate.*

Let us now give some examples.

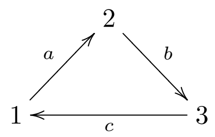
Example 8.3. Recall that a skew-symmetric integer matrix B is *acyclic* if the corresponding directed graph (with an arrow $i \to j$ associated with each entry $b_{i,j} > 0$) has no oriented cycles. If the matrix $B(A)$ given by (7.1) is acyclic, then $R\langle\langle A\rangle\rangle_{\mathrm{cyc}} = \{0\}$, and so the only QP associated with A is $(A, 0)$, which is clearly rigid.

Now suppose that A is 2-acyclic, and that $B(A)$ is not necessarily acyclic but is mutation-equivalent to an acyclic matrix (i.e., can be transformed to an acyclic matrix by a sequence of mutations). As a consequence of Corollary 6.11

and Theorem 5.7, there exists a potential S such that (A, S) is a rigid reduced QP; moreover, (A, S) is unique up to right-equivalences.

Example 8.4. For A arbitrary, the deformation space of a QP $(A, 0)$ is naturally identified with the space of potentials modulo cyclic equivalence, hence it is infinite-dimensional provided A has at least one oriented cycle.

Example 8.5 (Cyclic triangle). Let Q be the quiver with three vertices $1, 2, 3$ and three arrows $a : 1 \to 2$, $b : 2 \to 3$ and $c : 3 \to 1$:



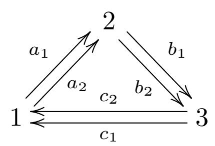
An arbitrary potential S is cyclically equivalent to the one of the form $S = F(cba)$, where $F \in K[[t]]$ is a formal power series. The deformation space $\mathrm{Def}(A, S)$ is naturally isomorphic to the quotient space of $tK[[t]]$ modulo the ideal generated by tdF/dt. If $\mathrm{char}\, K = 0$, and $n \geq 1$ is the smallest exponent such that t^n appears in F, then $\dim \mathrm{Def}(A, S) = n-1$. In particular, (A, S) is rigid if and only if $n = 1$.

Now let us consider the QP $(\widetilde{A}, \widetilde{S}) = \widetilde{\mu}_2(A, S)$; in view of (5.6), (5.7) and (5.8), $\widetilde{A}$ has four arrows $a^\star, b^\star, c, [ba]$, and

$$\widetilde{S} = F(c[ba]) + [ba]a^\star b^\star.$$

Thus, if $n \geq 2$ then $(\widetilde{A}, \widetilde{S})$ is reduced and so is equal to $\mu_2(A, S) = (\overline{A}, \overline{S})$. Since $\mu_2(A, S)$ has an oriented 2-cycle formed by the arrows c and $[ba]$, the mutations at vertices 1 and 3 cannot be applied. We see that the QP $(A, F(cba))$ is degenerate for $n \geq 2$.

Example 8.6 (Double cyclic triangle). Now consider the quiver with three vertices $1, 2, 3$ and six arrows $a_1, a_2 : 1 \to 2$, $b_1, b_2 : 2 \to 3$ and $c_1, c_2 : 3 \to 1$:



Any potential S on A is cyclically equivalent to the one whose degree 3 component belongs to the 8-dimensional space $A^3_{1,1} = A_{1,3}A_{3,2}A_{2,1}$. It is known that the diagonal action of the group GL_2^3 on $K^2 \otimes K^2 \otimes K^2$ has seven orbits (see e.g., [23, Chapter 14, Example 4.5]). Thus, by performing a change of arrows automorphism, we can assume that the degree 3 component of S is one of the representatives of these orbits. An easy case-by-case analysis shows that no potential can give rise to a rigid QP on A.

For instance, let

$$S = c_1b_1a_1 + c_2b_2a_2. \tag{8.2}$$

Then $J(S)$ is the closure of the ideal in $R\langle\langle A\rangle\rangle$ generated by six elements

$$c_1b_1, b_1a_1, a_1c_1, c_2b_2, b_2a_2, a_2c_2.$$

One checks easily that the cyclic path $c_1b_2a_1c_2b_1a_2$ is not cyclically equivalent to an element of $J(S)$, hence (A,S) is not rigid.

Now let us compute $\mu_2(A,S)$. Again setting $(\widetilde{A},\widetilde{S})=\widetilde{\mu}_2(A,S)$, we see that $\widetilde{A}$ has ten arrows

$$a_1^\star, a_2^\star, b_1^\star, b_2^\star, c_1, c_2, [b_1a_1], [b_1a_2], [b_2a_1], [b_2a_2],$$

and

$$\widetilde{S}=c_1[b_1a_1]+c_2[b_2a_2]+\sum_{i,j=1}^{2}[b_ia_j]a_j^\star b_i^\star.$$

To obtain the splitting (4.5) of $(\widetilde{A},\widetilde{S})$, we apply the automorphism φ of $R\langle\langle\widetilde{A}\rangle\rangle$ fixing all arrows except c_1 and c_2, and such that $\varphi(c_i)=c_i-a_i^\star b_i^\star$. An easy check shows that $\mu_2(A,S)=(\overline{A},\overline{S})$ can be described as follows: $\overline{A}$ is 6-dimensional with the arrows $a_1^\star, a_2^\star, b_1^\star, b_2^\star, [b_1a_2], [b_2a_1]$, and

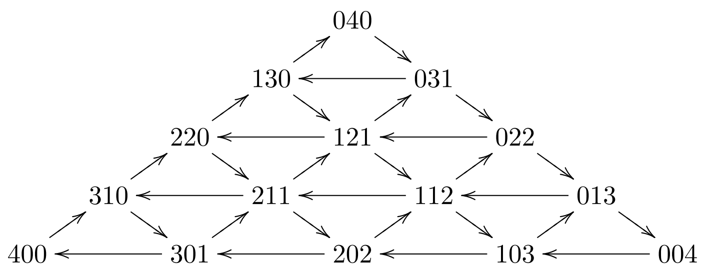
$$\overline{S}=[b_1a_2]a_2^\star b_1^\star+[b_2a_1]a_1^\star b_2^\star.$$

Thus, the mutated QP $(\overline{A},\overline{S})$ can be obtained from the initial QP (A,S) by a renumbering of the vertices. This implies that one can apply to (A,S) unlimited mutations at arbitrary vertices, so (A,S) is a nonrigid, nondegenerate QP.

Example 8.7. For each $n\geq 0$, let us consider the following quiver $Q(n)$, which we refer to as the *triangular grid of order* n. The vertex set of $Q(n)$ is

$$Q(n)_0=\{(p,q,r)\in\mathbb{Z}^3_{\geq 0}\mid p+q+r=n\};$$

and there is a single arrow $(p_1,q_1,r_1)\to(p_2,q_2,r_2)$ if and only if $(p_2,q_2,r_2)-(p_1,q_1,r_1)$ is one of the three vectors $(-1,1,0),(0,-1,1),(1,0,-1)$. Thus, the vertices of $Q(n)$ form a regular triangular grid with n^2 cyclically oriented unit triangles. For example, the quiver $Q(4)$ is



Let $A=A(n)$ be the arrow span of $Q(n)$, and let $a\in A$ (resp. $b\in A$, $c\in A$) denote the sum of all arrows of $Q(n)$ that are parallel translates of $(-1,1,0)$ (resp. $(0,-1,1)$, $(1,0,-1)$). Thus, every interior vertex i has three incoming arrows e_ia, e_ib, e_ic and three outgoing arrows ae_i, be_i, ce_i. Every path of length d can be uniquely written as $a_d\cdots a_2a_1e_j$, where each a_s is one of the elements a,b,c, and

j is a vertex; this expression is nonzero if and only if the polygonal line obtained by attaching to the vertex j the vectors corresponding to $a_1, \dots, a_d$ (in this order) is contained in our grid.

Define a potential $S \in A^3$ by setting

$$S = cba - bca.$$

Then the Jacobian ideal $J(S)$ is generated by the elements

$$(cb - bc)e_j,\ (ac - ca)e_j,\ (ba - ab)e_j$$

for all vertices j. It follows that the image of the path $a_d \cdots a_2 a_1 e_j$ in the Jacobian algebra $\mathcal{P}(A, S)$ does not change under any permutation of the factors $a_1, \dots, a_d$. In particular, we see that $\mathcal{P}(A, S)$ is spanned by the images of the paths $c^k b^l a^m e_j$ for all vertices j and all k, l, m such that $0 \le k, l, m \le n$; hence $\mathcal{P}(A, S)$ is finite-dimensional. By a similar argument, it is easy to see that (A, S) is rigid. Indeed, every potential on A is cyclically equivalent to an element of the closure of the span of the elements $(cba)^m e_j$ for all vertices j and all $m \ge 0$. Denoting by p the projection $R\langle\langle A\rangle\rangle \to \mathrm{Tr}(\mathcal{P}(A, S))$, we see that the rigidity of (A, S) follows from the fact that $p(cbae_j) = 0$ for all j. Now if $ae_j \ne 0$ and $h(ae_j) = i$ then

$$p(cbae_j) = p(acbe_i) = p(cabe_i) = p(cbae_i).$$

Continuing in the same way, we see that, for any $m \ge 1$ such that $a^m e_j \ne 0$, we have $p(cbae_j) = p(cbae_k)$, where k is the end-point of the path $a^m e_j$. Taking m the largest such that $a^m e_j \ne 0$, we conclude that $p(cbae_j) = 0$, as desired.

As shown in [27], the quiver $Q(3)$ in Example 8.7 is *not* mutation-equivalent to an acyclic one. So there exist QPs with finite-dimensional Jacobian algebras (and also rigid QPs), which are not mutation-equivalent to acyclic ones.

We now describe a procedure to obtain new QPs with finite-dimensional Jacobian algebras (and new rigid QPs) from old ones.

Definition 8.8. For a QP (A, S) and a subset I of the vertex set Q_0, we define the *restriction* of (A, S) to I as the QP $(A|_I, S|_I)$ given by

$$A|_I = \bigoplus_{i,j \in I} A_{i,j} \quad \text{and} \quad S|_I = \psi_I(S),$$

where $\psi_I : R\langle\langle A\rangle\rangle \to R\langle\langle A|_I\rangle\rangle$ is the algebra homomorphism such that $\psi_I(a) = a$ for $a \in A|_I$, and $\psi_I(b) = 0$ for any arrow b not belonging to $A|_I$.

Proposition 8.9. *The homomorphism ψ_I induces an epimorphism of Jacobian algebras $\mathcal{P}(A, S) \to \mathcal{P}(A|_I, S|_I)$ and an epimorphism of deformation spaces* $\mathrm{Def}(A, S) \to \mathrm{Def}(A|_I, S|_I)$*. Therefore, if $\mathcal{P}(A, S)$ is finite-dimensional, or if (A, S) is rigid, then the same is true for $(A|_I, S|_I)$.*

Proof. Remembering (3.1), it is easy to see that $\psi_I(\partial_a S) = \partial_a \psi_I(S)$ for any arrow $a \in A|_I$, and $\psi_I(\partial_b S) = 0$ for any arrow b not belonging to $A|_I$. Therefore, we have $\psi_I(J(S)) = J(\psi_I(S))$, implying all the assertions. □

Corollary 8.10. *Suppose that A and A' are 2-acyclic, and there is a rigid QP (A, S) on A. Let $B(A)$ and $B(A')$ be the corresponding skew-symmetric integer matrices given by* (7.1)*. Suppose that $B(A')$ can be obtained by a simultaneous permutation of rows and columns from some principal submatrix of a matrix mutation-equivalent to $B(A)$. Then there exists a rigid QP (A', S') on A'.*

Proof. In view of (7.1), the matrix $B(A|_I)$ is the principal submatrix of $B(A)$ involving rows and columns from I. Therefore, our assertion follows by combining Proposition 8.9 with Corollary 6.11 and Proposition 7.1. □

We conclude this section with a combinatorial application of Corollary 8.10.

