ANNALES

DE

L'INSTITUT FOURIER

Harm DERKSEN & Jerzy WEYMAN

The combinatorics of quiver representations

Tome 61, n° 3 (2011), p. 1061-1131.

<http://aif.cedram.org/item?id=AIF_2011__61_3_1061_0>

Ann. Inst. Fourier, Grenoble
61, 3 (2011) 1061-1131

THE COMBINATORICS OF QUIVER REPRESENTATIONS

by Harm DERKSEN & Jerzy WEYMAN (*)

Abstract. — We give a description of faces, of all codimensions, for the cones spanned by the set of weights associated to the rings of semi-invariants of quivers. For a triple flag quiver and its faces of codimension 1 this description reduces to the result of Knutson-Tao-Woodward on the facets of the Klyachko cone. We give new applications to Littlewood-Richardson coefficients, including a product formula for LR-coefficients corresponding to triples of partitions lying on a wall of the Klyachko cone. We systematically review and develop the necessary methods (exceptional and Schur sequences, orthogonal categories, semi-stable decompositions, GIT quotients for quivers). In an Appendix we include a variant of Belkale's geometric proof of a conjecture of Fulton that works for arbitrary quivers.

Résumé. — On donne une description des faces, des toutes codimensions, pour les cônes engendrés par l'ensemble des poids associés aux anneaux des semi-invariants des carquois. Pour un carquois de drapeaux triples et ses faces de codimension 1, la description est équivalente à un résultat de Knutson-Tao-Woodward sur les facettes du cône de Klyachko. On donne des nouvelles applications aux coefficients de Littlewood-Richardson, en particulier une formule pour les coefficients qui correspond à des triples de partitions sur un mur du cône de Klyachko. On commence par rappeler les méthodes utilisées (suites de Schur, les suites exceptionnelles, les catégories orthogonaux, les décompositions semi-stables, et les quotients GIT pour les carquois). Dans une appendice, on donne une variante d'une démonstration géométrique de Belkale d'une conjecture de Fulton qui est valable pour un carquois quelconque.

1. Introduction

1.1. Main results

A quiver is just a finite directed graph. If we attach vector spaces to the vertices and linear maps to the arrows, we get a representation of that graph.

Keywords: Quiver representations, Klyachko cone, Littlewood-Richardson coefficients.
Math. classification: 16G20, 05E10, 13A50.
(*) The first author was supported by NSF, grant DMS 0901298 and the second author was supported by NSF, grant DMS 0901185.

Let Q be a quiver without oriented cycles and K be an algebraically closed field. Suppose that $\alpha \in \mathbb{N}^{Q_0}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$, Q_0 is the set of vertices of the quiver and $\mathbb{N}^{Q_0}$ is the set of dimension vectors. In [11] the authors studied the set $\Sigma(Q, \alpha) \subseteq \mathbb{Z}^{Q_0}$ of weights occurring in the ring of semi-invariants $\mathrm{SI}(Q, \alpha)$ on the space of α-dimensional representations $\mathrm{Rep}(Q, \alpha)$ over the field K. We showed that this set is given by one linear homogeneous equation and a finite set of homogeneous linear inequalities. Thus the positive real span $\mathbb{R}_+\Sigma(Q, \alpha) \subseteq \mathbb{R}^{Q_0}$ forms a rational polyhedral cone in $\mathbb{R}^{Q_0}$.

Let α, β be dimension vectors for a quiver Q without oriented cycles. If $\langle \alpha, \beta \rangle_Q = 0$, where $\langle \cdot, \cdot \rangle_Q$ is the Euler form (or Ringel form), then we will study the numbers $\alpha \circ \beta$ which are defined as the *dimensions of the weight spaces* $\mathrm{SI}(Q, \beta)_{\langle \alpha, - \rangle}$ *of semi-invariants* (see Definition 2.5). It was shown in [10] that $\alpha \circ \beta$ can also be defined in terms of *Schubert calculus*. In the Schubert calculus approach, $\alpha \circ \beta$ counts the number of α-dimensional subrepresentations of a general $(\alpha + \beta)$-dimensional representation.

This interpretation allows a closer study whose main point is to understand the geometry of the cones $\Sigma(Q, \alpha)$. Our main result (Theorem 5.1) describes the faces of $\Sigma(Q, \alpha)$ of arbitrary codimension.

The new combinatorial tool we introduce to describe these faces are the *Schur sequences*. A Schur sequence is a sequence of Schur roots $\alpha_1, \ldots, \alpha_s$ such that $\alpha_i \circ \alpha_j = 1$ for all $i < j$. Schur sequences are a natural generalizations of exceptional sequences, allowing imaginary Schur roots instead of only real Schur roots. Schur sequences appear naturally as the dimension vectors appearing in the canonical decomposition of a dimension vector. In this paper we also study the dimension vectors of the factors in a Jordan-Hölder fitlration of a σ-semistable representation. This leads to the notion of the σ-stable decomposition of a dimension vector. Again, the dimensions in the σ-stable decomposition form a Schur sequence, and this result was the motivation for their definition sequences. A crucial result is that every Schur sequence can be refined to an exceptional sequence. This is important because it allows to use the induction on the number of vertices of a quiver.

Another key result is Theorem 2.22 stating that if $\alpha \circ \beta = 1$, then for arbitrary positive numbers M, N we have $M\alpha \circ N\beta = 1$. The proof of this result is virtually the same as the proof of P. Belkale of the special case for the triple flag quiver.

Before proving our main results, we will review various notions such as perpendicular categories, exceptional sequences and stability for quivers.

We also will use exceptional sequences to "embed" the category of representation of a quiver Q into the category of representations of another quiver Q'. This is sometimes possible even when Q is not a subquiver Q'.

Our approach is based on studying the notions of semistable filtrations from Geometric Invariant Theory in terms of quiver representations. As a result we obtain combinatorial description of faces of all codimensions in the cones $\mathbb{R}_+\Sigma(Q,\alpha)$. In the special case of extremal rays the results imply that the semi-invariants with weights lying on an extremal ray of $\mathbb{R}_+\Sigma(Q,\alpha)$ form a subring isomorphic to the ring of semi-invariants for quivers with two vertices and multiple arrows. This indicates that one needs to study *all* quivers, not just the special class of triple flag quivers, so the quiver technique is the right tool for studying similar questions.

Some of our results about $\Sigma(Q,\alpha)$ were obtained in the case where $\alpha = (1,1,\ldots,1)$ in [16]. Other results such as Theorem 2.9 or elements in Section 3 were obtained in [17]. Results about some small faces were obtain in [31, 29]. A new proof of Theorem 5.1 can be found in [30].

1.2. Horn's conjecture and related problems

In the special case of the triple flag quiver, and a special dimension vector β the number $\alpha\circ\beta$ turns out to a be Littlewood-Richardson (LR) coefficient $c^{\nu}_{\lambda,\mu}$ where the partitions $\lambda = \lambda(\alpha,\beta), \mu = \mu(\alpha,\beta), \nu = \nu(\alpha,\beta)$ depend on α.

This allows us to apply our technique to obtain many new results about LR-coefficients, and to extend results about LR coefficients to the more general setting of quiver representations. However several of our results were inspired by investigations of this important special case. Moreover, the cone occurring in this case remarkably also appears as a solution to other important problems.

Let us quickly review the relevant results.

A classical topic going back to Hermann Weyl [41] is to compare the eigenvalues of two Hermitian $n\times n$ matrices A, B with the eigenvalues of their sum $C := A+B$. For a Hermitian matrix with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ define $s(A) = (\lambda_1,\ldots,\lambda_n) \in \mathbb{R}^n$. One would like to understand the set

$$\mathcal{K}_n := \{(s(A), s(B), s(C)) \in \mathbb{R}^{3n} \mid A, B, C \text{ Hermitian, } C = A+B\}.$$

It turns out that $\mathcal{K}_n$ is given by the *trace equation*

$$\lambda_1 + \cdots + \lambda_n + \mu_1 + \cdots + \mu_n = \nu_1 + \cdots + \nu_n, \tag{1.1}$$

the *weakly decreasing conditions*

$$\lambda_1 \geqslant \cdots \geqslant \lambda_n, \quad \mu_1 \geqslant \cdots \geqslant \mu_n, \quad \nu_1 \geqslant \cdots \geqslant \nu_n, \tag{1.2}$$

and finitely many inequalities of the form

$$\sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j \geqslant \sum_{k \in K} \nu_k, \tag{1.3}$$

where I, J, K are subsets of $\{1, 2, \ldots, n\}$ of the same cardinality. We will denote the inequality (1.3) by $(\star_{I,J,K})$. Horn made in 1962 a precise conjecture about triples (I, J, K) for which the corresponding inequalities $(\star_{I,J,K})$ define $\mathcal{K}_n$ (see [18]). Horn's conjecture provides a recursive procedure to determine all those triples (I, J, K). This conjecture has been proven as a result of works by Klyachko, Totaro, Knutson and Tao and others. We will state here a closely related statement about the recursive nature of the inequalities defining $\mathcal{K}_n$. For

$$I = \{i_1, i_2, \ldots, i_r\} \subseteq \{1, 2, \ldots, n\}$$

with $i_1 < i_2 < \cdots < i_r$ we define

$$\lambda(I) = (i_r - r + 1, i_{r-1} - r + 2, \ldots, i_2 - 1, i_1).$$

Theorem 1.1. — *The set $\mathcal{K}_n \subseteq \mathbb{R}^{3n}$ is given by the trace equation (1.1), the weakly decreasing conditions (1.2) and all inequalities $(\star_{I,J,K})$ (see (1.3)) with $0 < r := |I| = |J| = |K| < n$, such that*

$$(\lambda(I), \lambda(J), \lambda(K)) \in \mathcal{K}_r.$$

The theorem reflects the recursive nature of the cones $\mathcal{K}_n$. Once we have determined the cones $\mathcal{K}_1, \ldots, \mathcal{K}_{n-1}$, we can determine a system of inequalities for the cone $\mathcal{K}_n$.

A crucial part of the solution of Horn's conjecture is its connection to the representation theory of $\mathrm{GL}_n(\mathbb{C})$. Irreducible representations V_λ of $\mathrm{GL}_n(\mathbb{C})$ are parameterized by nonincreasing integer sequences $\lambda = (\lambda_1, \ldots, \lambda_n)$. The *LR coefficient* $c^\nu_{\lambda,\mu}$ is defined as the multiplicity of V_ν inside the tensor product $V_\lambda \otimes V_\mu$, i.e.,

$$c^\nu_{\lambda,\mu} := \dim(V_\lambda \otimes V_\mu \otimes V_\nu^\star)^{\mathrm{GL}_n(\mathbb{C})}.$$

Here $V_\nu^\star$ denotes the dual space of V_ν and $(V_\lambda \otimes V_\mu \otimes V_\nu^\star)^{\mathrm{GL}_n(\mathbb{C})}$ denotes the $\mathrm{GL}_n(\mathbb{C})$-invariant tensors in $V_\lambda \otimes V_\mu \otimes V_\nu^\star$. We define $c^\nu_{\lambda,\mu} = 0$ if λ, μ, ν are not weakly decreasing. Let us define

$$\mathcal{L}R_n = \{(\lambda, \mu, \nu) \in (\mathbb{Z}^n)^3 \mid c^\nu_{\lambda,\mu} \neq 0\}.$$

The following results follow from Klyachko's paper [25].

Theorem 1.2. — *Let $\mathbb{R}_+$ be the set of nonnegative real numbers. The cone $\mathbb{R}_+\mathcal{L}R_n \subseteq \mathbb{R}^{3n}$ is equal to $\mathcal{K}_n$.*

Theorem 1.3. — *The set $\mathcal{K}_n \subseteq \mathbb{R}^{3n}$ is given by the trace equation (1.1), the weakly decreasing conditions (1.2) and all inequalities $(\star_{I,J,K})$ with $0 < r := |I| = |J| = |K| < n$, such that*

$$(\lambda(I), \lambda(J), \lambda(K)) \in \mathcal{L}R_r.$$

Finally, the missing link for Theorem 1.1 is proved by Knutson and Tao in [26].

Theorem 1.4 (Saturation Theorem). — *The set $\mathcal{L}R_n \subseteq \mathbb{Z}^{3n}$ is saturated, i.e.,*

$$\mathcal{L}R_n = \mathbb{R}_+\mathcal{L}R_n \cap \mathbb{Z}^{3n}.$$

The Saturation Theorem can also be formulated as follows: if $\lambda, \mu, \nu \in \mathbb{Z}^n$ such that $c^{N\nu}_{N\lambda,N\mu} \neq 0$ for some positive integer N, then $c^{\nu}_{\lambda,\mu} \neq 0$. By Theorems 1.2 and 1.4 we have $\mathcal{L}R_n = \mathcal{K}_n \cap \mathbb{Z}^{3n}$. Combining this with Theorem 1.3 implies Theorem 1.1. In [26], Knutson and Tao use their Honeycomb Model to prove Theorem 1.4. See also [3] for another version of the proof. A geometric proof of Theorem 1.4 was given by the authors in [11] and by Belkale in [1].