Corollary 8.11. *Let $B = B(A(n))$ be the matrix associated with the triangular grid of some order n (see Example* 8.7*). Then none of the matrices mutation-equivalent to B contains*

$$B' = \begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}$$

as a principal submatrix.

Proof. Note that $B' = B(A')$, where A' is the quiver in Example 8.6. We have seen that there exists a rigid QP on A, but not on A'. Thus our statement is a special case of Corollary 8.10. □

Remark 8.12. Using the results in [1, Section 2.6] (cf. also [22, Theorem 1]), it is easy to see that the quiver $Q(n)$ in Example 8.7 is naturally associated with the cluster algebra structure in the coordinate ring of the base affine space of the group SL_{n+3}. We have been informed by J. Schröer that, following his suggestion, A. Seven has shown that a skew-symmetric matrix B' associated with an arbitrary tree appears as a principal submatrix in some matrix mutation-equivalent to the matrix $B(A(n))$ for some n. J. Schröer also informed us that Corollary 8.11 has also been proved by B. Keller (using a different method).

9. Relation to cluster-tilted algebras

Let Q be a quiver with the vertex span R and the arrow span A. Assume that Q is 2-acyclic. Let $B(A)$ be the corresponding skew-symmetric integer matrix given by (7.1). As shown in [19], the matrix $B(A)$ gives rise to a cluster algebra of finite type if and only if Q is mutation-equivalent to a Dynkin quiver (that is, an arbitrary orientation of a Dynkin diagram of one of the types A_n, D_n, E_6, E_7, or E_8). In particular, as we have seen in Example 8.3, there is a rigid reduced QP (A, S), and it is unique up to a right-equivalence; in fact, (the right-equivalence class of) (A, S) is obtained by a sequence of mutations from a QP $(A_0, 0)$, where A_0 is associated to a Dynkin quiver. We now present an explicit choice of such a potential S. To do this, we need some preparation.

First note that, according to [19, Theorem 1.8], every quiver mutation-equivalent to a Dynkin quiver has no multiple edges, that is, $|b_{i,j}| \le 1$ for every entry of $B(A)$. In other words, we have

$$\dim A_{i,j} \le 1 \quad \text{for all } i \text{ and } j. \tag{9.1}$$

Therefore, we can unambiguously denote by $a_{i,j}$ the only arrow in a nonzero subspace $A_{i,j}$. We will also use the convention that $a_{i,j} = 0$ whenever $A_{i,j} = \{0\}$.

Second, we use the following terminology: a *chordless cycle* in (the underlying graph of) Q is a d-subset of vertices $I \subseteq Q_0$ that can be bijectively labeled by the elements of $\mathbb{Z}/d\mathbb{Z}$ so that the edges between them are precisely $\{i, i+1\}$ for $i \in \mathbb{Z}/d\mathbb{Z}$. According to [19, Proposition 9.7], if Q is mutation-equivalent to a Dynkin quiver then the arrows of every chordless cycle in Q must be cyclically oriented. In terms of the arrow span A, this condition can be stated as follows:

For any chordless d-cycle I, there exists a bijection $\nu : \mathbb{Z}/d\mathbb{Z} \to I$ such that the set of arrows in the restriction $A|_I$ is $\{a_{\nu(i),\nu(i+1)} \mid i \in \mathbb{Z}/d\mathbb{Z}\}$ (9.2)

(see Definition 8.8). Note that the choice of a bijection ν in (9.2) is unique up to a cyclic shift.

We call a potential S on A *primitive* if it has the form

$$S = \sum_I x_I a_{\nu(1),\nu(2)} \cdots a_{\nu(d-1),\nu(d)} a_{\nu(d),\nu(1)}, \tag{9.3}$$

where the (finite) sum is over all chordless cycles I in Q, for each I there is a bijection ν chosen as in (9.2), and the coefficients x_I are some nonzero elements of the base field K.

Proposition 9.1. *If a quiver Q with the arrow span A is mutation-equivalent to a Dynkin quiver, and S is a primitive potential on A, then the QP (A, S) is rigid.*

To prove Proposition 9.1, we impose on Q a weaker assumption that its arrow span A satisfies (4.13), (9.1) and (9.2). Choose some vertex k, and (as in Section 7) let $\mu_k(A)$ denote the 2-acyclic R-bimodule such that the skew-symmetric matrix $B(\mu_k(A))$ is obtained from $B(A)$ by the mutation at k. It is easy to see that $\mu_k(A)$ satisfies (9.1) but not necessarily (9.2). In view of Corollary 6.11, the assertion of Proposition 9.1 is a consequence of the following lemma.

Lemma 9.2. *Suppose that the arrow span A of a quiver Q satisfies* (4.13), (9.1) *and* (9.2)*, and that $\mu_k(A)$ also satisfies* (9.2) *for some vertex k. Let S be a primitive potential on A. Then the QP $(\overline{A}, \overline{S}) = \mu_k(A, S)$ is right-equivalent to a QP on $\mu_k(A)$ with a primitive potential.*

Proof. Let $\widetilde{\mu}_k(A, S) = (\widetilde{A}, \widetilde{S})$ be the QP given by (5.6)–(5.8). Denote by $\mathrm{In}(k)$ (resp. $\mathrm{Out}(k)$) the set of vertices j (resp. i) with $\dim A_{k,j} = 1$ (resp. $\dim A_{i,k} = 1$). In view of (9.2), every arrow a with both ends in $\mathrm{In}(k) \cup \mathrm{Out}(k)$ has $h(a) \in \mathrm{In}(k)$ and $t(a) \in \mathrm{Out}(k)$. We denote the set of these arrows by Q_1'. The arrows of $\widetilde{A}$ can be unambiguously denoted as follows:

- $\widetilde{a}_{i,j} = a_{i,j}$ for all i, j different from k and such that $a_{i,j} \neq 0$.
- $\widetilde{a}_{i,j} = [a_{i,k}a_{k,j}]$ for all $i \in \mathrm{Out}(k), j \in \mathrm{In}(k)$.
- $\widetilde{a}_{j,k} = a^{\star}_{k,j}$ for $j \in \mathrm{In}(k)$.
- $\widetilde{a}_{k,i} = a^{\star}_{i,k}$ for $i \in \mathrm{Out}(k)$.

We can split S into the sum of four terms

$$S = S_1 + S_2 + S_3 + S_4, \tag{9.4}$$

where

- S_1 involves (oriented) 3-cycles $a_{i,k}a_{k,j}a_{j,i}$;
- S_2 involves chordless d-cycles through k with $d \geq 4$;
- S_3 involves chordless cycles having an arrow $a_{j,i} \in Q'_1$ but not passing through k;
- S_4 involves chordless cycles having no arrows with both ends in $\mathrm{In}(k) \cup \{k\} \cup \mathrm{Out}(k)$.

Using (9.2), it is easy to see that every chordless cycle I involved in S_2 or S_3 has exactly one common point with each of the sets $\mathrm{In}(k)$ and $\mathrm{Out}(k)$. Also note that every term in S_1 or S_3 contains exactly one arrow from Q'_1, while none of the terms in S_2 or S_4 contain such arrows. Remembering (5.8), we write the potential $\widetilde{S}$ as follows:

$$\widetilde{S} = [S_1] + [S_2] + [S_3] + [S_4] + \Delta_k. \tag{9.5}$$

We have

$$[S_1] = \sum_{a_{j,i} \in Q'_1} x_{\{i,j,k\}} \widetilde{a}_{i,j} a_{j,i},$$

and this is the degree 2 component $\widetilde{S}^{(2)}$ of $\widetilde{S}$. In view of (5.16), the arrows in $\overline{A}$ are obtained from those in $\widetilde{A}$ by removing all arrows $a_{j,i} \in Q'_1$ and their opposites $\widetilde{a}_{i,j}$.

Inspecting the other terms in (9.5), it is easy to see that

$$[S_2] = \overline{S}_3, \quad [S_4] = \overline{S}_4, \quad \Delta_k = \overline{S}_1 + \sum_{a_{j,i} \in Q'_1} \widetilde{a}_{i,j}\widetilde{a}_{j,k}\widetilde{a}_{k,i},$$

$$[S_3] = \sum_{a_{j,i} \in Q'_1} f_{i,j} a_{j,i},$$

where $f_{i,j} = \partial_{a_{j,i}} S_3$, and the terms $\overline{S}_1, \overline{S}_3$ and $\overline{S}_4$ have the same meaning as in (9.4) with A replaced by $\overline{A}$. Let φ be the automorphism of $R\langle\langle \widetilde{A} \rangle\rangle$ acting on arrows as follows:

$$\varphi(a_{j,i}) = x^{-1}_{\{i,j,k\}}(a_{j,i} - \widetilde{a}_{j,k}\widetilde{a}_{k,i}), \quad \varphi(\widetilde{a}_{i,j}) = \widetilde{a}_{i,j} - x^{-1}_{\{i,j,k\}} f_{i,j}$$

for every $a_{j,i} \in Q'_1$, and φ fixes the rest of the arrows. Then we have

$$\begin{aligned}
\varphi(\widetilde{S}) &= \overline{S}_1 + \overline{S}_3 + \overline{S}_4 + \varphi\Big(\sum_{a_{j,i} \in Q'_1} (x_{\{i,j,k\}}\widetilde{a}_{i,j}a_{j,i} + \widetilde{a}_{i,j}\widetilde{a}_{j,k}\widetilde{a}_{k,i} + f_{i,j}a_{j,i})\Big) \\
&= \overline{S}_1 + \overline{S}_3 + \overline{S}_4 + \sum_{a_{j,i} \in Q'_1} (\widetilde{a}_{i,j}a_{j,i} - x^{-1}_{\{i,j,k\}} f_{i,j}\widetilde{a}_{j,k}\widetilde{a}_{k,i}).
\end{aligned}$$

We see that the degree 2 component of $\varphi(\widetilde{S})$ is

$$\varphi(\widetilde{S})^{(2)} = \sum_{a_{j,i} \in Q'_1} \widetilde{a}_{i,j} a_{j,i},$$

and a moment's reflection shows that

$$- \sum_{a_{j,i} \in Q'_1} x^{-1}_{\{i,j,k\}} f_{i,j} \widetilde{a}_{j,k} \widetilde{a}_{k,i}$$

can be viewed as the component $\overline{S}_2$ of a primitive potential on $\overline{A}$. We conclude that $\mu_k(A, S)$ is right-equivalent to $(\overline{A}, \overline{S}_1 + \overline{S}_2 + \overline{S}_3 + \overline{S}_4)$, finishing the proof. □

We conclude this section by briefly discussing a connection between Jacobian algebras of rigid QPs and cluster-tilted algebras introduced in [8]. We refer to [8] for precise definitions; roughly speaking, cluster-tilted algebras are endomorphism rings of tilting objects in cluster categories. One can associate such an algebra to any quiver Q which is mutation-equivalent to a Dynkin quiver. As shown in [13, Theorem 4.1] (for type A) and [9, Theorems 4.1, 4.2], the cluster-tilted algebra associated to Q is isomorphic to the path algebra of Q modulo an explicitly described ideal of relations. By inspection, this ideal is exactly the Jacobian ideal of a primitive potential S given by (9.3). Thus, Proposition 9.1 has the following corollary, which shows that the Jacobian algebras of QPs can be viewed as generalizations of cluster-tilted algebras.

Corollary 9.3. *If a quiver Q with the arrow span A is mutation-equivalent to a Dynkin quiver, then the Jacobian algebra $\mathcal{P}(A, S)$ of a rigid QP on A (explicitly given by* (9.3)*) is isomorphic to the cluster-tilted algebra associated to Q.*

10. Decorated representations and their mutations

The following definition is inspired by [28].

Definition 10.1. A *decorated representation* of a QP (A, S) is a pair $\mathcal{M} = (M, V)$, where V is a finite-dimensional (left) R-module, and M is a finite-dimensional $R\langle\langle A\rangle\rangle$-module annihilated by $J(S)$.

Equivalently, M is a finite-dimensional $\mathcal{P}(A, S)$-module. We will sometimes write $\mathcal{M} = (A, S, M, V)$ and refer to such a quadruple as a *QP-representation.*

We have $M = \bigoplus_{i \in Q_0} M_i$ and $V = \bigoplus_{i \in Q_0} V_i$, where $M_i = e_i M$ and $V_i = e_i V$. With some abuse of notation, for $u \in R\langle\langle A\rangle\rangle$ or $u \in \mathcal{P}(A, S)$, we denote the multiplication operator $m \mapsto um$ on M simply as $u : M \to M$; we write $u = u_M$ if we need to emphasize the dependence of this operator on M. In particular, for each arrow $a \in A$, we have $a : M_{t(a)} \to M_{h(a)}$, and $a|_{M_i} = 0$ for $i \neq t(a)$.

Note that every finite-dimensional $R\langle\langle A\rangle\rangle$-module M is *nilpotent*, i.e., M is annihilated by $\mathfrak{m}^n$ for $n \gg 0$. We thank Bill Crawley-Boevey for pointing this out to us; the following argument was also suggested by him. The above claim is a special case of the following more general fact: if a ring U with unit is complete in the I-adic topology for some two-sided ideal I, and M is a U-module of finite length n, then $I^n M = \{0\}$. Indeed, the element $1 + x$ is invertible in U for any $x \in I$, since it has inverse $1 - x + x^2 - x^3 + \cdots$. Thus I is contained in the Jacobson radical J (since J is the set of $x \in U$ such that $1 + xy$ is invertible for all $y \in U$). Thus $IS = \{0\}$ for any simple U-module S (since J is the intersection of annihilators of all simple modules). Now if M has composition series

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_n = M,$$

then for all $i \geq 1$, we have $I(M_i/M_{i-1}) = \{0\}$, so $IM_i \subseteq M_{i-1}$. It follows that $I^n M = \{0\}$, as claimed.

The aim of this section is to extend the definition of QP-mutations in Corollary 5.4 and Definition 5.5 to the level of QP-representations, and to prove a representation-theoretic extension of Theorem 5.7. To do this, we first introduce right-equivalence for QP-representations.

Definition 10.2. Let (A, S) and (A', S') be QPs on the same set of vertices, and let $\mathcal{M} = (M, V)$ (resp. $\mathcal{M}' = (M', V')$) be a decorated representation of (A, S) (resp. of (A', S')). A *right-equivalence* between $\mathcal{M}$ and $\mathcal{M}'$ is a triple (φ, ψ, η), where:

- $\varphi : R\langle\langle A\rangle\rangle \to R\langle\langle A'\rangle\rangle$ is a right-equivalence between (A, S) and (A', S') (see Definition 4.2);
- $\psi : M \to M'$ is a vector space isomorphism such that $\psi \circ u_M = \varphi(u)_{M'} \circ \psi$ for $u \in R\langle\langle A\rangle\rangle$;
- $\eta : V \to V'$ is an isomorphism of R-modules.

Remark 10.3. The usual notion of isomorphism of decorated representations $\mathcal{M} = (M, V)$ and $\mathcal{M}' = (M', V')$ of the same QP (A, S) (namely, that M and M' are isomorphic $\mathcal{P}(A, S)$-modules, and V and V' are isomorphic R-modules) is a special case of right-equivalence with the φ-component being the identity. The right-equivalence seems to be more relevant for applications to cluster algebras. To illustrate, consider the *Kronecker quiver* Q with two vertices 1 and 2, and two arrows a and b from 1 to 2. Since Q has no oriented cycles, it has only one QP (A, S): the one with $S = 0$. Thus, a decorated representation $\mathcal{M} = (M, V)$ with $V = 0$ is just a usual representation of the quiver Q, i.e., it consists of two vector spaces M_1 and M_2, and two linear maps a and b from M_1 to M_2. Such representations were classified by Kronecker. In particular, he proved that, for every $n \geq 1$, the isomorphism classes of indecomposable Q-representations with $\dim M_1 = \dim M_2 = n$ are naturally parameterized by the projective line. However all these representations are right-equivalent to each other. This is more in sync with the fact that, as discussed in [16], all these representations give rise to the same element of the corresponding cluster algebra.

We now present a representation-theoretic extension of Theorem 4.6. Let $\mathcal{M} = (A, S, M, V)$ be a QP-representation, and let $\varphi : R\langle\langle A_{\rm red} \oplus C\rangle\rangle \to R\langle\langle A\rangle\rangle$ be a right-equivalence of the QPs $(A_{\rm red}, S_{\rm red}) \oplus (C, T)$ and (A, S), where $(A_{\rm red}, S_{\rm red})$ is a reduced QP, and (C, T) is a trivial QP (see Theorem 4.6). We define an

$R\langle\langle A_{\rm red}\rangle\rangle$-module M' by setting $M' = M$ as a K-vector space with the action of $R\langle\langle A_{\rm red}\rangle\rangle$ given by $u_{M'} = \varphi(u)_M$. In view of Proposition 4.5, this makes a quadruple $\mathcal{M}_{\rm red} = (A_{\rm red}, S_{\rm red}, M', V)$ a well-defined QP-representation.

Definition 10.4. We call the QP-representation $\mathcal{M}_{\rm red}$ given by the above construction the *reduced part* of $\mathcal{M}$.

This terminology is justified by the following.

Proposition 10.5. *The right-equivalence class of $\mathcal{M}_{\rm red}$ is determined by the right-equivalence class of $\mathcal{M}$.*

Proof. To get more in sync with the notation in Proposition 4.9, we change our notation a little bit and assume that $\mathcal{M}$ is a decorated representation of a QP $(A \oplus C, S + T)$, where (A, S) is a reduced QP, and (C, T) a trivial one. Let φ be an auto-right-equivalence of $(A \oplus C, S + T)$, that is, an algebra automorphism of $R\langle\langle A \oplus C\rangle\rangle$ such that $\varphi(S + T)$ is cyclically equivalent to $S + T$. To prove Proposition 10.5, it suffices to show the following:

$$\text{there exists an algebra automorphism } \varphi' \text{ of } R\langle\langle A\rangle\rangle \text{ such that } \varphi'(S) \text{ is cyclically equivalent to } S, \text{ and } \varphi'(u)_M = \varphi(u)_M \text{ for } u \in R\langle\langle A\rangle\rangle. \quad (10.1)$$

As in the proof of Proposition 4.9, let L denote the closure of the two-sided ideal in $R\langle\langle A \oplus C\rangle\rangle$ generated by C. Recall from (4.10) that

$$R\langle\langle A \oplus C\rangle\rangle = R\langle\langle A\rangle\rangle \oplus L, \quad \text{and} \quad J(S + T) = J(S) \oplus L$$

(in the last equality, $J(S)$ is understood as the Jacobian ideal of S in $R\langle\langle A\rangle\rangle$). In particular, we have $u_M = 0$ for $u \in L$.

Let $\psi : R\langle\langle A\rangle\rangle \to R\langle\langle A\rangle\rangle$ denote the restriction to $R\langle\langle A\rangle\rangle$ of the composition $p \circ \varphi$, where p is the projection of $R\langle\langle A \oplus C\rangle\rangle$ onto $R\langle\langle A\rangle\rangle$ along L. Then we have $\psi(u)_M = \varphi(u)_M$ for $u \in R\langle\langle A\rangle\rangle$.

According to (4.12), ψ is an algebra automorphism of $R\langle\langle A\rangle\rangle$. Furthermore, using (4.12) in conjunction with Proposition 4.10, we conclude that $J(\psi(S)) = J(S)$, and that there exists an algebra automorphism η of $R\langle\langle A\rangle\rangle$ such that $\eta(\psi(S))$ is cyclically equivalent to S, and $\eta(u) - u \in J(S)$ for $u \in R\langle\langle A\rangle\rangle$. Taking $\varphi' = \eta \circ \psi$, we see that

$$\varphi'(u)_M = \eta(\psi(u))_M = \psi(u)_M = \varphi(u)_M$$

for $u \in R\langle\langle A\rangle\rangle$. Thus φ' satisfies all the conditions in (10.1), and we are done. □

We turn to the definition of *mutations* for a QP-representation $\mathcal{M} = (A, S, M, V)$. We fix a vertex k satisfying (5.1), and let $a_1, \ldots, a_s$ be all arrows with $h(a_p) = k$, and $b_1, \ldots, b_t$ be all arrows with $t(b_q) = k$. We define

$$M_{\rm in} = \bigoplus_{p=1}^{s} M_{t(a_p)}, \qquad M_{\rm out} = \bigoplus_{q=1}^{t} M_{h(b_q)}. \quad (10.2)$$

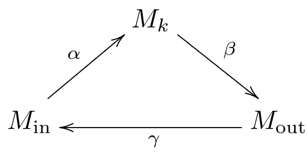
Let $\alpha = \alpha_M : M_{\text{in}} \to M_k$ and $\beta = \beta_M : M_k \to M_{\text{out}}$ be K-linear maps given in matrix form by

$$\alpha = (a_1 \quad \cdots \quad a_s), \qquad \beta = \begin{pmatrix} b_1 \\ \vdots \\ b_t \end{pmatrix}. \tag{10.3}$$

We also introduce a K-linear map $\gamma = \gamma_M : M_{\text{out}} \to M_{\text{in}}$ as follows. Replacing S if necessary by a cyclically equivalent potential, we may assume that $S \in R\langle\langle A\rangle\rangle_{\hat{k},\hat{k}}$ (see (6.1)). We identify $R\langle\langle A\rangle\rangle_{\hat{k},\hat{k}}$ with $R\langle\langle \widetilde{A}_{\hat{k},\hat{k}}\rangle\rangle$ as in Lemma 6.2. This allows us to define the component $\gamma_{p,q} : M_{h(b_q)} \to M_{t(a_p)}$ of γ by setting

$$\gamma_{p,q} = \partial_{[b_q a_p]} S. \tag{10.4}$$

Thus, we have constructed the following triangle of linear maps:



(10.5)

Lemma 10.6. *We have $\alpha\gamma = 0$ and $\gamma\beta = 0$.*

Proof. The q-th component of $\alpha\gamma$ is equal to

$$\sum_p a_p \partial_{[b_q a_p]} S = \partial_{b_q} S;$$

therefore, $\alpha\gamma = 0$, since M is a representation of $\mathcal{P}(A, S)$. Similarly, the p-th component of $\gamma\beta$ is equal to

$$\sum_q (\partial_{[b_q a_p]} S) b_q = \partial_{a_p} S,$$

implying that $\gamma\beta = 0$. □

Now let $(\widetilde{A}, \widetilde{S})$ be given by (5.4) and (5.8). We associate to a QP-representation $\mathcal{M} = (A, S, M, V)$ the QP-representation $\widetilde{\mu}_k(\mathcal{M}) = (\widetilde{A}, \widetilde{S}, \overline{M}, \overline{V})$ defined as follows. First, we set

$$\overline{M}_i = M_i, \qquad \overline{V}_i = V_i \qquad (i \neq k). \tag{10.6}$$

We define $\overline{M}_k$ and $\overline{V}_k$ by

$$\overline{M}_k = \frac{\ker \gamma}{\text{im } \beta} \oplus \text{im } \gamma \oplus \frac{\ker \alpha}{\text{im } \gamma} \oplus V_k, \qquad \overline{V}_k = \frac{\ker \beta}{\ker \beta \cap \text{im } \alpha}. \tag{10.7}$$

We now define the action on $\overline{M}$ of all arrows in $\widetilde{A}$ (recall that they are given by (5.6) and (5.7)). Thus, for every such arrow c, we need to define a linear map $c_{\overline{M}} : \overline{M}_{t(c)} \to \overline{M}_{h(c)}$.