By Theorem 1.3, the set $\mathcal{K}_n$ is defined by (1.1), (1.2) and all inequalities $(\star_{I,J,K})$ for which $c^{\lambda(K)}_{\lambda(I),\lambda(J)} \neq 0$ are nonzero. C. Woodward was first to note that some of these inequalities are redundant: they follow from the other inequalities. P. Belkale proved that all inequalities $(\star_{I,J,K})$ for which $c^{\lambda(K)}_{\lambda(I),\lambda(J)} > 1$ are redundant. This class includes the examples found by Woodward. As the following theorem by Knutson, Tao and Woodward ([27]) shows, none of the remaining inequalities can be omitted.

Theorem 1.5. — *For $n \geqslant 3$, $\mathcal{K}_n$ is defined by the equation (1.1), the inequalities (1.2) and all inequalities $(\star_{I,J,K})$ for which $c^{\lambda(K)}_{\lambda(I),\lambda(J)} = 1$. None of the inequalities the inequalities can be omitted.*

For $n = 2$, some of the inequalities (1.2) can be omitted. The cone $\mathcal{K}_n$ has dimension $3n - 1$. The inequality

$$(\star_{I,J,K}) : \sum_{i\in I} \lambda_i + \sum_{j\in J} \mu_j \geqslant \sum_{k\in K} \nu_k$$

is necessary if and only if the hyperplane section

$$\Big\{(\lambda, \mu, \nu) \in (\mathbb{R}^n)^3 \Big| \sum_{i\in I} \lambda_i + \sum_{j\in J} \mu_j = \sum_{k\in K} \nu_k\Big\} \cap \mathcal{K}_n$$

defines a *facet* (or *wall*) of the cone $\mathcal{K}_n$.

Along the way, Knutson, Tao and Woodward also proved (see [27]) the following Theorem, which was conjectured by W. Fulton.

Theorem 1.6. — *If $c^{\nu}_{\lambda,\mu} = 1$ for some λ, μ, ν, then $c^{N\nu}_{N\lambda,N\mu} = 1$ for all nonnegative integers N.*

A geometric proof of Theorem 1.6 using Schubert calculus was given by P. Belkale in [2], and it generalizes to the case of general quiver.

Generalizing the above properties to the cones $\mathbb{R}_+\Sigma(Q,\alpha)$, turns out to be fruitful. Our technique allows to prove stronger results about the cones $\mathbb{R}_+\mathcal{L}R_n$.

We construct a triple of partitions (λ, μ, ν) lying on an extremal ray of the Klyachko cone, such that $c^{\nu}_{\lambda,\mu} > 1$. We use the developed techniques to prove some new results on the faces of cones $\mathbb{R}_+\mathcal{L}R_n$, in particular a product formula for LR coefficients. The general approach explains why one could expect such formula. In [6], Calin Chindris and the authors constructed a counterexample to Okounkov's conjecture that LR-coefficients are log-concave functions of the partitions, using this embedding method. The Embedding Theorem was also used by the first author to give examples of small Galois groups in Schubert type problems (see [40, 2.10, 5.13]).

The cones $\mathbb{R}_+\mathcal{L}R_n$ are also related to a problem of existence of short exact sequences of abelian p-groups (see [15, 24]). Recent results of C. Chindris ([4, 5]) show that quivers can be successfully applied to get similar results for longer exact sequences.

1.3. Organization

The paper is organized as follows. In Section 2 we give the basic notation and review the needed results on quivers and their semi-invariants. In particular we review the content of papers [7], [11], [36] and [37], in particular the Saturation Theorem for the cones $\mathbb{R}_+\Sigma(Q,\alpha)$, the exceptional sequences and orthogonal categories.

There are some reformulations and extensions, notably the Embedding Theorem 2.38. Schofield's technique of orthogonal categories allows to embed the category of representations of a smaller quiver (not necessarily a subquiver) into the category of representations of the original quiver. Theorem 2.38 shows that this embedding respects semi-invariants. This gives us a method for proving results about the cones $\mathbb{R}_+\Sigma(Q,\alpha)$ by induction. This technique requires one to work with arbitrary quivers, not just triple flag quivers.

We also formulate the Generalized Fulton's Conjecture (Theorem 2.22) whose proof (essentially due to P. Belkale, see [2]) is given in the Appendix.

In Section 3 we study the notions of semi-stability and stability of quiver representations. We relate the geometric notion of $(\sigma:\tau)$-stability and the algebraic notion of σ-stability, using the approach of [35]. We introduce the notions of Harder-Narasimhan, Jordan-Hölder filtrations, and their combination - the HNJH filtrations. The key statement is Lemma 3.7 which shows that the subsets of the representation space $\operatorname{Rep}(Q,\alpha)$ where the dimensions of the factors of Harder-Narasimhan and HNJH filtrations are constant are locally closed. Then, filtrations are used to define the σ-stable decompositions and $(\sigma:\tau)$-stable decompositions of dimension vectors, and to prove their basic properties. In particular, Theorem 3.16 relates the semi-invariants in weights $m\sigma$ for a dimension vector α to those for the dimension vectors that are factors in σ-stable decomposition of α.

In Section 4 we introduce the key notions of Schur sequences and Schur quiver sequences, inspired by the notion of exceptional sequences. These notions provide the tools for the descriptions of the faces of the cones $\mathbb{R}_+\Sigma(Q,\alpha)$. Then we prove the Refinement Theorem (Theorem 4.11) which says that every Schur sequence can be refined to an exceptional sequence. This theorem makes it easy to understand Schur sequences in terms of exceptional sequences.

In Section 5 we prove the basic Theorem 5.1 giving a bijection between the faces of dimension $n-r$ in $\Sigma(Q,\alpha)$, and Schur quiver sequences of r dimension vectors summing to α. This is the main result of the paper. Then we draw consequences for faces of codimension 1 and for extremal rays.

In Section 6 we study the dual problem of how the σ-stable decomposition of α varies when α varies and σ is fixed. Theorem 6.4 gives a general combinatorial criterion of when α is σ-stable. We also extend the notion of σ-stable decomposition to quivers with oriented cycles.

In Section 7 we apply our results to triple flag quivers and the cones $\mathbb{R}_+\mathcal{L}R_n$. We recover Theorem 1.5 of Knutson, Tao and Woodward on faces of codimension one.

We also investigate the faces of $\mathbb{R}_+\mathcal{L}R_n$ of arbitrary codimension. In particular we show that for $n\leqslant 7$ all weight multiplicities along extremal rays of the cones $\mathbb{R}_+\mathcal{L}R_n$ are equal to 1, and for $n=8$ we give an example of an extremal ray with weight multiplicities bigger than 1.

Finally we prove the product formula for LR coefficients (Theorem 7.14). It shows that if a weight σ corresponding to a triple of partitions (λ,μ,ν)

lies on the face of $\Sigma(T_{n,n,n},\beta)$ of positive codimension, then the LR coefficient $c^{\nu}_{\lambda,\mu}$ decomposes to a product of smaller LR coefficients. Another proof of this result appeared in [23] which appeared in 2009, and the theorem was conjectured in 2006 in [22]. However, the result was already stated and proven in an earlier version of the present paper, which appeared on the math arXiv in 2006, see [14].

Acknowledgement. Both authors would like to thank Prakash Belkale for the permission to include his proof of the Fulton Conjecture, and to Calin Chindris for helpful discussions.

2. Preliminaries

2.1. Basic notions for quivers

A quiver Q is a quadruple $Q=(Q_0,Q_1,h,t)$ where Q_0 is a finite set of vertices, Q_1 is a finite set of arrows and $h,t:Q_1\to Q_0$ are maps. For each arrow $a\in Q_1$, its head is $ha:=h(a)\in Q_0$ and its tail is $ta:=t(a)\in Q_1$.

We fix an algebraically closed field K. A representation V of Q is a family of finite dimensional K-vector spaces

$$\{V(x)\mid x\in Q_0\}$$

together with a family of K-linear maps

$$\{V(a):V(ta)\to V(ha)\mid a\in Q_1\}.$$

The dimension vector of a representation V is the function $\underline{\dim}\,V:Q_0\to\mathbb{N}$ defined by

$$(\underline{\dim}\,V)(x):=\dim V(x),\quad x\in Q_0.$$

The dimension vectors $\mathbb{N}^{Q_0}$ are contained in the set $\Gamma:=\mathbb{Z}^{Q_0}$ of integer-valued functions on Q_0. A morphism $\phi:V\to W$ between two representations is the collection of linear maps

$$\{\phi(x):V(x)\to W(x)\mid x\in Q_0\}$$

such that for each $a\in Q_1$ we have

$$W(a)\phi(ta)=\phi(ha)V(a).$$

We denote the (finite dimensional) linear space of morphisms from V to W by $\operatorname{Hom}_Q(V,W)$.

A *nontrivial* path p in the quiver Q of length $n\geqslant 1$ is a sequence $p=a_na_{n-1}\cdots a_1$ of arrows, such that $ta_{i+1}=ha_i$ for $i=1,2,\dots,n-1$. We define the *head* and the *tail* of the path p as $hp := ha_n$ and $tp := ta_1$ respectively. An oriented cycle is a nontrivial path p satisfying $hp = tp$. *Throughout this paper, we will assume that Q has no oriented cycles, unless stated otherwise.*

The category $\mathrm{Rep}(Q) = \mathrm{Rep}_K(Q)$ of representations of Q over K is hereditary, i.e., any subobject of a projective object is projective. This means that every representation has projective dimension $\leqslant 1$, i.e., $\mathrm{Ext}^i_Q(V,W) = 0$ for all representations V, W and all $i > 1$.

Lemma 2.1 (See [32]). — *The spaces* $\mathrm{Hom}_Q(V,W)$ *and* $\mathrm{Ext}_Q(V,W) := \mathrm{Ext}^1_Q(V,W)$ *are the kernel and cokernel of the following linear map*

$$d^V_W : \bigoplus_{x\in Q_0} \mathrm{Hom}(V(x), W(x)) \longrightarrow \bigoplus_{a\in Q_1} \mathrm{Hom}(V(ta), W(ha)), \tag{2.1}$$

where d^V_W *is given by*

$$\sum_{x\in Q_0} \phi(x) \mapsto \{W(a)\phi(ta) - \phi(ha)V(a) \mid a \in Q_1\}.$$

Let α, β be two elements of Γ. We define the *Euler inner product* (or *Ringel form*) by

$$\langle \alpha, \beta\rangle_Q = \sum_{x\in Q_0} \alpha(x)\beta(x) - \sum_{a\in Q_1} \alpha(ta)\beta(ha). \tag{2.2}$$

It follows from Lemma 2.1 and (2.2) that

$$\langle \underline{\dim}\, V, \underline{\dim}\, W\rangle_Q = \dim \mathrm{Hom}_Q(V,W) - \dim \mathrm{Ext}_Q(V,W). \tag{2.3}$$

We will omit the subscript Q and just write $\langle\cdot,\cdot\rangle$ instead of $\langle\cdot,\cdot\rangle_Q$ if there is no chance of confusion.

2.2. Semi-invariants for quiver representations

Suppose that V is a β-dimensional representation. Choose a basis of each of the vector spaces $V(x)$, $x \in Q_0$. The matrix of $V(a)$ with respect to the bases in $V(ta)$ and $V(ha)$ lies in $\mathrm{Mat}_{\beta(ha),\beta(ta)}(K)$, where $\mathrm{Mat}_{p,q}(K)$ denotes the $p \times q$ matrices with entries in K. This way we can associate to V an element of the representation space

$$\mathrm{Rep}(Q,\beta) := \bigoplus_{a\in Q_1} \mathrm{Mat}_{\beta(ha),\beta(ta)}(K).$$

The group

$$\mathrm{GL}(Q,\beta) := \prod_{x\in Q_0} \mathrm{GL}_{\beta(x)}(K)$$

acts on $\mathrm{Rep}(Q,\beta)$ as follows

$$\{A(x)\mid x\in Q_0\}\cdot\{V(a)\mid a\in Q_1\}:=\{A(ha)V(a)A(ta)^{-1}\mid a\in Q_1\},$$

for $\{A(x)\mid x\in Q_0\}\in\mathrm{GL}(Q,\beta)$ and $\{V(a)\mid a\in Q_1\}\in\mathrm{Rep}(Q,\beta)$. The action of $\mathrm{GL}(Q,\beta)$ on $\mathrm{Rep}(Q,\beta)$ corresponds to base changes in each of the vector spaces $V(x)\cong K^{\beta(x)}$, $x\in Q_0$. The orbits of $\mathrm{GL}(Q,\beta)$ in $\mathrm{Rep}(Q,\beta)$ correspond to the isomorphism classes of β-dimensional representations of Q.

Define $\mathrm{SL}(Q,\beta)\subseteq\mathrm{GL}(Q,\beta)$ by

$$\mathrm{SL}(Q,\beta)=\prod_{x\in Q_0}\mathrm{SL}_{\beta(x)}(K).$$

We are interested in the ring of semi-invariants

$$\mathrm{SI}(Q,\beta)=K[\mathrm{Rep}(Q,\beta)]^{\mathrm{SL}(Q,\beta)}.$$

The ring $\mathrm{SI}(Q,\beta)$ has a weight space decomposition

$$\mathrm{SI}(Q,\beta)=\bigoplus_{\sigma}\mathrm{SI}(Q,\beta)_\sigma$$

where σ runs through the weights of $\mathrm{GL}(Q,\beta)$ and

$$\mathrm{SI}(Q,\beta)_\sigma=\{f\in K[\mathrm{Rep}(Q,\beta)]\mid g(f)=\sigma(g)f\ \forall g\in\mathrm{GL}(Q,\beta)\}.$$

Any *weight* of $\mathrm{GL}(Q,\beta)$ has the form

$$\{A(x)\mid x\in Q_0\}\mapsto\prod_{x\in Q_0}(\det A(x))^{\sigma(x)},\tag{2.4}$$

with $\sigma(x)\in\mathbb{Z}$ for all $x\in Q_0$. This way, the weight (2.4) of $\mathrm{GL}(Q,\beta)$ can be identified with $\sigma\in\Gamma:=\mathbb{Z}^{Q_0}$.