First, we set

$$c_{\overline{M}} = c_M$$

for every arrow c not incident to k, and

$$[b_q a_p]_{\overline{M}} = (b_q a_p)_M$$

for all p and q.

To define the action of the remaining arrows $a_p^\star$ and $b_q^\star$, we assemble them into the operators

$$\overline{\alpha} = (b_1^\star \ \cdots \ b_t^\star), \quad \overline{\beta} = \begin{pmatrix} a_1^\star \\ \vdots \\ a_s^\star \end{pmatrix}$$

in the same way as in (10.3). Thus, we need to define linear maps

$$\overline{\alpha} : M_{\text{out}} = \overline{M}_{\text{in}} \to \overline{M}_k, \quad \overline{\beta} : \overline{M}_k \to \overline{M}_{\text{out}} = M_{\text{in}}.$$

We will use the following notational convention: whenever we have a pair $U_1 \subseteq U_2$ of vector spaces, denote by $\iota : U_1 \to U_2$ the inclusion map, and by $\pi : U_2 \to U_2/U_1$ the natural projection. We now introduce the following *splitting data*:

Choose a linear map $\rho : M_{\text{out}} \to \ker \gamma$ such that $\rho\iota = \text{id}_{\ker \gamma}$. (10.8)

Choose a linear map $\sigma : \ker \alpha / \text{im}\, \gamma \to \ker \alpha$ such that $\pi\sigma = \text{id}_{\ker \alpha/\text{im}\,\gamma}$. (10.9)

Then we define:

$$\overline{\alpha} = \begin{pmatrix} -\pi\rho \\ -\gamma \\ 0 \\ 0 \end{pmatrix}, \quad \overline{\beta} = (0 \ \ \iota \ \ \iota\sigma \ \ 0). \tag{10.10}$$

Having defined the action of all arrows in $\widetilde{A}$ on $\overline{M}$, we can view $\overline{M}$ as a module over the path algebra $R\langle \widetilde{A} \rangle$. The property that M is annihilated by $\mathfrak{m}(A)^n$ for $n \gg 0$ implies that $\overline{M}$ is annihilated by $\widetilde{A}^n$ for $n \gg 0$. This allows us to view $\overline{M}$ as a module over the completed path algebra $R\langle\langle \widetilde{A} \rangle\rangle$.

Proposition 10.7. *The above definitions make $\widetilde{\mu}_k(\mathcal{M}) = (\overline{M}, \overline{V})$ a decorated representation of $(\widetilde{A}, \widetilde{S})$.*

Proof. We only need to show that $(\partial_c \widetilde{S})_{\overline{M}} = 0$ for every arrow c in $\widetilde{A}$. If c is not incident to k, the desired statement follows from (6.6). If c is one of the arrows $[b_q a_p]$, then, in view of (6.5) and (10.4), it is enough to show that

$$a_p^\star b_q^\star + \gamma_{p,q} = 0$$

for all p and q. In other words, we need to show that $\overline{\beta}\overline{\alpha} = -\gamma$; But this follows at once by multiplying the row and column given by (10.10).

It remains to show that $(\partial_{a_p^\star} \widetilde{S})_{\overline{M}} = 0$ and $(\partial_{b_q^\star} \widetilde{S})_{\overline{M}} = 0$ for all p and q. We first deal with the first equality. Remembering (5.8) and (5.9), we see that

$$(\partial_{a_p^\star} \widetilde{S})_{\overline{M}} = \Big(\sum_q b_q^\star [b_q a_p]\Big)_{\overline{M}} = \Big(\sum_q (b_q^\star)_{\overline{M}} (b_q)_M\Big)(a_p)_M.$$

Thus it suffices to show that

$$\sum_q (b_q^\star)_{\overline{M}} (b_q)_M = 0,$$

or equivalently, $\overline{\alpha}\beta = 0$. In view of (10.10), we have

$$\overline{\alpha}\beta = \begin{pmatrix} -\pi\rho\beta \\ -\gamma\beta \\ 0 \\ 0 \end{pmatrix} = 0,$$

as desired (the equality $\pi\rho\beta = 0$ is immediate from the definitions, while $\gamma\beta = 0$ by Lemma 10.6).

The remaining equality $(\partial_{b_\nu^\star}\widetilde{S})_{\overline{M}} = 0$ is proved in a similar way. We have

$$(\partial_{b_q^\star}\widetilde{S})_{\overline{M}} = \Big(\sum_p [b_q a_p] a_p^\star\Big)_{\overline{M}} = (b_q)_M \sum_p (a_p)_M (a_p^\star)_{\overline{M}}.$$

Thus, it suffices to observe that

$$\alpha\overline{\beta} = (0 \;\; \alpha\iota \;\; \alpha\iota\sigma \;\; 0) = 0. \qquad \square$$

Example 10.8. Let us illustrate the definition of the operation $\widetilde{\mu}_k$ by a special case where the vertex k is a sink or a source. First suppose k is a sink, that is, there are no arrows b with $t(b) = k$. Then we have $M_{\text{out}} = \{0\}$, hence $\beta = 0$ and $\gamma = 0$. Thus, (10.7) simplifies to

$$\overline{M}_k = \ker\alpha \oplus V_k, \qquad \overline{V}_k = \operatorname{coker}\alpha.$$

The arrow span $\widetilde{A}$ is obtained from A by reversing all arrows a with $h(a) = k$, that is, replacing every such arrow a with $a^\star$. Thus, k becomes a source for $\widetilde{A}$, hence $\overline{M}_{\text{in}} = \{0\}$ and $\overline{\alpha} = 0$. The choice of splitting data (10.8) and (10.9) becomes immaterial, and the second equality in (10.10) simplifies to

$$\overline{\beta} = (\iota \;\; 0).$$

Note that we have $\widetilde{A}_{\hat{k},\hat{k}} = A_{\hat{k},\hat{k}}$, and the potential $\widetilde{S} \in R\langle\langle \widetilde{A}_{\hat{k},\hat{k}}\rangle\rangle$ is naturally identified with S.

The case where k is a source is completely similar. In this case we have $M_{\text{in}} = \{0\}$, hence $\alpha = 0$ and $\gamma = 0$. It follows that

$$\overline{M}_k = \operatorname{coker}\beta \oplus V_k, \qquad \overline{V}_k = \ker\beta\ ,$$

and the map $\overline{\alpha} : \overline{M}_{\text{in}} = M_{\text{out}} \to \overline{M}_k$ is given by

$$\overline{\alpha} = \begin{pmatrix} -\pi \\ 0 \end{pmatrix}.$$

In both cases, $\widetilde{\mu}_k$ coincides with the "extended reflection functor" introduced in [28]; furthermore, if we ignore the decorations (and the potentials), it becomes the classical Bernstein–Gelfand–Ponomarev reflection functor at k (see [3]).

Now we return to the case of an arbitrary vertex k.

Proposition 10.9. *The isomorphism class of the decorated representation* $\widetilde{\mu}_k(\mathcal{M})$ *does not depend on the choice of the splitting data* (10.8)–(10.9).

Proof. We have the following freedom in choosing the splitting data: one can replace $\rho : M_{\mathrm{out}} \to \ker\gamma$ with $\rho' = \rho + \xi\gamma$ for some linear map $\xi : \mathrm{im}\,\gamma \to \ker\gamma$, and replace $\sigma : \ker\alpha/\mathrm{im}\,\gamma \to \ker\alpha$ by $\sigma' = \sigma + \eta$ for some linear map $\eta : \ker\alpha/\mathrm{im}\,\gamma \to \mathrm{im}\,\gamma$. Let $\overline{\alpha}'$ and $\overline{\beta}'$ be the maps obtained by replacing ρ with ρ', and σ with σ' in (10.10). It is enough to show that $\psi\overline{\alpha} = \overline{\alpha}'$ and $\overline{\beta}'\psi = \overline{\beta}$ for some linear automorphism $\psi : \overline{M}_k \to \overline{M}_k$. Decomposing $\overline{M}_k$ as in (10.7), we define ψ as the block-triangular matrix

$$\psi = \begin{pmatrix} I & \pi\xi & 0 & 0 \\ 0 & I & -\eta & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix},$$

where I stands for the identity transformation. The invertibility of ψ is obvious, and the desired equalities $\psi\overline{\alpha} = \overline{\alpha}'$ and $\overline{\beta}'\psi = \overline{\beta}$ are checked by direct matrix multiplication. □

Proposition 10.10. *The right-equivalence class of the representation* $\widetilde{\mu}_k(\mathcal{M})$ *is determined by the right-equivalence class of* $\mathcal{M}$.

Proof. Let φ be an automorphism of $R\langle\langle A\rangle\rangle$, and let $\mathcal{M}' = (A, \varphi(S), M', V')$ be the QP-representation defined as follows: $V' = V$ and $M' = M$ as R-modules, while the $R\langle\langle A\rangle\rangle$-actions in M and M' are related by

$$u_M = \varphi(u)_{M'} \quad (u \in R\langle\langle A\rangle\rangle) \tag{10.11}$$

(note that (10.11) indeed defines a representation of $(A, \varphi(S))$ in view of Proposition 3.7). To prove Proposition 10.10, it suffices to show that the representations $\widetilde{\mu}_k(\mathcal{M})$ and $\widetilde{\mu}_k(\mathcal{M}')$ are right-equivalent.

We retain all the above notation relating to $\mathcal{M}$ and $\widetilde{\mu}_k(\mathcal{M})$; in particular, α, β and γ stand for the linear maps in the triangle (10.5). Let α', β' and γ' denote the corresponding maps for the representation $\mathcal{M}'$. Our first order of business is to relate these maps to α, β and γ.

We can write the action of φ on the arrows $a_1, \dots, a_s$ as follows:

$$(\varphi(a_1) \ \cdots \ \varphi(a_s)) = (a_1 \ \cdots \ a_s)C, \tag{10.12}$$

where $C = C_0 + C_1$ is an invertible $s \times s$ matrix as in (5.14). Similarly, the action of φ on the arrows $b_1, \dots, b_s$ can be written as

$$\begin{pmatrix} \varphi(b_1) \\ \vdots \\ \varphi(b_t) \end{pmatrix} = D \begin{pmatrix} b_1 \\ \vdots \\ b_t \end{pmatrix}, \tag{10.13}$$

where $D = D_0 + D_1$ is an invertible $t \times t$ matrix as in (5.15). Therefore we have

$$\begin{aligned}\alpha &= (a_1 \ \cdots \ a_s)_M = (\varphi(a_1) \ \cdots \ \varphi(a_s))_{M'} \\ &= (a_1 \ \cdots \ a_s)C)_{M'} = \alpha' C_{M'}, \end{aligned} \tag{10.14}$$

and similarly,

$$\beta = D_{M'}\beta'; \tag{10.15}$$

here $C_{M'}$ (resp. $D_{M'}$) is understood as an R-module automorphism of $M'_{\mathrm{in}} = M_{\mathrm{in}}$ (resp. of $M'_{\mathrm{out}} = M_{\mathrm{out}}$).

Turning to the maps γ and γ', we claim that they are related by

$$\gamma' = C_{M'}\gamma D_{M'}. \tag{10.16}$$

To see this, we use (10.4) and (3.6) to write

$$\gamma'_{p,q} = (\partial_{[b_q a_p]}\varphi(S))_{M'} = \Big(\sum_c \Delta_{[b_q a_p]}(\varphi(c)) \square \varphi(\partial_c S)\Big)_{M'}, \tag{10.17}$$

where the sum is over all arrows c in $\widetilde{A}_{\hat{k},\hat{k}}$. If c is one of the arrows in A then by (10.11) we have

$$\varphi(\partial_c S)_{M'} = (\partial_c S)_M = 0;$$

remembering the definition (3.3), we see that c does not contribute to (10.17). Thus, we have

$$\gamma'_{p,q} = \Big(\sum_{p',q'} \Delta_{[b_q a_p]}(\varphi(b_{q'} a_{p'})) \square \varphi(\partial_{[b_{q'} a_{p'}]} S)\Big)_{M'}. \tag{10.18}$$

Remembering (3.2), and using (10.12) and (10.13), we see that the summand with $(p', q') = (p, q)$ in (10.18) contains among its terms the (p, q)-entry of the matrix

$$(C\varphi(\partial_{[b_q a_p]} S)D)_{M'} = C_{M'}\gamma D_{M'}.$$

Thus, to prove (10.16), it remains to show that the rest of the terms in (10.18) add up to 0. Again using the definitions (3.2) and (3.3), we can rewrite the rest of the sum in (10.18) as $S_1 + S_2$, where

$$S_1 = \Big(\sum_{p',q'} \Delta_{[b_q a_p]}(\varphi(b_{q'})) \square \varphi(a_{p'} \cdot \partial_{[b_{q'} a_{p'}]} S)\Big)_{M'},$$

$$S_2 = \Big(\sum_{p',q'} \Delta_{[b_q a_p]}(\varphi(a_{p'})) \square \varphi(\partial_{[b_{q'} a_{p'}]} S \cdot b_{q'})\Big)_{M'}.$$

It remains to observe that

$$S_1 = \Big(\sum_{q'} \Delta_{[b_q a_p]}(\varphi(b_{q'})) \square \varphi(\partial_{b_{q'}} S)\Big)_{M'} = 0$$

since $\varphi(\partial_{b_{q'}} S)_{M'} = (\partial_{b_{q'}} S)_M = 0$; and similarly,

$$S_2 = \Big(\sum_{p'} \Delta_{[b_q a_p]}(\varphi(a_{p'})) \square \varphi(\partial_{a_{p'}} S)\Big)_{M'} = 0.$$

In view of (10.14), (10.15) and (10.16), we have

$$\begin{aligned} \ker\alpha &= C_{M'}^{-1}(\ker\alpha'), & \operatorname{im}\alpha &= \operatorname{im}\alpha',\\ \ker\beta &= \ker\beta', & \operatorname{im}\beta &= D_{M'}(\operatorname{im}\beta'),\\ \ker\gamma &= D_{M'}(\ker\gamma'), & \operatorname{im}\gamma &= C_{M'}^{-1}(\operatorname{im}\gamma'). \end{aligned} \tag{10.19}$$

Recall that the spaces $\overline{M}$ and $\overline{V}$ in the decorated representation $\widetilde{\mu}_k(\mathcal{M}) = (\overline{M}, \overline{V})$ of $(\widetilde{A}, \widetilde{S})$ are given by (10.6) and (10.7). We express the decorated representation $\widetilde{\mu}_k(\mathcal{M}') = (\overline{M'}, \overline{V'})$ of $(\widetilde{A}, \widetilde{\varphi(S)})$ in the same way, with the maps α, β and γ replaced by α', β' and γ'. In particular, we have $\overline{V'} = \overline{V}$, and $\overline{M'}_i = \overline{M}_i = M_i$ for $i \neq k$. To specify the actions of $R\langle\langle A\rangle\rangle$ in $\overline{M}$ and $\overline{M'}$, we need to choose the splitting data (ρ, σ) and (ρ', σ') as in (10.8) and (10.9). Note that, in view of (10.19), we can choose

$$\rho' = D_{M'}^{-1}\rho D_{M'}, \qquad \sigma' = C_{M'}\sigma C_{M'}^{-1}; \tag{10.20}$$

here with some abuse of notation we use the same notation $C_{M'}^{-1}$ for the isomorphism $\ker\alpha' \to \ker\alpha$ and the induced isomorphism $\ker\alpha'/\operatorname{im}\gamma' \to \ker\alpha/\operatorname{im}\gamma$.

Everything is now in place for defining the desired right-equivalence $(\widehat{\varphi}, \psi, \eta)$ between $\widetilde{\mu}_k(\mathcal{M})$ and $\widetilde{\mu}_k(\mathcal{M}')$ (see Definition 10.2). First of all, we define $\widehat{\varphi} : R\langle\langle\widetilde{A}\rangle\rangle \to R\langle\langle\widetilde{A}\rangle\rangle$ as the right-equivalence between $(\widetilde{A}, \widetilde{S})$ and $(\widetilde{A}, \widetilde{\varphi(S)}$ constructed in the proof of Lemma 5.3. In particular, we have

$$\begin{pmatrix}\widehat{\varphi}(a_1^\star)\\ \vdots\\ \widehat{\varphi}(a_s^\star)\end{pmatrix} = C^{-1}\begin{pmatrix}a_1^\star\\ \vdots\\ a_s^\star\end{pmatrix}, \qquad (\widehat{\varphi}(b_1^\star) \ \cdots \ \widehat{\varphi}(b_t^\star)) = (b_1^\star \ \cdots \ b_t^\star)D^{-1}.$$

Next we define $\psi : \overline{M} \to \overline{M'}$ as the identity map on $\bigoplus_{i\neq k}\overline{M}_i = \bigoplus_{i\neq k} M_i = \bigoplus_{i\neq k}\overline{M'}_i$, and the restriction $\psi|_{\overline{M}_k} : \overline{M}_k \to \overline{M'}_k$ given by the block-diagonal matrix

$$\psi|_{\overline{M}_k} = \begin{pmatrix} D_{M'}^{-1} & 0 & 0 & 0\\ 0 & C_{M'} & 0 & 0\\ 0 & 0 & C_{M'} & 0\\ 0 & 0 & 0 & I\end{pmatrix} \tag{10.21}$$

(this is well-defined in view of (10.19)). Finally, we define $\eta : \overline{V} \to \overline{V'}$ simply as the identity map.

The only thing to check is the equality $\psi \circ c_{\overline{M}} = \widehat{\varphi}(c)_{\overline{M'}} \circ \psi$ for any arrow $c \in \widetilde{A}$. And the only case that may require some consideration is when c is one of the arrows $a_p^\star$ or $b_q^\star$. Unraveling the definitions, it suffices to show that

$$\overline{\beta} = C_{M'}^{-1}\overline{\beta'} \circ \psi|_{\overline{M}_k}, \qquad \psi|_{\overline{M}_k} \circ \overline{\alpha} = \overline{\alpha'} D_{M'}^{-1}.$$

But this is an immediate consequence of the definitions (10.21) and (10.10) (we also need an analogue of (10.10) for the maps $\overline{\beta'}$ and $\overline{\alpha'}$, using the splitting data (10.20)). This completes the proof of Proposition 10.10. $\square$

Note that in the above treatment of the operation $\mathcal{M} \mapsto \widetilde{\mu}_k(\mathcal{M})$ for a QP-representation $\mathcal{M} = (A, S, M, V)$, the QP (A, S) was not assumed to be reduced. Recall from Proposition 10.5 that we have a well-defined operation $\mathcal{M} \mapsto \mathcal{M}_{\text{red}}$ on (right-equivalence classes of) QP-representations. The following property is immediate from definitions.

Proposition 10.11. *Let (A, S) be a QP satisfying* (5.1). *Then, for every representation $\mathcal{M}$ of (A, S), the representation $\widetilde{\mu}_k(\mathcal{M})_{\text{red}}$ is right-equivalent to $\widetilde{\mu}_k(\mathcal{M}_{\text{red}})_{\text{red}}$.*

Recall that, according to Corollary 5.4 and Definition 5.5, the correspondence $\widetilde{\mu}_k : (A, S) \mapsto \widetilde{\mu}_k(A, S) = (\widetilde{A}, \widetilde{S})$ gives rise to the *mutation* $(A, S) \mapsto \mu_k(A, S) = (\overline{A}, \overline{S})$, which is a well-defined bijective transformation on the set of right-equivalence classes of reduced QPs satisfying (5.1). Here $(\overline{A}, \overline{S})$ is the reduced part of $(\widetilde{A}, \widetilde{S})$. Now for every QP-representation $\mathcal{M} = (A, S, M, V)$ of a reduced QP (A, S) we define

$$\mu_k(\mathcal{M}) = \widetilde{\mu}_k(\mathcal{M})_{\text{red}}; \tag{10.22}$$

thus, $\mu_k(\mathcal{M})$ is a decorated representation $(\overline{A}, \overline{S}, \overline{M}, \overline{V})$ of a reduced QP $(\overline{A}, \overline{S})$. Combining Propositions 10.5 and 10.10, we obtain the following important corollary.

Corollary 10.12. *The correspondence $\mathcal{M} \mapsto \mu_k(\mathcal{M})$ is a well-defined transformation on the set of right-equivalence classes of decorated representations of reduced QPs satisfying* (5.1).

We refer to the transformation $\mathcal{M} \mapsto \mu_k(\mathcal{M})$ in Corollary 10.12 as the *mutation at vertex k*. With some abuse of terminology, we will talk about mutations of decorated representations (rather than their right-equivalence classes).

The following result naturally extends Theorem 5.7.

Theorem 10.13. *The mutation μ_k of decorated representations is an involution; that is, for every decorated representation $\mathcal{M}$ of a reduced QP (A, S) satisfying* (5.1)*, the decorated representation $\mu_k^2(\mathcal{M})$ of a QP $\mu_k^2(A, S)$ is right-equivalent to $\mathcal{M}$.*

Proof. In view of Proposition 10.11, $\mu_k^2(\mathcal{M})$ is right-equivalent to $\widetilde{\mu}_k^2(\mathcal{M})_{\text{red}}$. Therefore, it suffices to show that $\widetilde{\mu}_k^2(\mathcal{M})_{\text{red}}$ is right-equivalent to $\mathcal{M}$.