If $\alpha\in\Gamma$ then we define

$$\sigma(\alpha)=\sum_{x\in Q_0}\sigma(x)\alpha(x).$$

We will identify the set of weights with $\Gamma^\star=\mathrm{Hom}_{\mathbb{Z}}(\Gamma,\mathbb{Z})\cong\mathbb{Z}^{Q_0}$. Note that $\Gamma^\star$ and Γ are canonically isomorphic, but we still would like to distinguish between them.

Let us choose the dimension vectors $\alpha,\beta\in\mathbb{N}^{Q_0}$ such that $\langle\alpha,\beta\rangle=0$. If $V\in\mathrm{Rep}(Q,\alpha)$ and $W\in\mathrm{Rep}(Q,\beta)$, then the matrix of d^V_W in (2.1) is a square matrix. Following [36] we define the semi-invariant

$$c(V,W):=\det d^V_W$$

of the action of $\mathrm{GL}(Q,\alpha)\times\mathrm{GL}(Q,\beta)$ on $\mathrm{Rep}(Q,\alpha)\times\mathrm{Rep}(Q,\beta)$. Note that the function $c(V,W)$ is well-defined up to a constant. For a fixed V the restriction of c to $\{V\}\times\mathrm{Rep}(Q,\beta)$ defines a semi-invariant c^V in $\mathrm{SI}(Q,\beta)$.

Schofield proved ([36, Lemma 1.4]) that the weight of c^V equals $\langle \alpha, \cdot \rangle$. Note that $\langle \alpha, \cdot \rangle$ can be viewed as an element in $\Gamma^\star$. Similarly, for a fixed W the restriction of c to $\mathrm{Rep}(Q, \alpha) \times \{W\}$ defines a semi-invariant c_W in $\mathrm{SI}(Q, \alpha)$ of weight $-\langle \cdot, \beta \rangle$ ([36, Lemma 1.4]).

Lemma 2.2 (Lemma 1 of [11]). — *Suppose that*

$$0 \to V_1 \to V \to V_2 \to 0$$

is an exact sequence of representations of Q *and* $\langle \underline{\dim}\, V_1, \beta \rangle = \langle \underline{\dim}\, V_2, \beta \rangle = 0$, *then as a function on* $\mathrm{Rep}(Q, \beta)$, c^V *is, up to a scalar, equal to* $c^{V_1} \cdot c^{V_2}$.

Theorem 2.3 (Theorem 1 of [11]). — *As a vector space, the ring* $\mathrm{SI}(Q, \beta)$ *is spanned by semi-invariants of the form* c^V *for which* $\langle \underline{\dim}\, V, \beta \rangle = 0$. *It is also spanned by semi-invariants of the form* c_W *for which* $\langle \beta, \underline{\dim}\, W \rangle = 0$.

For a more general statement for quivers with oriented cycles, see [13, 39].

Remark 2.4. — If $\langle \underline{\dim}\, V, \underline{\dim}\, W \rangle = 0$ then we have $c(V, W) = c^V(W) = c_W(V) = 0$ if and only if $\mathrm{Hom}_Q(V, W) \neq 0$ which is equivalent to $\mathrm{Ext}_Q(V, W) \neq 0$ by Lemma 2.1.

It was also shown in [11] that

$$\dim \mathrm{SI}(Q, \beta)_{\langle \alpha, \cdot \rangle} = \dim \mathrm{SI}(Q, \alpha)_{-\langle \cdot, \beta \rangle}.$$

Definition 2.5. — *For dimension vectors* α, β *we define*

$$(\alpha \circ \beta)_Q := \dim \mathrm{SI}(Q, \beta)_{\langle \alpha, \cdot \rangle} = \dim \mathrm{SI}(Q, \alpha)_{-\langle \cdot, \beta \rangle}.$$

Again, we will drop the subscript Q *most of the time and write* $\alpha \circ \beta$ *instead of* $(\alpha \circ \beta)_Q$.

2.3. Representations in general position

A representation V is called indecomposable if it is not isomorphic to the direct sum of two nonzero representations. The set of dimension vectors α for which there exists an α-dimensional indecomposable representation can be identified with the set of positive roots for the Kac-Moody algebra associated with the graph Q (where we forget the orientation). This was proven in [19]. We will call a dimension vector α a *root* if there exists an indecomposable representation of dimension α. Kac proved that $\langle \alpha, \alpha \rangle \leqslant 1$ for every root α. If α is a root, then we will call α *real* if $\langle \alpha, \alpha \rangle = 1$ and *imaginary* if $\langle \alpha, \alpha \rangle \leqslant 0$. We call α *isotropic* if $\langle \alpha, \alpha \rangle = 0$.

A representation V is called a Schur representation (or a *brick*) if $\mathrm{Hom}_Q(V,V)\cong K$. Note that every Schur representation must be indecomposable. If $\mathrm{Rep}(Q,\alpha)$ contains a Schur representation, then α is called a *Schur root.*

A representation V is called in general position of dimension α if $V\in\mathrm{Rep}(Q,\alpha)$ lies in a sufficiently small Zariski open subset ("sufficient" here depends on the context). Suppose that α is a Schur root. Since $V\mapsto \dim\mathrm{Hom}_Q(V,V)$ depends upper semi-continuously on $V\in\mathrm{Rep}(Q,\alpha)$, its minimal value 1 is attained on some open dense subset $U\subseteq\mathrm{Rep}(Q,\alpha)$. This shows that a general representation of $\mathrm{Rep}(Q,\alpha)$ is indecomposable. Conversely, if a general representation of dimension α is indecomposable, then α must be a Schur root (see [20, Proposition 1]).

We define

$$\hom(\alpha,\beta)=\min\{\dim\mathrm{Hom}_Q(V,W)\mid V\in\mathrm{Rep}(Q,\alpha),W\in\mathrm{Rep}(Q,\beta)\},$$

where min denotes the minimum. The function $(V,W)\mapsto\dim\mathrm{Hom}_Q(V,W)$ is upper semi-continuous, so $\dim\mathrm{Hom}_Q(V,W)=\hom(\alpha,\beta)$ if $(V,W)\in\mathrm{Rep}(Q,\alpha)\times\mathrm{Rep}(Q,\beta)$ is in general position (see [37]). Similarly, we define

$$\mathrm{ext}(\alpha,\beta)=\min\{\dim\mathrm{Ext}_Q(V,W)\mid V\in\mathrm{Rep}(Q,\alpha),W\in\mathrm{Rep}(Q,\beta)\}.$$

We will drop the subscript and write $\hom(\alpha,\beta)$ and $\mathrm{ext}(\alpha,\beta)$ if there is no confusion. From (2.2) follows that

$$\langle\alpha,\beta\rangle=\hom(\alpha,\beta)-\mathrm{ext}(\alpha,\beta). \tag{2.5}$$

Definition 2.6. — *If* $\hom(\alpha,\beta)=\mathrm{ext}(\alpha,\beta)=0$, *then we write* $\alpha\perp\beta$ *and we will say that* α *is left perpendicular to* β.

By Remark 2.4 we have $\alpha\perp\beta$ if and only if $\alpha\circ\beta\neq 0$. Following Schofield, we write $\alpha\hookrightarrow\beta$ if a general representation of dimension β contains a subrepresentation of dimension α. We write $\alpha\twoheadrightarrow\beta$ if a general representation of dimension α has a factor of dimension β. The proof of the following theorem can be found in [37] for a base field of characteristic 0. For a proof that also works in positive characteristic, see [8].

Theorem 2.7 (Theorem 3.3 of [37]). — *We have*

$$\alpha\hookrightarrow\alpha+\beta\quad\Leftrightarrow\quad\mathrm{ext}(\alpha,\beta)=0\quad(\Leftrightarrow\alpha+\beta\twoheadrightarrow\beta).$$

Definition 2.8. — *For a dimension vector* β, *we define*

$$\Sigma(Q,\beta)=\{\sigma\in\Gamma^\star\mid\mathrm{SI}(Q,\beta)_\sigma\neq 0\}.$$

Theorem 2.9 (see [11]). — *We have*

$$\Sigma(Q,\beta)=\{\sigma\in\Gamma^\star\mid\sigma(\beta)=0\text{ and }\sigma(\gamma)\leqslant 0\text{ for all }\gamma\hookrightarrow\beta\}.$$

For some $\gamma \hookrightarrow \beta$, the inequality $\sigma(\gamma) \leqslant 0$ can be omitted because it follows from the other inequalities. Later, we will describe a minimal list of inequalities for $\Sigma(Q,\beta)$.

Theorem 2.10 (see [10]). — *Suppose that α, β are dimension vectors satisfying $\alpha \perp \beta$. Then a general representation of dimension $\alpha+\beta$ has exactly $\alpha \circ \beta$ subrepresentations of dimension α.*

Lemma 2.11. — *Under the assumptions of Theorem 2.10, if $V \in \operatorname{Rep}(Q, \alpha+\beta)$ is arbitrary such that V has exactly r subrepresentations, where r is finite, then $r \leqslant \alpha \circ \beta$.*

Proof. — Schofield constructs a variety $Z := R(Q, \alpha \subset \alpha+\beta)$ (see [37]) and a projective morphism $p : Z \to \operatorname{Rep}(Q, \alpha+\beta)$ such that the fiber $p^{-1}(V)$ of $V \in \operatorname{Rep}(Q, \alpha+\beta)$ can be identified with the set of all subrepresentations of V of dimension α. Let $U \subseteq \operatorname{Rep}(Q, \alpha+\beta)$ be the set of all $V \in \operatorname{Rep}(Q, \alpha+\beta)$ such that the fiber $p^{-1}(V)$ is finite. Because p is projective it follows by the semicontinuity of the dimension of a fiber that U is open. Let us restrict p to $p^{-1}(U) \to U$. Now $p : p^{-1}(U) \to U$ is a projective, quasi-finite map, hence it is finite. It follows that all fibers have the same cardinality if counted with multiplicity. It was shown in [8] that a general fiber of p is reduced (this is not immediately clear in positive characteristic). Therefore, a general fiber is, set-theoretically, the largest among all fibers $p^{-1}(V)$, $V \in U$.

□

2.4. The canonical decomposition

Following Kac, we make the following definition.

Definition 2.12 (Section 4 of [20]). — *We call*

$$\alpha = \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_s$$

the canonical decomposition of α if a general representation of dimension α decomposes into indecomposable representations of dimensions $\alpha_1, \alpha_2, \ldots, \alpha_s$.

For more details on the canonical decomposition, see [20, 12].

Theorem 2.13 (Proposition 3 of [20]). — *The expression*

$$\alpha = \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_s$$

is the canonical decomposition if and only if $\alpha_1, \ldots, \alpha_s$ are Schur roots, and for all $i \neq j$ we have $\operatorname{ext}(\alpha_i, \alpha_j) = 0$.

Lemma 2.14 (Lemma 5.2 of [38]). — *Suppose that*

$$\alpha = \alpha_1^{\oplus r_1} \oplus \alpha_2^{\oplus r_2} \oplus \cdots \oplus \alpha_s^{\oplus r_s}$$

is the canonical decomposition of α, where $\alpha_1, \alpha_2, \dots, \alpha_s$ are distinct dimension vectors and $r_1, \dots, r_s$ are positive integers. Then we may assume, after rearranging $\alpha_1, \dots, \alpha_s$, that $\hom(\alpha_i, \alpha_j) = 0$ *for all $i < j$.*

In [12] an efficient algorithm was given to compute the canonical decomposition of a given dimension vector. A similar recursive procedure was given in [38]. Lemma 2.14 follows immediately from the correctness of the algorithm in [12], because the output of the algorithm has the desired property.

For a representation $V \in \operatorname{Rep}(Q, \alpha)$, we have

(2.6)
$$\langle \alpha, \alpha \rangle = \sum_{x \in Q_0} \alpha(x)^2 - \sum_{a \in Q_1} \alpha(ta)\alpha(ha) = \dim \operatorname{GL}(Q, \alpha) - \dim \operatorname{Rep}(Q, \alpha).$$

On the other hand,

$$\dim \operatorname{GL}(Q, \alpha) = \dim \operatorname{GL}(Q, \alpha)_V + \dim \operatorname{GL}(Q, \alpha) \cdot V, \tag{2.7}$$

where $\operatorname{GL}(Q, \alpha)_V$ is the stabilizer of V and $\operatorname{GL}(Q, \alpha) \cdot V$ is the orbit of V. Because $\operatorname{GL}(Q, \alpha)_V$ is equal to the invertible elements of $\operatorname{Hom}_Q(V, V)$, it follows from (2.3) that

$$\dim \operatorname{GL}(Q, \alpha)_V = \dim \operatorname{Hom}_Q(V, V) = \langle \alpha, \alpha \rangle + \dim \operatorname{Ext}_Q(V, V). \tag{2.8}$$

Adding (2.6), (2.7) and (2.8) yields

$$\dim \operatorname{Rep}(Q, \alpha) - \dim \operatorname{GL}(Q, \alpha) \cdot V = \dim \operatorname{Ext}(V, V) \tag{2.9}$$

(see also [20, Lemma 4] and [32]).