We write the QP-representation $\widetilde{\mu}_k^2(\mathcal{M})$ as $\widetilde{\mu}_k^2(\mathcal{M}) = (\widetilde{\widetilde{A}}, \widetilde{\widetilde{S}}, \overline{\overline{M}}, \overline{\overline{V}})$. The QP $(\widetilde{\widetilde{A}}, \widetilde{\widetilde{S}})$ is given by (5.18) and (5.19). In particular, there is a natural embedding of A into $\widetilde{\widetilde{A}}$ identifying A with the reduced part $\widetilde{\widetilde{A}}_{\text{red}}$. Furthermore, as shown in the proof of Theorem 5.7, an automorphism of $R\langle\langle \widetilde{\widetilde{A}} \rangle\rangle$ that establishes the right-equivalence in (5.17) can be chosen so that it restricts to an automorphism $\varphi_0 : R\langle\langle A \rangle\rangle \to R\langle\langle A \rangle\rangle$ acting as follows:

$$\begin{aligned}&\varphi_0 \text{ multiplies each of the arrows } b_1, \ldots, b_t \text{ by } -1,\\ &\text{and fixes the rest of the arrows in } A.\end{aligned} \tag{10.23}$$

In view of Definition 10.4, the QP-representation $\mu_k^2(\mathcal{M}) = \widetilde{\mu}_k^2(\mathcal{M})_{\mathrm{red}}$ can be realized as (A, S, M', V'), where $M' = \overline{\overline{M}}$ and $V' = \overline{\overline{V}}$ as vector spaces, and the action of $R\langle\langle A\rangle\rangle$ in M' is given by

$$u_{M'} = \varphi_0(u)_{\overline{\overline{M}}} \qquad (u \in R\langle\langle A\rangle\rangle). \tag{10.24}$$

To prove Theorem 10.13, it suffices to show that the decorated representation (M', V') of (A, S) is isomorphic to (M, V).

We first compute $M' = \overline{\overline{M}}$ and $V' = \overline{\overline{V}}$ as vector spaces. According to (10.6),

$$M'_i = \overline{\overline{M}}_i = \overline{M}_i = M_i, \qquad V'_i = \overline{\overline{V}}_i = \overline{V}_i = V_i$$

for all $i \neq k$. As for the spaces M'_k and V'_k, they are given as in (10.7), with the maps α, β, and γ replaced by $\overline{\alpha}, \overline{\beta}$, and $\overline{\gamma}$, respectively. Recall that $\overline{\alpha}$ and $\overline{\beta}$ are given by (10.10). As for $\overline{\gamma}$, by applying the definition (10.4) to the potential $\widetilde{S}$ given by (5.8) and (5.9), we see that

$$\overline{\gamma} = \beta\alpha. \tag{10.25}$$

As a direct consequence of the definitions, we conclude that

$$\begin{gathered}\ker\overline{\alpha} = \mathrm{im}\,\beta, \qquad \mathrm{im}\,\overline{\alpha} = \frac{\ker\gamma}{\mathrm{im}\,\beta} \oplus \mathrm{im}\,\gamma \oplus \{0\} \oplus \{0\},\\ \ker\overline{\beta} = \frac{\ker\gamma}{\mathrm{im}\,\beta} \oplus \{0\} \oplus \{0\} \oplus V_k, \qquad \mathrm{im}\,\overline{\beta} = \ker\alpha,\\ \ker\overline{\gamma} = \ker(\beta\alpha), \qquad \mathrm{im}\,\overline{\gamma} = \mathrm{im}(\beta\alpha).\end{gathered} \tag{10.26}$$

It follows that

$$V'_k = \overline{\overline{V}}_k = \frac{\ker\overline{\beta}}{\ker\overline{\beta} \cap \mathrm{im}\,\overline{\alpha}} = V_k,$$

and so $V' = V$, as desired. We also have

$$M'_k = \frac{\ker(\beta\alpha)}{\ker\alpha} \oplus \mathrm{im}(\beta\alpha) \oplus \frac{\mathrm{im}\,\beta}{\mathrm{im}(\beta\alpha)} \oplus \frac{\ker\beta}{\ker\beta \cap \mathrm{im}\,\alpha}.$$

We now make the following easy observations:

- the map α induces an isomorphism $\ker(\beta\alpha)/\ker\alpha \to \ker\beta \cap \mathrm{im}\,\alpha$;
- the map β induces an isomorphism $\mathrm{im}\,\alpha/(\ker\beta \cap \mathrm{im}\,\alpha) \to \mathrm{im}(\beta\alpha)$;
- the map β induces an isomorphism $M_k/(\ker\beta + \mathrm{im}\,\alpha) \to \mathrm{im}\,\beta/\mathrm{im}(\beta\alpha)$.
- there is a natural isomorphism $\ker\beta/(\ker\beta \cap \mathrm{im}\,\alpha) \to (\ker\beta + \mathrm{im}\,\alpha)/\mathrm{im}\,\alpha$.

Using these isomorphisms, we can identify M'_k with the space

$$M'_k = (\ker\beta \cap \mathrm{im}\,\alpha) \oplus \frac{\mathrm{im}\,\alpha}{\ker\beta \cap \mathrm{im}\,\alpha} \oplus \frac{M_k}{\ker\beta + \mathrm{im}\,\alpha} \oplus \frac{\ker\beta + \mathrm{im}\,\alpha}{\mathrm{im}\,\alpha}. \tag{10.27}$$

To describe the action of $R\langle\langle A\rangle\rangle$ in M', we only need to describe the maps $\alpha' : M_{\mathrm{in}} \to M'_k$ and $\beta' : M'_k \to M_{\mathrm{out}}$ constructed in the same way as in (10.3). As in (10.10), the definition of α' and β' involves splitting data (10.8)–(10.9). In view of (10.26), in the current situation the splitting data take the following form:

Choose a linear map $\overline{\rho} : M_{\mathrm{in}} \to \ker(\beta\alpha)$ such that $\overline{\rho}\iota = \mathrm{id}_{\ker(\beta\alpha)}$.

Choose a linear map $\overline{\sigma} : \operatorname{im}\beta/\operatorname{im}(\beta\alpha) \to \operatorname{im}\beta$ such that $\pi\overline{\sigma} = \mathrm{id}_{\operatorname{im}\beta/\operatorname{im}(\beta\alpha)}$.

Adapting (10.10) to the current situation (in particular, realizing M'_k as in (10.27)), we see that the maps α' and β' take the following form:

$$\alpha' = \begin{pmatrix} -\alpha\overline{\rho} \\ -\pi\alpha \\ 0 \\ 0 \end{pmatrix}, \qquad \beta' = (0 \ \ -\beta \ \ -\iota\overline{\sigma}\beta \ \ 0); \tag{10.28}$$

here with some abuse of notation we denote by the same symbol β the two maps $\operatorname{im}\alpha/\ker\beta\cap\operatorname{im}\alpha \to M_{\mathrm{out}}$ and $M_k/(\ker\beta + \operatorname{im}\alpha) \to \operatorname{im}\beta/\operatorname{im}(\beta\alpha)$ induced by β. Note that the appearance of the minus sign in β' is caused by the minus sign in (10.23).

To complete the proof of Theorem 10.13, it remains to construct an isomorphism of vector spaces $\psi : M'_k \to M_k$ such that

$$\psi\alpha' = \alpha, \qquad \beta\psi = \beta'. \tag{10.29}$$

To do this, notice that the four direct summands in (10.27) are the factors in the filtration

$$\{0\} \subseteq \ker\beta\cap\operatorname{im}\alpha \subseteq \operatorname{im}\alpha \subseteq \ker\beta + \operatorname{im}\alpha \subseteq M_k.$$

Choose some sections

$$\begin{aligned} &\sigma_1 : \operatorname{im}\alpha/(\ker\beta\cap\operatorname{im}\alpha) \to \operatorname{im}\alpha, \\ &\sigma_2 : (\ker\beta + \operatorname{im}\alpha)/\operatorname{im}\alpha \to (\ker\beta + \operatorname{im}\alpha), \\ &\sigma_3 : M_k/(\ker\beta + \operatorname{im}\alpha) \to M_k \end{aligned}$$

for the three factors of this filtration, so that they satisfy:

$$\operatorname{im}\sigma_1 = \alpha(\ker\overline{\rho}), \qquad \operatorname{im}\sigma_2 \subseteq \ker\beta, \qquad \operatorname{im}(\beta\sigma_3) = \operatorname{im}\overline{\sigma}.$$

Now define an an isomorphism $\psi : M'_k \to M_k$ by setting

$$\psi = (-\iota \ \ -\iota\sigma_1 \ \ -\iota\sigma_3 \ \ -\iota\sigma_2).$$

The equalities (10.29) are checked by a direct inspection, finishing the proof. □

Note that there is an obvious way to define direct sums for decorated representations of a given QP (A, S). Hence we can talk about *indecomposable* QP-representations. Clearly, the right-equivalence relation respects direct sums and indecomposability. It is also immediate from the definitions that any mutation μ_k of QP-representations sends direct sums to direct sums. Combining this with Theorem 10.13, we obtain the following corollary.

Corollary 10.14. *Any mutation μ_k is an involution on the set of right-equivalence classes of indecomposable decorated representations of reduced QPs satisfying* (5.1).

We call a QP-representation (A, S, M, V) *positive* if $V = \{0\}$. Thus, indecomposable positive representations are just indecomposable $\mathcal{P}(A, S)$-modules. In particular, for every vertex k, the *simple* representation $\mathcal{S}_k(A, S)$ is the indecomposable positive representation of (A, S) such that $\dim M_i = \delta_{i,k}$. We denote by $\mathcal{S}_k^-(A, S)$ the indecomposable representation (M, V) of (A, S) such that $M = \{0\}$ and $\dim V_i = \delta_{i,k}$. We refer to $\mathcal{S}_k^-(A, S)$ as the *negative simple* representation at k. The following proposition is immediate from the definitions.

Proposition 10.15. *Any indecomposable QP-representation is either positive, or negative simple. If* $\mu_k(A, S) = (\overline{A}, \overline{S})$ *then*

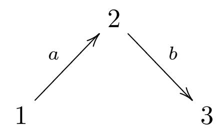
$$\mu_k(\mathcal{S}_k(A, S)) = \mathcal{S}_k^-(\overline{A}, \overline{S}), \qquad \mu_k(\mathcal{S}_k^-(A, S)) = \mathcal{S}_k(\overline{A}, \overline{S}); \tag{10.30}$$

and this is the only mutation that interchanges positive and negative indecomposable representations.

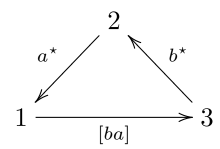
11. Some three-vertex examples

In this section we illustrate the action of mutations on QP-representations by some examples dealing with three-vertex quivers. All the representations (M, V) considered below will be positive, i.e., $V = \{0\}$.

Example 11.1. Let Q be the quiver with three vertices $1, 2, 3$ and two arrows $a : 1 \to 2$ and $b : 2 \to 3$:



Since Q is acyclic, the only QP on it is $(A, 0)$. We have $\mu_2(A, 0) = (\overline{A}, \overline{S})$, where $\overline{A}$ is the arrow span of the quiver $\overline{Q}$ given by



and $\overline{S} = b^\star [ba] a^\star$. Thus, positive representations of $(A, 0)$ are the representations of the quiver Q, while positive representations of $(\overline{A}, \overline{S})$ are the representations of the quiver $\overline{Q}$ satisfying the relations

$$b^\star [ba] = [ba] a^\star = a^\star b^\star = 0. \tag{11.1}$$

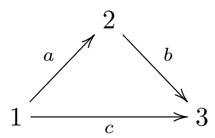
In view of Corollary 10.14 and Proposition 10.15, the mutation μ_2 establishes a bijection between the set of right-equivalence classes of indecomposable positive representations of $(A, 0)$ different from the simple representation $\mathcal{S}_2$, and the same set for $(\overline{A}, \overline{S})$. Since Q is a Dynkin quiver of type A_3, by Gabriel's theorem, an indecomposable positive representation M of $(A, 0)$ is uniquely up to an isomorphism determined by its dimension vector $\mathbf{dim}\, M = (\dim M_1, \dim M_2, \dim M_3)$, and these dimension vectors are the positive roots of type A_3 (note that in this case, the right-equivalence classes are the same as isomorphism classes). Computing the images of these representations under μ_2, we obtain the correspondence between the dimension vectors given in Table 1.

TABLE 1. Indecomposable representations for A_3 and the cyclic triangle.

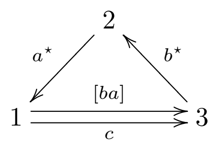
$\mathbf{dim}\, M$	$(1,0,0)$	$(0,0,1)$	$(1,1,0)$	$(0,1,1)$	$(1,1,1)$
$\mathbf{dim}\, \mu_2(M)$	$(1,1,0)$	$(0,1,1)$	$(1,0,0)$	$(0,0,1)$	$(1,0,1)$

We conclude that an indecomposable positive representation of $(\overline{A}, \overline{S})$ is uniquely up to right-equivalence determined by its dimension vector, and these dimension vectors are given in the second line of Table 1, with the exception of $\mathbf{dim}\, \mathcal{S}_2 = (0, 1, 0)$.

Example 11.2. Now let Q be the quiver with three vertices $1, 2, 3$ and three arrows $a : 1 \to 2$, $b : 2 \to 3$, and $c : 1 \to 3$:



Again, the only QP on Q is $(A, 0)$. We have $\mu_2(A, 0) = (\overline{A}, \overline{S})$, where $\overline{A}$ is the arrow span of the quiver $\overline{Q}$ given by



and $\overline{S} = b^\star [ba] a^\star$. Again, positive representations of $(A, 0)$ are the representations of the quiver Q, while positive representations of $(\overline{A}, \overline{S})$ are the representations of the quiver $\overline{Q}$ satisfying the relations (11.1).

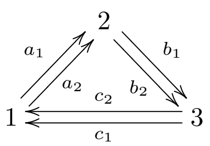
We consider indecomposable positive representations of $(A, 0)$ with the dimension vector (n, n, n) for some $n \geq 1$. Assume that K is algebraically closed. Since Q is an extended Dynkin quiver of type $A_2^{(1)}$, and (n, n, n) is an isotropic imaginary root, by Kac's extension of Gabriel's theorem, the isomorphism classes of indecomposable Q-representations of this dimension form a 1-parameter family. An easy check shows that these representations break into three right-equivalence classes. Their representatives can be described as follows. For each of them we have $M_1 = M_2 = M_3 = K^n$, and two of the maps a_M, b_M, c_M are equal to the identity map I, while the third one is the nilpotent Jordan block N. If $a_M = N$ (resp. $b_M = N$, $c_M = N$) then we denote the corresponding Q-representation by

$M(a)$ (resp. $M(b)$, $M(c)$). In view of (10.7), if M is one of these representations then $\mu_2(M) = \overline{M}$ is positive, and we have

$$\overline{M}_2 = \operatorname{coker} b_M \oplus \ker a_M$$

(note that $\gamma = 0$ since $S = 0$). It follows that $\overline{M(c)}$ has dimension vector $(n, 0, n)$, with the maps $[ba]_{\overline{M(c)}}, c_{\overline{M(c)}} : K^n \to K^N$ given by $[ba]_{\overline{M(c)}} = I, c_{\overline{M(c)}} = N$. Also both representations $\overline{M(a)}$ and $\overline{M(b)}$ have dimension vector $(n, 1, n)$. In each of them, the arrows $[ba]$ and c act as $[ba] = N, c = I$. We also have $b^\star_{\overline{M(a)}} = 0$, while the map $a^\star_{\overline{M(a)}} : K \to K^n$ has $\operatorname{im} a^\star_{\overline{M(a)}} = \ker N$; similarly, $a^\star_{\overline{M(b)}} = 0$, while the map $b^\star_{\overline{M(b)}} : K^n \to K$ has $\ker b^\star_{\overline{M(a)}} = \operatorname{im} N$.

Example 11.3. Our last example deals with the QP (A, S) from Example 8.6. Thus, the quiver in question has three vertices $1, 2, 3$ and six arrows $a_1, a_2 : 1 \to 2$, $b_1, b_2 : 2 \to 3$ and $c_1, c_2 : 3 \to 1$; and the potential S is given by (8.2).



To specify a positive representation M of (A, S), we need to define three vector spaces M_1, M_2, M_3, and six linear maps $(a_1)_M, (a_2)_M : M_1 \to M_2$, $(b_1)_M, (b_2)_M : M_2 \to M_3$, and $(c_1)_M, (c_2)_M : M_3 \to M_1$. In our case, $J(S)$ is the closure of the ideal in $R\langle\langle A \rangle\rangle$ generated by six elements

$$c_1 b_1, b_1 a_1, a_1 c_1, c_2 b_2, b_2 a_2, a_2 c_2.$$

Thus, all the compositions $(c_1)_M (b_1)_M, \ldots, (a_2)_M (c_2)_M$ must be equal to 0.

We first consider the indecomposable positive representation M of (A, S) given by

$$M_1 = M_2 = K, \quad M_3 = 0; \quad (a_1)_M = (a_2)_M = 1. \tag{11.2}$$

Let us compute $\mu_2(M) = (\overline{M}, \overline{V})$. First of all, the QP $\mu_2(A, S) = (\overline{A}, \overline{S})$ was computed in Example 8.6: recall that the arrows in $\overline{A}$ are $a_1^\star, a_2^\star, b_1^\star, b_2^\star, [b_1 a_2], [b_2 a_1]$, and the potential $\overline{S}$ is given by

$$\overline{S} = [b_1 a_2] a_2^\star b_1^\star + [b_2 a_1] a_1^\star b_2^\star.$$

To compute $\overline{M}$ and $\overline{V}$, we apply (10.6) and (10.7) to the triangle (10.5) given by

$$M_{\text{in}} = K^2, \quad M_k = M_2 = K, \quad M_{\text{out}} = \{0\}, \quad \alpha = (1 \ \ 1)$$

(so we have $\beta = 0$ and $\gamma = 0$). It follows that $\overline{V} = \{0\}$, i.e., $\mu_2(M)$ is positive; we also have $\overline{M}_1 = M_1 = K$, $\overline{M}_3 = M_3 = \{0\}$, and

$$\overline{M}_2 = \ker \alpha = K \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(this is the third term in the decomposition of $\overline{M}_k$ in (10.7)). Since $M_3 = 0$, the arrows $b_1^\star, b_2^\star, [b_1 a_2], [b_2 a_1]$ act as 0 in $\overline{M}$. As for $a_1^\star$ and $a_2^\star$, their action is given by the second equality in (10.10) (note that the choice of a splitting (10.9) is immaterial here). Namely, identifying $\overline{M}_2$ with K via choosing $\left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)$ as the standard basis vector, we obtain

$$(a_1^\star)_{\overline{M}} = 1, \qquad (a_2^\star)_{\overline{M}} = -1$$

as maps $\overline{M}_2 = K \to K = \overline{M}_1$.

Note that the resulting representation $\mu_2(M)$ can be conveniently described as follows: by renumbering the vertices of our quiver via

$$1' = 2, \qquad 2' = 1, \qquad 3' = 3, \tag{11.3}$$

and setting

$$a_1' = -a_2^\star,\ a_2' = a_1^\star,\ b_1' = [b_1a_2],\ b_2' = [b_2a_1],\ c_1' = -b_1^\star,\ c_2' = b_2^\star, \tag{11.4}$$

the representation $\mu_2(M)$ gets identified with the initial representation M of the initial QP (A,S).

The mutation $\mu_1(M)$ can be computed in a similar way. But since we have already computed the QP $\mu_2(A,S) = (\overline{A},\overline{S})$, we find it more convenient to renumber the vertices via $1' = 3$, $2' = 1$, $3' = 2$, so that $\mu_1(M)$ gets identified with $\mu_2(M')$, where M' is given by

$$M_1' = 0, \qquad M_2' = M_3' = K; \qquad (b_1)_{M'} = (b_2)_{M'} = 1. \tag{11.5}$$

Now the triangle (10.5) is given by

$$M_{\text{in}}' = \{0\}, \qquad M_k' = M_2' = K, \qquad M_{\text{out}}' = K^2, \qquad \beta = \begin{pmatrix}1\\1\end{pmatrix}$$

(so we have $\alpha = 0$ and $\gamma = 0$). It follows that $\mu_2(M')$ is positive, and we have $\overline{M'}_1 = M_1' = \{0\}$, $\overline{M'}_3 = M_3' = K$, and

$$\overline{M'}_2 = K^2/K \cdot \begin{pmatrix}1\\1\end{pmatrix}$$

(this is the first term in the decomposition of $\overline{M}_k$ in (10.7)). Since $M_1' = 0$, the arrows $a_1^\star, a_2^\star, [b_1a_2], [b_2a_1]$ act as 0 in $\overline{M'}$. As for $b_1^\star$ and $b_2^\star$, their action is given by the first equality in (10.10) (note that the choice of a splitting (10.8) is immaterial here). Namely, identifying $\overline{M'}_2$ with K via choosing $\pi\big(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\big) = -\pi\big(\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\big)$ as the standard basis vector, we obtain

$$(b_1^\star)_{\overline{M'}} = -1, \qquad (b_2^\star)_{\overline{M'}} = 1$$

as maps $\overline{M'}_3 = K \to K = \overline{M'}_2$.

As above, by renumbering the vertices of our quiver via

$$1' = 1, \qquad 2' = 3, \qquad 3' = 2, \tag{11.6}$$

and setting

$$a_1' = [b_2a_1],\ a_2' = [b_1a_2],\ b_1' = b_2^\star,\ b_2' = -b_1^\star,\ c_1' = a_1^\star,\ c_2' = -a_2^\star, \tag{11.7}$$

the resulting representation $\mu_2(M')$ gets identified with the initial representation M'.

We now include the representations M and M' given by (11.2) and (11.5) into a family of positive representations of (A, S) defined as follows: for every pair $(m, n) \neq (0, 0)$ of nonnegative integers, we define the positive representation $M = M(m, n)$ of (A, S) by setting

$$M_1 = K^m, \quad M_2 = K^{m+n}, \quad M_3 = K^n \tag{11.8}$$

and

$$(a_1)_M = \begin{pmatrix} I_m \\ 0 \end{pmatrix}, \quad (a_2)_M = \begin{pmatrix} 0 \\ I_m \end{pmatrix}, \quad (b_1)_M = (0 \ \ I_n), \tag{11.9}$$
$$(b_2)_M = (I_n \ \ 0), \quad (c_1)_M = 0, \quad (c_2)_M = 0,$$

where I_n is the $n \times n$ identity matrix.

We refer to the representations $M(m, n)$ as well as those obtained from them by renumbering the vertices as *band representations*; they are a special case of band modules studied in [11, 21] in the context of string algebras. Note that both representations M and M' treated above are indeed special cases of band representations: we have $M = M(1, 0)$, $M' = M(0, 1)$. By a direct generalization of the above computations, we obtain the following proposition.

Proposition 11.4.

1. *If $m \geq n$ then after renumbering of vertices as in* (11.3) *and the change of arrows as in* (11.4)*, the representation $\mu_2(M(m, n))$ can be identified with $M(m - n, n)$.*
2. *If $m \leq n$ then after renumbering of vertices as in* (11.6) *and the change of arrows as in* (11.7)*, the representation $\mu_2(M(m, n))$ can be identified with $M(m, n - m)$.*

Remembering Theorem 10.13, we obtain the following corollary.

Corollary 11.5.