Let us prove the following known fact.

Lemma 2.15. — *If α is a real Schur root, then* $\operatorname{Rep}(Q, \alpha)$ *has a dense* $\operatorname{GL}(Q, \alpha)$*-orbit.*

Proof. — Suppose that α is a real Schur root and $V \in \operatorname{Rep}(Q, \alpha)$ is a Schur representation. From $\langle \alpha, \alpha \rangle = 1$ and $\dim \operatorname{Hom}_Q(V, V) = 1$ follows that $\dim \operatorname{Ext}_Q(V, V) = 0$ by (2.2) and (2.3). So the orbit is dense by (2.9). □

Theorem 2.16 (See Proposition 4 in [20]). — *Suppose that*

$$\alpha = \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_s$$

is the canonical decomposition of α. Then α is prehomogeneous if and only if $\alpha_1, \dots, \alpha_s$ are real Schur roots.

2.5. The combinatorics of dimension vectors

We point out that many of the notions we just introduced can be defined combinatorially. For example the quantity $\operatorname{ext}(\alpha,\beta)$ can be, in principle, computed using a recursive procedure using the following result.

Theorem 2.17 (Theorem 5.4 of [37]). — *We have*

$$\operatorname{ext}(\alpha,\beta) = \max\{-\langle\alpha',\beta'\rangle \mid \alpha' \hookrightarrow \alpha, \beta \twoheadrightarrow \beta'\} =$$
$$= \max\{-\langle\alpha,\beta'\rangle \mid \beta \twoheadrightarrow \beta'\} = \max\{-\langle\alpha',\beta\rangle \mid \alpha' \hookrightarrow \alpha\}.$$

By Theorem 2.7, the conditions $\alpha' \hookrightarrow \alpha$ and $\beta \twoheadrightarrow \beta'$ can be verified by computing ext-numbers for smaller dimension vectors.

Using (2.5) we can also compute $\hom(\alpha,\beta)$ recursively.

Corollary 2.18. — *The numbers* $\hom(\alpha,\beta), \operatorname{ext}(\alpha,\beta)$ *do not depend on the base field* K*.*

Corollary 2.19 (Generalized Saturation Theorem). — *For dimension vectors* α,β *and positive integers* p,q*, we have*

$$\alpha\circ\beta > 0 \Leftrightarrow p\alpha \circ q\beta > 0$$

Proof. — If $\langle\alpha,\beta\rangle \neq 0$ then $\alpha\circ\beta = p\alpha\circ q\beta = 0$. If $\langle\alpha,\beta\rangle = 0$, then we get $\alpha\circ\beta > 0 \Leftrightarrow \operatorname{ext}(\alpha,\beta) = 0$ and $p\alpha\circ q\beta > 0 \Leftrightarrow \operatorname{ext}(p\alpha,q\beta) = 0$. From Theorem 2.17 follows that $\operatorname{ext}(p\alpha,q\beta) = pq\operatorname{ext}(\alpha,\beta)$. □

Proposition 2.20. — *(see* [10]*) The numbers* $\alpha\circ\beta$ *do not depend on the base field* K*.*

The numbers $\alpha\circ\beta$ can be computed either in terms of Schur functors, or equivalently, in terms of Schubert calculus. This way, $\alpha\circ\beta$ can be expressed as a (perhaps large) sum of products of Littlewood-Richardson coefficients. See [10] for more details.

The following definition will be important later.

Definition 2.21. — *Suppose that* $\alpha,\beta \in \mathbb{N}^{Q_0}$*. We say that* α *is strongly left perpendicular to* β *if*

$$\alpha\circ\beta = 1.$$

We will denote this by $\alpha \perp\!\!\!\perp \beta$*.*

We have

$$\alpha \perp\!\!\!\perp \beta \Rightarrow \alpha \perp \beta \Rightarrow \langle\alpha,\beta\rangle = 0,$$

and none of the implications can be reversed. The following result will be crucial for this paper.

THEOREM 2.22 (Generalized Fulton Conjecture, Belkale, see the Appendix). — *If* $\alpha \circ \beta = 1$, *then*

$$p\alpha \circ q\beta = 1$$

for all $p, q \in \mathbb{N}$.

Remark 2.23. — Theorem 2.22 can be thought of as a generalization of Fulton's Conjecture (Theorem 1.6). For partitions λ, μ, ν one can construct a quiver Q and dimension vectors α, β such that $\alpha \circ \beta = c^{\nu}_{\lambda,\mu}$, and $\alpha \circ (n\beta) = c^{n\nu}_{n\lambda,n\mu}$. We will explain this in more detail in Section 7.

A dimension vector α is a Schur root if and only if there exist no nonzero dimension vectors β, γ with $\alpha = \beta + \gamma$ and $\operatorname{ext}(\beta, \gamma) = \operatorname{ext}(\gamma, \beta) = 0$. Therefore, the set of Schur roots does not depend on the base field.

2.6. Perpendicular categories

DEFINITION 2.24. — *A representation* V *is called exceptional if* $\operatorname{Hom}_Q(V, V) \cong K$ *and* $\operatorname{Ext}_Q(V, V) = 0$.

If $V \in \operatorname{Rep}(Q, \alpha)$ is exceptional, then V is a Schur representation and α is a Schur root. Moreover,

$$\langle \alpha, \alpha \rangle = \dim \operatorname{Hom}_Q(V, V) - \dim \operatorname{Ext}_Q(V, V) = 1 - 0 = 1,$$

so α is a real Schur root. Conversely, if α is a real Schur root, then there exists a Schur representation $V \in \operatorname{Rep}(Q, \alpha)$. From

$$1 = \langle \alpha, \alpha \rangle = \dim \operatorname{Hom}_Q(V, V) - \dim \operatorname{Ext}_Q(V, V) = 1 - \dim \operatorname{Ext}_Q(V, V)$$

follows that $\operatorname{Ext}_Q(V, V) = 0$. This means that the orbit $\operatorname{GL}(Q, \alpha) \cdot V$ is open and dense in $\operatorname{Rep}(Q, \alpha)$. Therefore, a general representation of dimension α is isomorphic to V. This shows that there is a natural bijection between real Schur roots and exceptional representations.

DEFINITION 2.25. — *Suppose that* V *is a representation. The right perpendicular category* $V^{\perp}$ *is the full subcategory of* $\operatorname{Rep}_K(Q)$ *whose objects are all representations* W *such that* $\operatorname{Hom}_Q(V, W) = \operatorname{Ext}_Q(V, W) = 0$. *Similarly, we define the left perpendicular category* ${}^{\perp}V$. *as the full subcategory of* $\operatorname{Rep}_K(Q)$ *whose objects are all representations* W *for which* $\operatorname{Hom}_Q(W, V) = \operatorname{Ext}_Q(W, V) = 0$.

Note that if $V \perp W$ then $\underline{\dim}\, V \perp \underline{\dim}\, W$ (see Definition 2.6). Conversely, if α, β are dimension vectors with $\alpha \perp \beta$ then there exist $V \in \operatorname{Rep}(Q, \alpha)$ and $W \in \operatorname{Rep}(Q, \beta)$ with $V \perp W$.

The subcategory $V^\perp$ (respectively ${}^\perp V$) are closed under taking kernels, cokernels, direct sums, images and extensions.

Theorem 2.26 (Theorem 2.2 of [36]). — *Suppose that V is a sincere representation, i.e., $V(x) \neq 0$ for all $x \in Q_0$. Then the categories $V^\perp$ and ${}^\perp V$ are equivalent.*

The equivalence in the theorem is given by the Auslander-Reiten transform. If W is an object of ${}^\perp V$, then W does not contain any projective summands. Then one can define the Auslander-Reiten translate $\tau(W)$ of W. From Auslander-Reiten duality (see properties (5), (6), (7) on pages 75–76 in [33]) follows that $\tau(W)$ lies in the right perpendicular category. The Auslander-Reiten transform induces an equivalence of categories.

Theorem 2.27 (Theorem 2.3 of [36]). — *Suppose that V is an exceptional representation of a quiver Q and with $n = \#Q_0$ vertices. Then $V^\perp$ (resp. ${}^\perp V$) is equivalent to $\operatorname{Rep}_K(Q')$ where Q' is a quiver without oriented cycles such that $\#Q'_0 = n - 1$.*

Suppose we are in the situation of Theorem 2.27. The category $V^\perp \cong \operatorname{Rep}_K(Q')$ has exactly $n-1$ simple objects, say $E_1, E_2, \dots, E_{n-1}$. Now Q' is the graph with vertices $1, 2, \dots, n-1$ and $r_{i,j} := \dim \operatorname{Ext}_{Q'}(E_i, E_j)$ arrows from i to j for all i, j. We have

$$\operatorname{Hom}_{Q'}(E_i, E_i) \cong \operatorname{Hom}_Q(E_i, E_i) \cong K$$

for all i. This shows that $E_1, \dots, E_{n-1}$ are Schur representations. The category $V^\perp$ is closed under extensions, so every nontrivial extension of E_i with itself in the category $\operatorname{Rep}(Q)$ would yield a nontrivial extension of E_i with itself in the category $\operatorname{Rep}(Q')$. Since $\operatorname{Ext}_{Q'}(E_i, E_i) = 0$, we have $\operatorname{Ext}_Q(E_i, E_i) = 0$. Therefore, $E_1, \dots, E_{n-1}$ are exceptional representations for Q. Let W be an object of $V^\perp \cong \operatorname{Rep}_K(Q')$. Suppose that, as a representation of Q', its dimension vector is $\alpha' = (\alpha'_1, \dots, \alpha'_{n-1})$. Then W can be build up from extensions using α'_i copies of E_i for $i = 1, 2, \dots, n-1$. This shows that the dimension vector of W, as a representation of Q is equal to

$$\alpha := \sum_{i=1}^{n-1} \alpha'_i \varepsilon_i,$$

where $\varepsilon_i = \underline{\dim}_Q E_i$, the dimension vector of E_i seen as a representation of Q. Let us define

$$I : \mathbb{N}^{Q'_0} \cong \mathbb{N}^{n-1} \to \mathbb{N}^{Q_0}$$

by

$$I(\beta_1,\dots,\beta_{n-1})=\sum_{i=1}^{n-1}\beta_i\varepsilon_i.$$

So if W is a representation of Q' of dimension β, then W, viewed as a representation of Q, has dimension $I(\beta)$.

If W and Z are representations of Q', then

$$\operatorname{Hom}_Q(W,Z)\cong\operatorname{Hom}_{Q'}(W,Z)$$

because $\operatorname{Rep}_K(Q')$ is a full subcategory of $\operatorname{Rep}_K(Q)$. Since $V^\perp$ is closed under extensions, we also have

$$\operatorname{Ext}_Q(W,Z)\cong\operatorname{Ext}_{Q'}(W,Z).$$

From this follows that

$$\hom_{Q'}(\beta,\gamma)=\hom_Q(I(\beta),I(\gamma)) \tag{2.10}$$

and

$$\operatorname{ext}_{Q'}(\beta,\gamma)=\operatorname{ext}_Q(I(\beta),I(\gamma)). \tag{2.11}$$

Now we also get

$$\begin{aligned}\langle\beta,\gamma\rangle_{Q'}&=\hom_{Q'}(\beta,\gamma)-\operatorname{ext}_{Q'}(\beta,\gamma)=\\&=\hom(I(\beta),I(\gamma))-\operatorname{ext}(I(\beta),I(\gamma))=\langle I(\beta),I(\gamma)\rangle_Q.\end{aligned} \tag{2.12}$$

Lemma 2.28. — *Suppose that $\beta\in\mathbb{N}^{Q_0'}$. Then β is a Schur root (for Q') if and only if $I(\beta)$ is a Schur root (for Q).*

Proof. — If W is a Schur representation of dimension β for the quiver Q', then W is also a Schur representation of dimension $I(\beta)$ as a representation of Q.

Conversely, suppose that $I(\beta)$ is a Schur root. Then a general representation of dimension $I(\beta)$ is a Schur representation. Since there exists a representation of dimension $I(\beta)$ in $V^\perp$, we have that a general representation of dimension $I(\beta)$ lies in $V^\perp$ (because $W\mapsto\dim\operatorname{Hom}_Q(V,W)$ and $W\mapsto\dim\operatorname{Ext}_Q(V,W)$ are upper semi-continuous). A general representation of dimension $I(\beta)$, can be seen as a β-dimensional Schur representation for Q'. □

Theorem 2.29. — *If $\beta,\gamma\in\mathbb{N}^{Q_0'}$ and $\beta\perp\gamma$, then*

$$(I(\beta)\circ I(\gamma))_Q=(\beta\circ\gamma)_{Q'}.$$

Proof. — Choose $W \in \operatorname{Rep}(Q', \beta + \gamma)$ in general position. So W has $(\beta \circ \gamma)_{Q'}$ subrepresentations of dimension β. These subrepresentations correspond to $I(\beta)$-dimensional subrepresentations of W, seen as representations of Q. Suppose that Z is an $I(\beta)$-dimensional subrepresentation of W. Since $\operatorname{Hom}_Q(V, W) = 0$ and Z is a subrepresentation of W, we have $\operatorname{Hom}_Q(V, Z) = 0$. Since $\langle \beta, \gamma \rangle = 0$ we get $\operatorname{Ext}_Q(V, Z) = 0$ as well. This implies that Z lies in $V^{\perp}$, so Z may be viewed as a representation of Q'. As a representation of Q, W has exactly $(\beta \circ \gamma)_{Q'}$ subrepresentations. By Lemma 2.11 we obtain

$$(I(\beta) \circ I(\gamma))_Q \geqslant (\beta \circ \gamma)_{Q'}.$$

Choose $W \in \operatorname{Rep}(Q, I(\beta) + I(\gamma))$ in general position. Then W has exactly $(I(\beta) \circ I(\gamma))_Q$ subrepresentations of dimension $I(\beta)$. We have $\operatorname{Hom}_Q(V, W) = \operatorname{Ext}_Q(V, W) = 0$ (by semicontinuity) so W lies in $V^{\perp}$. We may view W as a representation of Q' of dimension $\beta + \gamma$. Again, the $I(\beta)$-dimensional subrepresentations of W, seen as a representation of Q are exactly the β-dimensional subrepresentations of W, seen as a representation of Q'. So as a representation of Q', W has exactly

$$(I(\beta) \circ I(\gamma))_Q$$

subrepresentations of dimension β. Again, Lemma 2.11 implies that

$$(I(\beta) \circ I(\gamma))_Q \leqslant (\beta \circ \gamma)_{Q'}.$$

We conclude that

$$(I(\beta) \circ I(\gamma))_Q = (\beta \circ \gamma)_{Q'}.$$

□

2.7. Exceptional Sequences

We will introduce exceptional sequences and their basic properties. For more details, see [7, 34].

Definition 2.30. — *An exceptional sequence is a sequence $E_1, \dots, E_r$ of exceptional representations such that $E_i \perp E_j$ for $i < j$.*

Define $\varepsilon_i := \underline{\dim}\, E_i$ for all i. The matrix $(\langle \varepsilon_i, \varepsilon_j \rangle)_{i,j}$ is lower triangular with 1's on the diagonal, and is therefore invertible. It follows that $\varepsilon_1, \dots, \varepsilon_r$ are linearly independent, hence $r \leqslant n := \# Q_0$.

Definition 2.31. — *An exceptional sequence $E_1, \dots, E_r$ is called maximal or complete if $r = n$.*

Lemma 2.32 (Lemma 1 of [7]). — *If $E_1, E_2, \dots, E_i, E_j, E_{j+1}, \dots, E_n$ is an exceptional sequence ($i < j$) then there exist $E_{i+1}, \dots, E_{j-1}$ such that $E_1, E_2, \dots, E_n$ is an exceptional sequence.*

In particular, every exceptional sequence $E_1, \dots, E_r$ can be extended to a maximal exceptional sequence $E_1, \dots, E_n$. To see this, consider the full subcategory of all representations V such that

$$E_i \perp V \text{ for } i = 1, 2, \dots, r.$$

Let us denote this category by

$$E_1^\perp \cap E_2^\perp \cap \cdots \cap E_r^\perp$$

or simply $E^\perp$ where $E = (E_1, \dots, E_r)$. Using Theorem 2.27 and induction on r we see that this category $E^\perp$ is equivalent to the category of representations $\mathrm{Rep}(Q')$ of a quiver Q' with $n - r$ vertices and with no oriented cycles. Let $E_{r+1}, \dots, E_n$ be the simple representations (pairwise nonisomorphic) in $E^\perp$ corresponding to the $n - r$ vertices of Q'. We can order $E_{r+1}, \dots, E_n$ in such a way that $E_j \perp E_k$ for all j, k with $r+1 \leqslant j < k \leqslant n$, because Q' has no oriented cycles. Then $E_1, \dots, E_n$ is a (maximal) exceptional sequence. Lemma 2.32 is proven in a similar fashion (see [7]).

Definition 2.33. — *For an exceptional sequence $E_1, E_2, \dots, E_r$, we define $\mathcal{C}(E_1, \dots, E_r)$ as the full subcategory of $\mathrm{Rep}(Q)$ which contains $E_1, \dots, E_r$ and is closed under extentions, kernels of epimorphisms, and cokernels of monomorphisms.*

Lemma 2.34 (Lemma 4 of [7]). — *If $E_1, \dots, E_n$ is a maximal exceptional sequence, then $\mathcal{C}(E_1, \dots, E_r)$ is equivalent to the category*

$$^\perp E_{r+1} \cap \cdots \cap {}^\perp E_n.$$

Lemma 2.35. — *Suppose that $E_1, E_2, \dots, E_r$ is exceptional, and $\mathrm{Hom}_Q(E_i, E_j) = 0$ for all $i \neq j$. Then $E_1, \dots, E_r$ are exactly all simple objects in $\mathcal{C}(E_1, E_2, \dots, E_r)$.*

Proof. — Let $\mathcal{D}(E_1, E_2, \dots, E_r)$ be the the smallest full subcategory of $\mathrm{Rep}(Q)$ which contains $E_1, \dots, E_r$ and which is closed under extensions. The objects of $\mathcal{D}(E_1, \dots, E_r)$ are all representations which allow a filtration such that each factor is isomorphic to one of the representations $E_1, \dots, E_r$. We claim that $\mathcal{D}(E_1, \dots, E_r) = \mathcal{C}(E_1, \dots, E_r)$. To show this, it suffices to show that the category $\mathcal{D}(E_1, \dots, E_r)$ is closed under taking kernels of epimorphisms and taking cokernels of monomorphisms. We will show that $\mathcal{D}(E_1, \dots, E_r)$ is closed under taking cokernels of monomorphisms.

Dualizing the arguments one can then show that $\mathcal{D}(E_1, \ldots, E_r)$ is also closed under taking kernels of epimorphisms. Suppose that $\phi : V \to W$ is a monomorphism and V, W are objects of $\mathcal{D}(E_1, \ldots, E_r)$. We have filtrations

$$V = F^0(V) \supset F^1(V) \supset \cdots \supset F^s(V) = \{0\}.$$

$$W = F^0(W) \supset F^1(W) \supset \cdots \supset F^t(W) = \{0\}$$

such that all quotients $F^i(V)/F^{i+1}(V)$, $F^i(W)/F^{i+1}(W)$ are isomorphic to one of the representations $E_1, \ldots, E_r$.

The case $s = 0$ is trivial. Suppose that $s = 1$. Then we have that $V = F^0(V)$ is isomorphic to one of the representations $E_1, \ldots, E_r$. We prove the statement by induction on t. If $t = 1$, then ϕ must be an isomorphism because V and W are simple. So $V/W = 0$ and we are done. Suppose that $t > 0$. Let ψ be the composition $V \to W \to W/F^1(W)$. Suppose that $\psi = 0$. Then V is a subrepresentation of $F^1(W)$ By induction $F^1(W)/V$ is an object of $\mathcal{D}(E_1, \ldots, E_r)$, and $W/F^1(W)$ is also an object of $\mathcal{D}(E_1, \ldots, E_r)$. From the exact sequence

$$0 \to F^1(W)/V \to W/V \to W/F^1(W) \to 0$$

follows that W/V is an object of $\mathcal{D}(E_1, \ldots, E_r)$.

Suppose that $\psi \neq 0$. Both V and $W/F^1(W)$ are isomorphic to one of the $E_1, \ldots, E_r$. Because V and $W/F^1(W)$ are simple, they are isomorphic to each other and to E_i for some i. Since E_i is exceptional and ψ is nonzero, we must have that ψ is an isomorphism. It follows that $V + F^1(W) = W$ and $V \cap F^1(W) = 0$. But then

$$F^1(W) = F^1(W)/(V \cap F^1(W)) \cong (F^1(W) + V)/V = W/V.$$

This shows that W/V is an object of $\mathcal{D}(E_1, \ldots, E_r)$. We have proven the case $s = 1$.

Suppose now that $s > 1$. We will prove the theorem by induction on s. By the above, we know that $W/F^{s-1}(V)$ and $V/F^{s-1}(V)$ are objects of $\mathcal{D}(E_1, \ldots, E_r)$. From induction and the exact sequence

$$0 \to V/F^{s-1}(V) \to W/F^{s-1}(V) \to W/V \to 0,$$

we conclude that W/V also is an object of $\mathcal{D}(E_1, \ldots, E_r)$. We have proven that $\mathcal{D}(E_1, \ldots, E_r) = \mathcal{C}(E_1, \ldots, E_r)$.

Since every object in $\mathcal{D}(E_1, \ldots, E_r) = \mathcal{C}(E_1, \ldots, E_r)$ has a subobject isomorphic to one of the representations $E_1, \ldots, E_r$, the only possible simple objects of $\mathcal{C}(E_1, \ldots, E_r)$ are $E_1, \ldots, E_r$. It is also easy to see that each E_i is simple. Indeed, if W is a proper subrepresentation of E_i, then W has a proper subrepresentation isomorphic to E_k for some $k \neq i$. But then E_k

is a proper subrepresentation of E_i which contradicts the assumption that $\mathrm{Hom}_Q(E_k, E_i) = 0$.

□

As we have noted before, there is a bijection between real Schur roots and exceptional representations. Suppose that $E_1, \ldots, E_r$ are exceptional representations and let $\varepsilon_i := \underline{\dim}\, E_i$. We have seen that the orbit of E_i in $\mathrm{Rep}(Q, \varepsilon_i)$ is dense. From this follows that

$$\dim \mathrm{Hom}(E_i, E_j) = \hom(\varepsilon_i, \varepsilon_j)$$

and

$$\dim \mathrm{Ext}(E_i, E_j) = \mathrm{ext}(\varepsilon_i, \varepsilon_j)$$

for all i, j. This allows us to give a more combinatorial definition of an exceptional sequence.

Definition 2.36. — *A sequence of dimension vectors $\varepsilon_1, \ldots, \varepsilon_r$ is called an exceptional sequence if $\varepsilon_1, \ldots, \varepsilon_r$ are real Schur roots, and $\varepsilon_i \perp \varepsilon_j$ for all $i < j$.*

So if $E_1, \ldots, E_r$ is an exceptional sequence of quiver representations, then $\varepsilon_1, \ldots, \varepsilon_r$ is an exceptional sequence of dimension vectors, where $\varepsilon_i = \underline{\dim}\, E_i$ for all i. Conversely, suppose that $\varepsilon_1, \ldots, \varepsilon_r$ is an exceptional sequence of dimension vector. Since ε_i is a real Schur root, the exists a unique dense orbit in $\mathrm{Rep}(Q, \varepsilon_i)$. Let E_i be a representation that lies in that orbit (E_i is unique op to isomorphism). Then $E_1, \ldots, E_r$ is an exceptional sequence of representations.

Theorem 2.37 (Theorem 4.1 of [37]). — *Suppose that α, β are Schur roots such that $\mathrm{ext}(\alpha, \beta) = 0$. Then $\hom(\beta, \alpha) = 0$ or $\mathrm{ext}(\beta, \alpha) = 0$. Moreover, if both α and β are imaginary, then $\hom(\beta, \alpha) = 0$.*

Theorem 2.38 (Embedding Theorem). — *Suppose that $\varepsilon_1, \ldots, \varepsilon_r$ is an exceptional sequence for the quiver Q. Suppose that $\langle \varepsilon_i, \varepsilon_j \rangle \leqslant 0$ for all $i > j$. We define a quiver Q' with vertices $Q'_0 = \{1, 2, \ldots, r\}$ and without oriented cycles. We draw $-\langle \varepsilon_i, \varepsilon_j \rangle$ arrows from i to j for all $i > j$. We define*

$$I : \mathbb{N}^{Q'_0} \cong \mathbb{N}^r \to \mathbb{N}^{Q_0}$$

by

$$I(\beta_1, \ldots, \beta_r) = \sum_{i=1}^{r} \beta_i \varepsilon_i$$

for all $\beta = (\beta_1, \ldots, \beta_r) \in \mathbb{N}^{Q'_0} \cong \mathbb{N}^r$.

(a) *For all* $\beta, \gamma \in \mathbb{N}^{Q'_0}$ *we have*

$$\hom_Q(I(\beta), I(\gamma)) = \hom_{Q'}(\beta, \gamma),$$

$$\mathrm{ext}_Q(I(\beta), I(\gamma)) = \mathrm{ext}_{Q'}(\beta, \gamma),$$

$$\langle I(\beta), I(\gamma)\rangle_Q = \langle \beta, \gamma\rangle_{Q'}.$$

(b) *The dimension vector* β *is a Schur root (for* Q'*) if and only if* $I(\beta)$ *is a Schur root (for* Q*).*

(c) *For all* $\beta, \gamma \in \mathbb{N}^{Q'_0}$ *with* $\beta \perp \gamma$ *we have*

$$(I(\beta) \circ I(\gamma))_Q = (\beta \circ \gamma)_{Q'}.$$

Proof. — Let E_i be a representation corresponding to the dense orbit in $\mathrm{Rep}(Q, \varepsilon_i)$. Then $E_1, E_2, \dots, E_r$ is an exceptional sequence. For $i > j$ we have

$$\langle \varepsilon_i, \varepsilon_j\rangle_Q = \hom(\varepsilon_i, \varepsilon_j)_Q - \mathrm{ext}(\varepsilon_i, \varepsilon_j)_Q \leqslant 0.$$

Since $\hom(\varepsilon_i, \varepsilon_j)_Q = 0$ or $\mathrm{ext}(\varepsilon_i, \varepsilon_j)_Q = 0$ by Theorem 2.37, we must have $\hom(\varepsilon_i, \varepsilon_j)_Q = 0$. It follows that $\mathrm{Hom}_Q(E_i, E_j) = 0$ for $i > j$. From Lemma 2.35 we see that $E_1, E_2, \dots, E_r$ are exactly all simple objects in $\mathcal{C}(E_1, \dots, E_r)$. We can extend $E_1, \dots, E_r$ to a maximal exceptional sequence $E_1, E_2, \dots, E_n$ using Lemma 2.32. Now, we have equality

$$\mathcal{C}(E_1, \dots, E_r) = {}^{\perp}E_{r+1} \cap \cdots \cap {}^{\perp}E_n$$

by Lemma 2.34.