1. *After renumbering of vertices as in* (11.3)*, the representation $\mu_1(M(m, n))$ becomes right-equivalent to $M(m + n, n)$.*
2. *After renumbering of vertices as in* (11.6)*, the representation $\mu_3(M(m, n))$ becomes right-equivalent to $M(m, m + n)$.*

Corollary 11.6. *The class of band representations is closed under mutations.*

Note that if we iterate the mutations in Proposition 11.4, the pair (m, n) gets transformed according to the Euclid algorithm for finding $\gcd(m, n)$. Thus, after a sequence of mutations (and appropriate renumberings of vertices), every $M(m, n)$ can be transformed into $M(\gcd(m, n), 0)$. Since $M(d, 0)$ is obviously isomorphic to the direct sum of d copies of $M(1, 0)$, by backtracking this sequence of mutations, we obtain the following well-known corollary.

Corollary 11.7. *The representation $M(m, n)$ is indecomposable if and only if m and n are relatively prime. Furthermore, if* $\gcd(m, n) = d$ *then $M(m, n)$ is right-equivalent to the direct sum of d copies of $M(m/d, n/d)$.*

Remark 11.8. By the same methods as above, one can compute all the mutations for another family of representations of the QP (A,S) in Example 11.3: *string modules* introduced and studied in [11, 21].

12. Some open problems

Here we collect some natural questions that we find important for better understanding of QPs and their representations. In what follows, suppose that (A,S) is a reduced QP with the Jacobian algebra $\mathcal{P}(A,S)$. Let $\mathcal{M}(A,S)$ denote the category of finite-dimensional $\mathcal{P}(A,S)$-modules. Suppose also that $k \in Q_0$ is a vertex satisfying (5.1), so that the mutated reduced QP $\mu_k(A,S)$ is well-defined.

Question 12.1. Is the isomorphism class of $\mathcal{P}(A,S)$ determined by the equivalence class of the category $\mathcal{M}(A,S)$?

Question 12.2. Is the isomorphism class of $\mathcal{P}(\mu_k(A,S))$ determined by the isomorphism class of $\mathcal{P}(A,S)$?

Question 12.3. Is the category $\mathcal{M}(\mu_k(A,S))$ determined up to equivalence by $\mathcal{M}(A,S)$?

Note that the right-equivalence class of (A,S) is *not* determined by the isomorphism class of the Jacobian algebra $\mathcal{P}(A,S)$. In fact, we can construct a QP (A,S) which is *not* right-equivalent to (A,cS) for some nonzero $c \in K$, while we obviously have $\mathcal{P}(A,S) = \mathcal{P}(A,cS)$ (the possibility of such an example was brought to our attention by Bill Crawley-Boevey).

We conclude with the following intriguing question.

Question 12.4. Is there a proper analogue of the cluster category for a nonacyclic quiver with potential?

13. Appendix. Proof of Lemma 4.12

We include Lemma 4.12 into a more general setup. We call a K-vector space V a *C-space* (for lack of a better term) if V has an increasing filtration $\{0\} = V_0 \subseteq V_1 \subseteq \cdots$ such that all V_n are finite-dimensional, and $V = \bigcup_{n\geq 0} V_n$. (Equivalently, V is either finite-dimensional, or it has countable dimension.) The class of C-spaces is clearly closed under taking subspaces, quotient spaces, finite direct sums, and finite tensor products. We always consider C-spaces equipped with discrete topology; in particular, this applies to the base field K.

We refer to the dual space $V^\star$ of a C-space V as a *D-space* (the dual is understood as the space of all linear forms $V \to K$). Most of the properties of D-spaces discussed below are undoubtedly well-known; for the convenience of the reader, we provide a self-contained treatment.

Example 13.1. The complete path algebra $R\langle\langle A\rangle\rangle$ can be naturally viewed as a D-space $V^\star$, corresponding to the C-space $V = \bigoplus_{d=0}^{\infty}(A^d)^\star$, and the filtration (V_n) given by

$$V_n = \bigoplus_{d=0}^{n-1}(A^d)^\star \quad (n \geq 1).$$

For a subspace W of V, we denote by $W^\perp \subset V^\star$ its orthogonal complement $\{f \in V^\star \mid f(W) = 0\}$. We make $V^\star$ into a topological vector space by taking the sets $V_n^\perp$ for all $n \geq 0$ as a basic system of open neighborhoods of 0. In particular, in Example 13.1, we have $V_n^\perp = \mathfrak{m}(A)^n$, so the D-space topology on $R\langle\langle A\rangle\rangle$ coincides with the topology introduced in Section 2.

Since every $v \in V$ belongs to some V_n, a sequence $f_1, f_2, \ldots$ converges in $V^\star$ if and only if, for every $v \in V$, the sequence $(f_k(v))$ stabilizes as $k \to \infty$. This implies in particular that $W^\perp$ is a closed subspace of $V^\star$ for every subspace W of V. In fact, the converse is also true.

Lemma 13.2. *A vector subspace Z of $V^\star$ is closed if and only if $Z = W^\perp$ for some subspace W of V.*

Proof. Let Z be a vector subspace of $V^\star$. Let

$$W = \{v \in V \mid f(v) = 0 \text{ for } f \in Z\}.$$

It suffices to show that $W^\perp$ is contained in the closure $\overline{Z}$ of Z. Let $f \in W^\perp$. Restricting f to each finite-dimensional subspace V_n of V, we conclude that $f\,|_{V_n} = h_n\,|_{V_n}$ for some $h_n \in Z$. Thus, the sequence $h_1, h_2, \ldots$ converges to f, implying that $f \in \overline{Z}$, as required. □

In view of Lemma 13.2, for every closed subspace Z of $V^\star$, the spaces Z and $V^\star/Z$ can be naturally viewed as D-spaces: indeed, we have

$$Z = W^\perp = (V/W)^\star, \qquad V^\star/Z = V^\star/W^\perp = W^\star$$

for some subspace W of V. The following lemma is immediate from the definitions.

Lemma 13.3. *For every closed subspace $Z \subseteq V^\star$, the D-space topologies on Z and $V^\star/Z$ coincide with the topologies induced from $V^\star$. In particular, the embedding $Z \to V^\star$ and the projection $V^\star \to V^\star/Z$ are continuous.*

Lemma 13.4. *If Z_1 and Z_2 are closed subspaces of $V^\star$, then $Z_1 + Z_2$ is a closed subspace of $V^\star$ as well.*

Proof. By Lemma 13.2, $Z_1 = W_1^\perp$ and $Z_2 = W_2^\perp$ for some subspaces W_1 and W_2 of V. Choosing some direct complements of $W_1 \cap W_2$ in W_1 and W_2, and a direct complement of $W_1 + W_2$ in V, it is easy to see that

$$Z_1 + Z_2 = W_1^\perp + W_2^\perp = (W_1 \cap W_2)^\perp,$$

proving that $Z_1 + Z_2$ is closed. □

Lemma 13.5. *Let U and V be C-spaces, and $U^\star$ and $V^\star$ be the corresponding D-spaces. A linear map $\alpha : U^\star \to V^\star$ is continuous if and only if $\alpha = \beta^\star$ for some linear map $\beta : V \to U$.*

Proof. First let us show that $\alpha = \beta^\star$ is continuous. By the definition, it is enough to show that, for every n, there exists an index k such that $U_k^\perp \subset \alpha^{-1}(V_n^\perp)$. Since the subspace $\beta(V_n) \subset U$ is finite-dimensional, it is contained in some U_k, implying the desired inclusion $U_k^\perp \subset \alpha^{-1}(V_n^\perp)$.

Conversely, suppose $\alpha : U^\star \to V^\star$ is a continuous linear map. Let $v \in V$. Then the linear form $f \mapsto \alpha(f)(v)$ is a continuous linear map $U^\star \to K$, and so its kernel is a closed subspace of $U^\star$. Using Lemma 13.2, we conclude that there exists a unique $u \in U$ such that $\alpha(f)(v) = f(u)$ for all $f \in U^\star$. The correspondence $v \mapsto u$ is the desired linear map $\beta : V \to U$ such that $\alpha = \beta^\star$. □

Lemma 13.6. *Any continuous linear map of D-spaces $\alpha : U^\star \to V^\star$ sends closed vector subspaces of $U^\star$ to closed vector subspaces of $V^\star$.*

Proof. Let $Z \subseteq U^\star$ be a closed vector subspace. By Lemma 13.2, $Z = W^\perp$ for some vector subspace $W \subset U$. Also by Lemma 13.5, we have $\alpha = \beta^\star$ for a linear map $\beta : V \to U$. The definitions imply that $\alpha(Z) = \beta^\star(W^\perp) = (\beta^{-1}(W))^\perp$, hence $\alpha(Z)$ is a closed subspace of $V^\star$, as claimed. □

We will call a D-space $V^\star$ a *D-algebra* if it has a structure of an associative K-algebra such that $V_m^\perp V_n^\perp \subset V_{m+n}^\perp$ for all $m, n \geq 0$. In particular, $R\langle\langle A\rangle\rangle$ is a D-algebra.

Lemma 13.7. *If $I_1, \dots, I_N$ are closed subspaces in a D-algebra $V^\star$, then the subspace $I_1 f_1 + \cdots + I_N f_N$ is closed for any $f_1, \dots, f_N \in V^\star$. In particular, finitely generated left ideals in $V^\star$ are closed.*

Proof. By the definition of a D-algebra, the operator of right multiplication with any $f \in V^\star$ is continuous. Thus each subspace $I_k f_k$ is closed by Lemma 13.6, and our assertion follows from Lemma 13.4. □

Recall from Definition 3.4 that the trace space of a D-algebra $V^\star$ is the quotient $\mathrm{Tr}(V^\star) = V^\star/\{V^\star, V^\star\}$, where $\{V^\star, V^\star\}$ is the closure of the vector subspace in $V^\star$ spanned by all commutators. We denote by $\pi : V^\star \to \mathrm{Tr}(V^\star)$ the canonical projection. By Lemma 13.3, π is continuous with respect to the D-space topologies.

In view of Proposition 3.5, the assertion of Lemma 4.12 is a special case of the following.

Lemma 13.8. *Let I be a closed (two-sided) ideal of a D-algebra $V^\star$, and J be the closure of an ideal generated by finitely many elements $f_1, \dots, f_N$. Then the subspace $\pi(IJ) \subseteq \mathrm{Tr}(V^\star)$ is equal to $\pi(If_1 + \cdots + If_N)$.*

Proof. Let J^0 be the ideal generated by $f_1, f_2, \dots, f_N$, that is, the linear span of elements of the form uf_kv with $u, v \in V^\star$ and $k = 1, \dots, N$. Thus the ideal IJ^0 is the linear span of elements of the form guf_kv with $g \in I$. By the definition, we have $\pi(guf_k v) = \pi(vguf_k)$, and so $\pi(IJ^0) = \pi(If_1 + \cdots + If_N)$. Since IJ^0 is dense in IJ, it follows that $\pi(IJ^0)$ is dense in $\pi(IJ)$. On the other hand, the subspace $\pi(If_1 + \cdots + If_N) \subseteq \mathrm{Tr}(V^\star)$ is closed by Lemmas 13.7 and 13.6. We conclude that $\pi(IJ^0) = \pi(If_1 + \cdots + If_N) = \pi(IJ)$, as required. □

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Harm Derksen
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109, USA
e-mail: `hderksen@umich.edu`

Jerzy Weyman and Andrei Zelevinsky
Department of Mathematics
Northeastern University
Boston, MA 02115, USA
e-mail: `j.weyman@neu.edu, andrei@neu.edu`