Suppose that $r = n - 1$. Then we are exactly in the situation of Theorem 2.27. We note that the definition of the quiver Q' and the map I coincide with the definitions in the discussion after Theorem 2.27. In particular, we get (see (2.10), (2.11), (2.12))

$$\hom_{Q'}(\beta, \gamma) = \hom_Q(I(\beta), I(\gamma))$$

$$\mathrm{ext}_{Q'}(\beta, \gamma) = \mathcal{E}_Q(I(\beta), I(\gamma)).$$

and

$$\langle \beta, \gamma\rangle_{Q'} = \langle I(\beta), I(\gamma)\rangle_Q.$$

Lemma 2.28 implies that $\beta \in \mathbb{N}^{Q'_0}$ is a Schur root if and only if $I(\beta)$ is a Schur root. Finally, Theorem 2.29 implies that

$$(\beta \circ \gamma)_{Q'} = (I(\beta) \circ I(\gamma))_Q.$$

If $r < n$, the theorem can be proven in a similar way, using induction on $n - r$. □

Theorem 2.38 allows us to "embed" the combinatorics of a quiver Q' into the combinatorics of another quiver Q, using exceptional sequences. This is a very useful tool. Of course, we would like to be able to construct such exceptional sequences. Crawley-Boevey defined an action of the braid group on the set of all maximal exceptional sequences and he proved that this action is transitive (see [7]).

Assume that the vertices are labeled by $1, 2, \dots, n$. Since there are no oriented cycles, we can arrange the vertices in such a way that for every arrow $i \to j$ in the quiver we have $i > j$. If S_i is the simple representation corresponding to the vertex i, then $S_1, S_2, \dots, S_n$ is a maximal exceptional sequence.

In [7], Crawley-Boevey defines a group action of the Braid group $\mathcal{B}_n$ on the set of all maximal exceptional sequences and he proves:

Theorem 2.39 ([7]). — *The action of $\mathcal{B}_n$ on $\mathcal{E}_Q$ is transitive.*

2.8. Stability and GIT-quotients

King studied in [21] moduli spaces for representations of quivers (and more generally, finite dimensional algebras). Here we will study the moduli space of a quiver Q. Suppose that $\sigma \in \Gamma^\star$ is a weight. A representation $V \in \operatorname{Rep}(Q, \alpha)$ is called σ-semistable if there exists a positive integer m and a semi-invariant $f \in \operatorname{SI}(Q, \alpha)_{m\sigma}$ such that $f(V) \neq 0$. King gave the following nice characterization of σ-stable and σ-semistable representations.

Theorem 2.40 (Proposition 3.1 of [21]). —

(1) *A representation V is σ-semistable if $\sigma(\underline{\dim}\, V) = 0$ and $\sigma(\underline{\dim}\, W) \leqslant 0$ for all subrepresentations $W \subseteq V$.*

(2) *representation V is σ-stable if $\sigma(\underline{\dim}\, V) = 0$ and $\sigma(\underline{\dim}\, W) < 0$ for all nonzero, proper subrepresentations $W \subset V$.*

In King's setup the inequalities goes the other way. The reason for that is that his convention for writing weights is opposite of our.

Example 2.41. — Suppose that $\sigma = 0$. Then every representation is σ-semistable. A representation is σ-stable if and only if it is simple.

In general, σ-stable representations are indecomposable. Indeed, if $V = V_1 \oplus V_2$ for some nontrivial representations V_1 and V_2, then $0 = \sigma(\underline{\dim}\, V) = \sigma(\underline{\dim}\, V_1) + \sigma(\underline{\dim}\, V_2)$, so $\sigma(\underline{\dim}\, V_1) \leqslant 0$ or $\sigma(\underline{\dim}\, V_2) \leqslant 0$. If d is a positive integer, then a representation V is σ-stable if and only if it is $(d\sigma)$-stable. The set of σ-semistable representations in $\operatorname{Rep}(Q, \alpha)$ is denoted by

$\mathrm{Rep}(Q,\alpha)^{\mathrm{ss}}_{\sigma}$. We will assume that $\mathrm{Rep}(Q,\alpha)^{\mathrm{ss}}_{\sigma}$ is nonempty. In other words $\mathrm{SI}(Q,\alpha)_{n\sigma} \neq 0$ for some positive integer n.

Recall that σ defines a weight $\mathrm{GL}(Q,\alpha) \to K^{\star}$. Let $\mathrm{GL}(Q,\alpha)_{\sigma}$ be the kernel of this weight.

Lemma 2.42. — *Suppose that σ is not divisible by the characteristic of K. The invariant ring of $K[\mathrm{Rep}(Q,\alpha)]$ with respect to $\mathrm{GL}(Q,\alpha)_{\sigma}$ is equal to*

$$K[\mathrm{Rep}(Q,\alpha)]^{\mathrm{GL}(Q,\alpha)_{\sigma}} = \bigoplus_{n\geqslant 0} \mathrm{SI}(Q,\alpha)_{n\sigma}.$$

Proof. — The inequality $\supseteq$ is easy to see. We will prove $\subseteq$. Suppose that f is an element of $K[\mathrm{Rep}(Q,\alpha)]^{\mathrm{GL}(Q,\alpha)_{\sigma}}$. Then in particular

$$f \in \mathrm{SI}(Q,\alpha) = \bigoplus_{\tau\in\Gamma^{\star}} \mathrm{SI}(Q,\alpha)_{\tau}.$$

Write $f = \sum_{\tau} f_{\tau}$ with $f_{\tau} \in \mathrm{SI}(Q,\alpha)_{\tau}$ for all $\tau \in \Gamma^{\star}$. Suppose that $f_{\eta} \neq 0$. If η is not an integer multiple of σ, then there exists an $A \in \mathrm{GL}(Q,\alpha)_{\sigma}$ such that $\eta(A) \neq 1$. But then

$$\sum_{\tau} f_{\tau} = f = A\cdot f = \sum_{\tau} A\cdot f_{\tau} = \sum_{\tau} \tau(A) f_{\tau}$$

so $f_{\eta} = \eta(A) f_{\eta}$ and $f_{\eta} = 0$. So $f_{\eta} = 0$ unless η is an integer multiple of σ. This shows that $f \in \bigoplus_{n\in\mathbb{Z}} \mathrm{SI}(Q,\alpha)_{n\sigma}$. We have assumed that there exists a nonzero $t \in \mathrm{SI}(Q,\alpha)_{n\sigma} \neq 0$ for some positive n. This implies that $\mathrm{SI}(Q,\alpha)_{-m\sigma} = 0$ for all $m > 0$, for if $u \in \mathrm{SI}(Q,\alpha)_{-m\sigma}$ is nonzero, then $u^n t^m \in K[\mathrm{Rep}(Q,\alpha)]^{\mathrm{GL}(Q,\alpha)} = K$, but this is not possible because $u^n t^m$ is a polynomial of positive degree. □

Let $X = \mathrm{Spec}(R)$ be the affine variety corresponding to the ring

$$R = K[\mathrm{Rep}(Q,\alpha)]^{\mathrm{GL}(Q,\alpha)_{\sigma}} = \bigoplus_{n\in\mathbb{N}} \mathrm{SI}(Q,\alpha)_{n\sigma}.$$

(Here Spec denotes the spectrum of *maximal* ideals.) The inclusion $R \subseteq K[\mathrm{Rep}(Q,\alpha)]$ corresponds to a morphism

$$\psi : \mathrm{Rep}(Q,\alpha) \twoheadrightarrow X.$$

This morphism is an affine quotient with respect to a reductive group. Such a quotient map is known to be surjective. Note that $\psi^{-1}(0)$ is exactly the complement of $\mathrm{Rep}(Q,\alpha)^{\mathrm{ss}}_{\sigma}$ in $\mathrm{Rep}(Q,\alpha)$. Here $0 \in X$ is the point corresponding to the homogeneous maximal ideal $\mathfrak{m} = \bigoplus_{n>0} \mathrm{SI}(Q,\alpha)_{n\sigma}$. Define

$$Y = \mathrm{Proj}\Big(\bigoplus_{n\geqslant 0} \mathrm{SI}(Q,\sigma)_{n\sigma}\Big).$$

There is a natural surjective map $X \setminus \{0\} \to Y$. So we have a diagram

$$\begin{array}{ccc} \operatorname{Rep}(Q,\alpha)^{\mathrm{ss}}_{\sigma} & \longrightarrow & X \setminus \{0\} \\ & \searrow & \downarrow \\ & & Y \end{array}$$

The map $\pi : \operatorname{Rep}(Q,\alpha)^{\mathrm{ss}}_{\sigma} \to Y$ is a GIT-quotient of $\operatorname{Rep}(Q,\alpha)$. Note that this quotient depends on σ. Again, the quotient π is surjective.

3. Stability

3.1. Harder-Narasimhan and Jordan-Hölder filtrations

The notion of stability was compared (see [35, Section 3]) to the stability defined through a slope of two additive functions. In the quiver setting, the set of additive functions on the category can be identified with $\Gamma^\star$. Suppose that $\sigma, \tau \in \Gamma^\star$ and $\tau(\alpha) > 0$ for all nonzero dimension vectors α. The *slope* of a representation V is defined by

$$\mu(V) := \frac{\sigma(\underline{\dim}\, V)}{\tau(\underline{\dim}\, V)}.$$

Definition 3.1. — *A representation V is called $(\sigma : \tau)$-semistable if*

$$\mu(W) \leqslant \mu(V)$$

for every proper subrepresentation $W \subsetneq V$. It is called $(\sigma : \tau)$-stable if

$$\mu(W) < \mu(V)$$

for every proper subpresentation $W \subset V$.

Rudakov showed (see [35, Lemma 3.2]) that the order $\prec$ defined by

$$V \prec W \Leftrightarrow \mu(V) < \mu(W)$$

and equivalence relation defined by

$$V \asymp W \Leftrightarrow \mu(V) = \mu(W)$$

defines a stability order in the sense of [35, Section 1].

Remark 3.2. — The notions of σ-stability and $(\sigma : \tau)$-stability are closely related.

Suppose that V is a representation of Q and choose τ such that $\tau(\alpha) > 0$ for every dimension vector α. For example, define $\tau(\alpha) := \sum_{x \in Q_0} \alpha(x)$. If V is σ-(semi-)stable, then V is also $(\sigma : \tau)$-(semi-)stable.

Also, if V is $(\sigma : \tau)$-(semi-)stable, and $\sigma(\underline{\dim}\, V)/\tau(\underline{\dim}\, V) = a/b$ where $a, b \in \mathbb{Z}$ and b is positive, then V is $(b\sigma - a\tau)$-(semi-)stable.

Theorems 2 and 3 from [35] give immediately the following results.

Proposition 3.3. — *With the assumptions of Definition 3.1 we have:*

(1) *(Harder-Narasimhan filtration) Every object V has a filtration*

$$V = F_H^0(V) \supset F_H^1(V) \supset \ldots \supset F_H^m(V) \supset F_H^{m+1}(V) = 0$$

such that

(a) *each factor $G_H^i(V) = F_H^i(V)/F_H^{i+1}(V)$ is $(\sigma:\tau)$-semistable;*

(b)

$$G_H^0(V) \prec G_H^1(V) \prec \ldots \prec G_H^m(V).$$

The filtration with properties (a), (b) is unique.

(2) *(Jordan-Hölder filtration) Every $(\sigma:\tau)$-semistable object V has a filtration*

$$V = F_J^0(V) \supset F_J^1(V) \supset \ldots \supset F_J^m(V) \supset F_J^{m+1}(V) = 0$$

such that

(a) *each factor $G_J^i(V) = F_J^i(V)/F_J^{i+1}(V)$ is $(\sigma:\tau)$-stable;*

(b)

$$G_J^0(V) \asymp G_J^1(V) \asymp \ldots \asymp G_J^m(V).$$

The set of factors $\{G_J^i(V)\}$ is uniquely determined by the properties (a), (b).

In part (2), the Jordan-Hölder filtration itself may not be unique.

Remark 3.4. — It follows from (2) that a σ-semistable representation has a Jordan-Hölder filtration whose factors are σ-stable.

Lemma 3.5. — *Let $\alpha_0 = \alpha, \alpha_1, \alpha_2, \ldots, \alpha_m, \alpha_{m+1} = 0$ be dimension vectors. The set of all representations $V \in \mathrm{Rep}(Q,\alpha)$ which allow a filtration*

$$V = F^0(V) \supset F^1(V) \supset \cdots \supset F^{m+1}(V) = 0,$$

with $\underline{\dim}\, F^i(V) = \alpha_i$ is Zariski closed.

Proof. — The proof goes exactly as in [37, Section 3]. □

The following lemma follows immediately from Theorem 2.40.

Lemma 3.6. — *Suppose that V is a σ-semistable representation, and W is a subrepresentation with $\sigma(\underline{\dim}\, W) = 0$, then W is σ-semistable as well.*

Lemma 3.7. — *Let $\alpha_0 = \alpha, \alpha_1, \alpha_2, \ldots, \alpha_m, \alpha_{m+1} = 0$ be dimension vectors. Define $U = U_{JH}(\alpha_0, \alpha_1, \ldots, \alpha_{m+1})$ as the set of all representations*

$V \in \mathrm{Rep}(Q,\alpha)$ *which are* σ*-semistable and which have a Jordan-Hölder filtration*

$$V = F^0(V) \supset F^1(V) \supset \cdots \supset F^{m+1}(V) = 0,$$

with $\underline{\dim}\, F^i(V) = \alpha_i$ *for all* i*. Then* U *is locally closed.*

Proof. —
Suppose that U is not locally closed. Then $\overline{U} \setminus U$ is not Zariski closed. Now $\overline{U} \setminus U$ is contained in the union of all $U_{JH}(\beta_0, \ldots, \beta_{r+1})$. Since this is only a finite union, there exist $\beta_0, \beta_1, \ldots, \beta_{r+1}$ such that the Zariski closure of $U' \cap (\overline{U} \setminus U)$ is not contained in $\overline{U} \setminus U$, where $U' = U_{JH}(\beta_0, \ldots, \beta_{r+1})$. So we have $U' \cap \overline{U} \neq \emptyset$ and $U \cap \overline{U'} \neq \emptyset$. Let $V' \in U' \cap \overline{U}$. Now V' has also a filtration $\{F^i(V')\}$ with dimensions $\alpha_0, \alpha_1, \ldots, \alpha_{m+1}$ by Lemma 3.5. The quotients in this filtration are σ-semistable. A Jordan-Hölder filtration of V' can be obtained by refining the filtration $\{F^i(V')\}$. It follows that each α_i is a sum of 1 or more of the β_j's.

The same reasoning for a representation $V \in U \cap \overline{U'}$ shows that each β_j is a sum of 1 or more of the α_i's.

We conclude that

$$\{\alpha_1, \ldots, \alpha_{m+1}\} = \{\beta_1, \ldots, \beta_{r+1}\}.$$

Therefore $U = U'$ which gives a contradiction.

□

Proposition 3.8. — *Let* Q *be a quiver,* σ *be a weight and* α *be a dimension vector. If a general representation of dimension* α *is* σ*-semistable, then there exists a nonempty open set* $U \subseteq \mathrm{Rep}_K(Q,\alpha)$ *such that for* $V \in U$ *the dimensions of the factors of a Jordan-Hölder-filtration of* V *are constant.*

Proof. —
There exist dimension vectors $\alpha_0 = \alpha, \alpha_1, \ldots, \alpha_m, \alpha_{m+1} = 0$ such that

$$U = U_{JH}(\alpha_0, \ldots, \alpha_{m+1})$$

lies dense in $\mathrm{Rep}(Q,\beta)$. Since U is locally closed by Lemma 3.7, we conclude that U is open and dense in $\mathrm{Rep}(Q,\beta)$.

□

Lemma 3.9. — *Suppose* $\sigma \in \Gamma^\star$ *is a weight and* V, W *are* σ*-stable representations of the quiver. If* $\phi : V \to W$ *is a morphism then either* $\phi = 0$ *or* ϕ *is an isomorphism.*

Proof. — This follows immediately from [35, Theorem 1(d)]. □

3.2. The σ-stable decomposition

Definition 3.10. — *Suppose that α is a dimension vector and σ is a weight such that $\sigma(\alpha) = 0$. A dimension vector α is called σ-(semi-)stable if a general representation of dimension α is σ-(semi-)stable. The expression*

$$\alpha = \alpha_1 \dotplus \alpha_2 \dotplus \ldots \dotplus \alpha_s$$

is called the σ-stable decomposition *of a σ-semistable dimension vector α if a general representation V of dimension α has a Jordan-Hölder filtration with factors of dimension $\alpha_1, \alpha_2, \ldots, \alpha_s$ (in some order).*

Proposition 3.11. — *A dimension vector α is a Schur root if and only if α is σ-stable for some weight σ.*

Proof. — If α is σ-stable, then a general α-dimensional representation is σ-stable, hence indecomposable. Conversely, if α is a Schur root, then α is σ-stable where

$$\sigma = \langle \alpha, \cdot \rangle - \langle \cdot, \alpha \rangle.$$

This follows from [37, Theorem 6.1]. □

Lemma 3.12. — *Let σ be a weight. Suppose that $\alpha \in \mathbb{N}^{Q_0}$ has σ-stable decomposition*

$$\alpha_1 \dotplus \alpha_2 \dotplus \cdots \dotplus \alpha_s.$$

For any indices $i_1 < \cdots < i_r$ the σ-stable decomposition of $\beta = \alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_r}$ is

$$\alpha_{i_1} \dotplus \alpha_{i_2} \dotplus \cdots \dotplus \alpha_{i_r}.$$

Proof. — After rearranging $\alpha_1, \ldots, \alpha_s$ we may assume that

$$U_\alpha := U_{JH}(\alpha_1 + \cdots + \alpha_s, \alpha_1 + \cdots + \alpha_{s-1}, \ldots, \alpha_1, 0)$$

lies dense in $\mathrm{Rep}(Q, \alpha)$. By Lemma 3.7, U_α is locally closed. Therefore, U_α is dense and *open.*

Let $j_1 < j_2 < \cdots < j_{s-r}$ such that

$$\{1, 2, \ldots, s\} = \{i_1, \ldots, i_r\} \cup \{j_1, \ldots, j_{s-r}\}.$$

Then we have

$$\alpha - \beta = \alpha_{j_1} + \cdots + \alpha_{j_{s-r}}$$

Suppose that the σ-stable decomposition of β is

$$\beta_1 \dotplus \beta_2 \dotplus \cdots \dotplus \beta_t.$$

Choose $V_i \in \operatorname{Rep}(Q, \alpha_i)$ σ-stable for $i = 1, 2, \dots, s$. By Lemma 3.7, there exists an open dense set $U_\beta \subseteq \operatorname{Rep}(Q, \beta)$ and dimension vectors $\gamma_0 = \alpha, \gamma_1, \dots, \gamma_{l+1} = 0$ such that

$$V' \oplus V_{j_1} \oplus \cdots \oplus V_{j_{s-r}} \in U'_\alpha := U_{JH}(\gamma_0, \dots, \gamma_{l+1})$$

for all $V' \in U_\beta$. Note that

$$\gamma_0 - \gamma_1, \gamma_1 - \gamma_2, \dots, \gamma_l - \gamma_{l+1}$$

is a rearrangement of

$$\beta_1, \dots, \beta_t, \alpha_{j_1}, \dots, \alpha_{j_{s-r}}.$$

We have

$$V_1 \oplus V_2 \oplus \cdots \oplus V_s \in \overline{U_\beta \oplus V_{j_1} \oplus \cdots \oplus V_{j_{s-r}}} \subseteq \overline{U'_\alpha}.$$

This shows that $\overline{U'_\alpha} \cap U_\alpha \neq \emptyset$. Since U_α is open, we have $U'_\alpha \cap U_\alpha \neq \emptyset$. Therefore $U'_\alpha = U_\alpha$. Hence

$$\alpha_1, \alpha_2, \dots, \alpha_r$$

is a rearrangement of

$$\alpha_{j_1}, \dots, \alpha_{j_{r-s}}, \beta_1, \dots, \beta_t$$

and

$$\beta_1, \dots, \beta_t$$

is a rearrangmement of

$$\alpha_{i_1}, \dots, \alpha_{i_r}.$$

□

Proposition 3.13. — *Suppose that α is a σ-stable dimension vector and $p \in \mathbb{N}$. Then the σ-stable decomposition of $p\alpha$ is*

$$\{p\alpha\} := \begin{cases} p \cdot \alpha := \underbrace{\alpha \dotplus \cdots \dotplus \alpha}_{p} & \text{if } \alpha \text{ is real or isotropic} \\ p\alpha & \text{if } \alpha \text{ is imaginary and nonisotropic.} \end{cases}$$

Proof. — Suppose that the σ-stable decomposition of $p\alpha$ is

$$\gamma_1 \dotplus \gamma_2 \dotplus \cdots \dotplus \gamma_r$$

Let $W \in \operatorname{Rep}(Q, \alpha)$ be a σ-stable representation. Then $V = W^p = W \oplus \cdots \oplus W$ is σ-semistable and it has a filtration with factors of dimensions $\gamma_1, \dots, \gamma_r$ (see Lemma 3.5). This filtration can be refined to a Jordan-Hölder filtration of V whose factors are all isomorphic to the α-dimensional representation W. This shows that $\gamma_1, \dots, \gamma_r$ are multiples of α. If α is real or isotrophic, then the canonical decomposition of $p\alpha$ is $\alpha^{\oplus p}$. In particular, a general representation Z of dimension $p\alpha$ has a filtration $\{F^i(Z)\}$ with factors of dimension α. This filtration can be refined to a Jördan Hölder filtration. But since the factors of the Jördan Hölder filtration have dimensions that are *multiples* of α, the filtration $\{F^i(Z)\}$ is already a Jordan-Hölder filtration. Therefore $r = p$ and $\gamma_1, \ldots, \gamma_r = \alpha$.

Suppose α is imaginary and not isotropic. Then $\gamma_i \hookrightarrow p\alpha$ for some i. Assume that $\gamma_i = q\alpha$. Then $q\alpha \hookrightarrow p\alpha$, so $\operatorname{ext}(q\alpha, (p-q)\alpha) = 0$. It follows that

$$q(p-q)\langle\alpha, \alpha\rangle = \langle q\alpha, (p-q)\alpha\rangle \geqslant 0.$$

Since $\langle\alpha, \alpha\rangle < 0$, we must have $p = q$. It follows that $i = 1$ and $\gamma_1 = p\alpha$. □

Proposition 3.14. — *If the σ-stable decomposition of α is*

$$\alpha = \alpha_1 \dotplus \alpha_2 \dotplus \cdots \dotplus \alpha_s$$

then the σ-stable decomposition of $p\alpha$ is

$$p\alpha = \{p\alpha_1\} \dotplus \{p\alpha_2\} \dotplus \cdots \dotplus \{p\alpha_s\} \tag{3.1}$$

for every positive integer p.

Proof. — If $\beta \hookrightarrow \alpha$ then $\operatorname{ext}(\beta, \alpha - \beta) = 0$. It follows that $\operatorname{ext}(p\alpha, p(\alpha - \beta)) = 0$ and $m\beta \hookrightarrow m\alpha$. Suppose that $V \in \operatorname{Rep}(Q, p\alpha)$. Then V has a filtration with factors $V_1, V_2, \ldots, V_s$ of dimensions $p\alpha_1, \ldots, p\alpha_s$. From Proposition 3.13 and Lemma 3.5 follows that if α_i is isotropic or real, then each V_i has a filtration such that all quotients have dimension α_i. This shows that V has a filtration $\{F^i(V)\}$ such that the dimensions of the factors are exactly all dimensions appearing on the right-hand side of (3.1). Assume now that V is a general representation of dimension $p\alpha$. Then all factors of the filtration $\{F^i(V)\}$ are σ-semistable. There exists a Jordan-Hölder filtration $\{(F')^i(V)\}$ which is a refinement of the filtration $\{F^i(V)\}$. If W is any representation of dimension $p\alpha$, then W has a filtration $\{(F')^i(W)\}$ such that $\underline{\dim}\,(F')^i(W) = \underline{\dim}\,(F')^i(V)$ by Lemma 3.5. Take now $W = W_1 \oplus W_2 \oplus \cdots \oplus W_s$ with $W_1, \ldots, W_s$ in general position of dimension $p\alpha_1, \ldots, p\alpha_s$. The filtration $\{(F')^i(W)\}$ can be refined to a Jordan-Hölder filtration of W. But we know that the dimensions of the factors of the Jordan-Hölder filtration of W are exactly the dimensions appearing on the right-hand side of (3.1) by Proposition 3.13. It follows that the filtrations $\{F^i(W)\}$ and $\{(F')^i(W)\}$ are the same, so the filtrations $\{F^i(V)\}$ and $\{(F')^i(V)\}$ are the same. This shows that the factors of the Jordan-Hölder filtration of V are exactly given by the dimensions on the right-hand side of (3.1). □

Proposition 3.15. — *Let Q be a quiver, σ be a weight and α be a σ-semistable dimension vector. Let*

$$\alpha = c_1 \cdot \alpha_1 \dotplus c_2 \cdot \alpha_2 \dotplus \cdots \dotplus c_s \cdot \alpha_s$$

be the σ-stable decomposition where the α_i are distinct. Then we have

(a) *All α_i are Schur roots;*
(b) *if $c_i > 1$ then α_i must be a real or an isotropic root;*
(c) $\hom(\alpha_i, \alpha_j) = 0$ *if $i \neq j$;*
(d) *after rearranging one may assume that $\alpha_i \perp\!\!\!\perp \alpha_j$ for all $i < j$.*

Proof. —
(a) If general representations of dimension α_i are decomposable, then a general representation of dimension α_i is not σ-stable. Since a general representation of dimension α_i has a non-trivial Jordan-Hölder filtration, *every* representation of dimension α_i has a nontrivial Jordan-Hölder filtration by Lemma 3.5. It follows that there are no σ-stable representations in dimension α_i.

(b) If $c_i \geqslant 2$, then by Lemma 3.12, the σ-stable decomposition of $2\alpha_i$ is $\alpha_i \dotplus \alpha_i$. In particular we have $\alpha_i \hookrightarrow 2\alpha_i$, so $\operatorname{ext}(\alpha_i, \alpha_i) = 0$ and $\langle \alpha_i, \alpha_i \rangle \geqslant 0$.

(c) Let V_i and V_j be σ-stable representations of dimensions α_i and α_j respectively. From Lemma 3.9 follows that any nonzero homomorphism between V_i and V_j must be an isomorphism. Since $\alpha_i \neq \alpha_j$, we have $\operatorname{Hom}_Q(V_i, V_j) = 0$ and therefore $\hom(\alpha_i, \alpha_j) = 0$.

(d) Suppose that there exists an r-cycle:

$$\operatorname{ext}(\alpha_{i_1}, \alpha_{i_2}) \neq 0,\ \operatorname{ext}(\alpha_{i_2}, \alpha_{i_3}) \neq 0, \dots,\ \operatorname{ext}(\alpha_{i_{r-1}}, \alpha_{i_r}) \neq 0,\ \operatorname{ext}(\alpha_{i_r}, \alpha_{i_1}) \neq 0.$$

We assume that $r \geqslant 2$ is minimal such that an r-cycle exists. After rearranging $\alpha_1, \alpha_2, \dots, \alpha_r$ we may assume that

$$\operatorname{ext}(\alpha_1, \alpha_2) \neq 0,\ \operatorname{ext}(\alpha_2, \alpha_3) \neq 0, \dots,\ \operatorname{ext}(\alpha_{r-1}, \alpha_r) \neq 0,\ \operatorname{ext}(\alpha_r, \alpha_1) \neq 0.$$

Also, by the minimality of r, $\operatorname{ext}(\alpha_i, \alpha_j) = 0$ if $1 \leqslant i, j \leqslant r$, unless $j = i$ or $j \equiv i + 1 \pmod{r}$. The σ-stable decomposition of $\beta = \alpha_1 + \alpha_2 + \cdots + \alpha_r$ is

$$\alpha_1 \dotplus \alpha_2 \dotplus \cdots \dotplus \alpha_r$$

by Lemma 3.12. We have $\alpha_i \hookrightarrow \beta$ for some i, and after cyclic relabeling we may assume that $\alpha_1 \hookrightarrow \beta$, so $\operatorname{ext}(\alpha_1, \beta - \alpha_1) = 0$. If $3 \leqslant i \leqslant r$, then we have $\operatorname{ext}(\alpha_i, \alpha_2) = 0$ by minimality of r. It follows that $\operatorname{ext}(\alpha_3 + \alpha_4 + \cdots + \alpha_r, \alpha_2) = 0$ or equivalently, $\beta - \alpha_1 \twoheadrightarrow \alpha_2$. Consider an exact sequence

$$0 \to V' \to V \to V'' \to 0 \tag{3.2}$$

with V', V, V'' of dimension $\beta - \alpha_1 - \alpha_2, \beta - \alpha_1, \alpha_2$ respectively, and V is in general position. Let W be a general representation of dimension α_1 and apply the functor $\operatorname{Hom}_Q(W, \cdot)$ to (3.2) to obtain a long exact sequence

$$\cdots \to \operatorname{Ext}_Q(W, V') \to \operatorname{Ext}_Q(W, V) \to \operatorname{Ext}_Q(W, V'') \to 0.$$

Since $\operatorname{Ext}_Q(W, V) = 0$, we have $\operatorname{Ext}_Q(W, V'') = 0$. It follows that $\operatorname{ext}(\alpha_1, \alpha_2) = 0$. Contradiction.

Therefore, there are no r-cycles. So it is possible to rearrange $\alpha_1, \ldots, \alpha_s$ such that $\operatorname{ext}(\alpha_i, \alpha_j) = 0$ for all $i < j$. Together with $\hom(\alpha_i, \alpha_j) = 0$ by (c) we get $\alpha_i \perp \alpha_j$ for $i < j$.

The number $p = \alpha_i \circ \alpha_j$ is finite and nonzero. We would like to show that $p = 1$. Suppose that $p \geqslant 2$. Let V be a general representation of dimension $\alpha_i + \alpha_j$. Then V has $p \geqslant 2$ subrepresentations of dimension α_i (see Theorem 2.10). Let V_1 and V_2 be two distinct subrepresentations of V of dimension α_i. The σ-stable decomposition of $\alpha_i + \alpha_j$ is $\alpha_i \dotplus \alpha_j$ by Lemma 3.12. This implies that $V_1, V_2, V/V_1, V/V_2$ are σ-stable. Let $\varphi : V_1 \to V/V_2$ be the composition $V_1 \to V \to V/V_2$. Since V_1 and V_2 are distinct of the same dimension, φ is not equal to 0. Since V_1, V/V_2 are σ-stable φ must be an isomorphism. This implies that $\alpha_i = \alpha_j$. Contradiction. So we conclude that $p = \alpha_i \circ \alpha_j = 1$ and $\alpha_i \perp\!\!\!\perp \alpha_j$. ☐

Theorem 3.16. — *Suppose that σ is an indivisible weight. If*

$$\alpha = c_1 \cdot \alpha_1 \dotplus c_2 \cdot \alpha_2 \dotplus \cdots \dotplus c_r \cdot \alpha_r$$

is the σ-stable decomposition of α, then there exists an isomorphism

$$\operatorname{SI}(Q, \alpha)_{m\sigma} \cong S^{c_1}(\operatorname{SI}(Q, \alpha_1)_{m\sigma}) \otimes S^{c_2}(\operatorname{SI}(Q, \alpha_2)_{m\sigma}) \otimes \cdots \otimes S^{c_r}(\operatorname{SI}(Q, \alpha_r)_{m\sigma}).$$

Proof. — First we assume that the base field K has characteristic 0. Let

$$X := \operatorname{Rep}(Q, \alpha_1)^{c_1} \oplus \operatorname{Rep}(Q, \alpha_2)^{c_2} \oplus \cdots \oplus \operatorname{Rep}(Q, \alpha_r)^{c_r}.$$

We have a natural embedding

$$\varphi : X \hookrightarrow \operatorname{Rep}(Q, \alpha).$$

Let $\operatorname{GL}(Q, \alpha)_\sigma$ be the kernel of σ, where σ is interpreted as a weight $\operatorname{GL}(Q, \alpha) \to K^\star$. Let G be the stabilizer of X within $\operatorname{GL}(Q, \alpha)_\sigma$. This group G is isomorphic to the intersection of

$$(\Sigma_{c_1} \ltimes \operatorname{GL}(Q, \alpha_1)^{c_1}) \times (\Sigma_{c_2} \ltimes \operatorname{GL}(Q, \alpha_2)^{c_2}) \times \cdots \times (\Sigma_{c_r} \ltimes \operatorname{GL}(Q, \alpha_r)^{c_r})$$

and $\operatorname{GL}(Q, \alpha)_\sigma$. Here Σ_c is the symmetric group on c elements. Let $\pi_X : X \to X/\!/G$ and $\pi : \operatorname{Rep}(Q, \alpha) \to \operatorname{Rep}(Q, \alpha)/\!/ \operatorname{GL}(Q, \alpha)_\sigma$ be the categorical quotients. The embedding φ induces a morphism between categorical quotients

$$\psi : X/\!/G \to \operatorname{Rep}(Q,\alpha)/\!/\operatorname{GL}(Q,\alpha)_\sigma.$$

We will show that ψ is an isomorphism.

First we will show that ψ is dominant. Let $V \in \operatorname{Rep}(Q,\alpha)$ be a general representation and suppose V is σ-semistable, i.e., $\pi(V) \neq 0$. Let

$$0 = F_J^0(V) \subset F_J^1(V) \subset \cdots \subset F_J^s(V) = V$$

be a Jordan-Hölder filtration with σ-stable quotients

$$G^i(V) = F^i(V)/F^{i-1}(V).$$

Now $W := \bigoplus_i G^i(V) \in X$ lies in the $\operatorname{GL}(Q,\alpha)_\sigma$ closure of V. In particular $\psi(\pi_X(W)) = \pi(V)$. This proves that ψ is dominant.

Note that $W \in X$ is G-semistable if and only if all the summands in $\operatorname{Rep}(Q,\alpha_i)$ are σ-semistable. In particular, if $W \in X$ is G-semistable, then W σ-semistable, so W is $\operatorname{GL}(Q,\alpha)_\sigma$-stable. This shows that $\psi^{-1}(0) = \{0\}$, so ψ is a finite map.

For a general representation $V \in \operatorname{Rep}(Q,\alpha)$, the quotients of a Jordan-Hölder filtration corresponding to σ are unique up to permutation. This shows that a generic fiber of ψ consists of only one point. So ψ is birational.

Because ψ is birational and finite, and $\operatorname{Rep}(Q,\alpha)/\!/\operatorname{GL}(Q,\alpha)_\sigma$ is normal, ψ must be an isomorphism. Now the graded coordinate ring of $\operatorname{Rep}(Q,\alpha)/\!/\operatorname{GL}(Q,\alpha)_\sigma$ is $\bigoplus_{m\geqslant 0}\operatorname{SI}(Q,\alpha)_{m\sigma}$ and $X/\!/G$ has graded coordinate ring

$$\bigoplus_{m\geqslant 0} S^{c_1}(\operatorname{SI}(Q,\alpha_1)_{m\sigma}) \otimes S^{c_2}(\operatorname{SI}(Q,\alpha_2)_{m\sigma}) \otimes \cdots \otimes S^{c_r}(\operatorname{SI}(Q,\alpha_r)_{m\sigma}).$$

In particular we get

$$\operatorname{SI}(Q,\alpha)_{m\sigma} \cong S^{c_1}(\operatorname{SI}(Q,\alpha_1)_{m\sigma}) \otimes S^{c_2}(\operatorname{SI}(Q,\alpha_2)_{m\sigma}) \otimes \cdots \otimes S^{c_r}(\operatorname{SI}(Q,\alpha_r)_{m\sigma})$$

for all $m \geqslant 0$.

Suppose that K has arbitrary characteristic. We note that $\dim \operatorname{SI}(Q,\beta)_\tau$ is independent of the base field K (see [10]) for any dimension vector β and any weight τ. The vector spaces $\operatorname{SI}(Q,\alpha)_{m\sigma}$ and

$$S^{c_1}(\operatorname{SI}(Q,\alpha_1)_{m\sigma}) \otimes S^{c_2}(\operatorname{SI}(Q,\alpha_2)_{m\sigma}) \otimes \cdots \otimes S^{c_r}(\operatorname{SI}(Q,\alpha_r)_{m\sigma})$$

have the same dimension if K has characteristic 0. But then they have the same dimension even in the case where K has positive characteristic. We have an isomorphism

$$\operatorname{SI}(Q,\alpha)_{m\sigma} \cong S^{c_1}(\operatorname{SI}(Q,\alpha_1)_{m\sigma}) \otimes S^{c_2}(\operatorname{SI}(Q,\alpha_2)_{m\sigma}) \otimes \cdots \otimes S^{c_r}(\operatorname{SI}(Q,\alpha_r)_{m\sigma})$$

although this isomorphism may not be canonical. If we replace the symmetric powers S^c by the divided powers D^c, then the argument in characteristic 0 still shows that we have a canonical injective map

$$\mathrm{SI}(Q,\alpha)_{m\sigma} \hookrightarrow D^{c_1}(\mathrm{SI}(Q,\alpha_1)_{m\sigma}) \otimes D^{c_2}(\mathrm{SI}(Q,\alpha_2)_{m\sigma}) \otimes \cdots \otimes D^{c_r}(\mathrm{SI}(Q,\alpha_r)_{m\sigma}).$$

Comparing dimensions shows that this map is an isomorphism. □

4. Schur sequences

Proposition 3.15 motivates us to define Schur sequences. The notion of Schur sequences is closely related to the notion of exceptional sequences (see Section 2.7). A Schur sequence is similar to an exceptional sequence, but also imaginary Schur roots are allowed.

DEFINITION 4.1. — *A sequence $\underline{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_s)$ is called a Schur sequence if*

(1) *γ_i is a Schur root for every i;*
(2) *$\gamma_i \perp\!\!\!\perp \gamma_j$ for all $i < j$.*

LEMMA 4.2. — *If* $\mathrm{Rep}(Q,\alpha)$ *has a dense* $\mathrm{GL}(Q,\alpha)$*-orbit or* $\mathrm{Rep}(Q,\beta)$ *has a dense* $\mathrm{GL}(Q,\beta)$*-orbit, then $\alpha \circ \beta \leqslant 1$.*

Proof. — If $\mathrm{Rep}(Q,\beta)$ has a dense $\mathrm{GL}(Q,\beta)$-orbit, then there are no rational $\mathrm{GL}(Q,\beta)$-invariants in $K(\mathrm{Rep}(Q,\beta))$, the quotient field of $K[\mathrm{Rep}(Q,\beta)]$. In particular, any quotient of two semi-invariants of the same weight must be constant. This shows that

$$\alpha \circ \beta = \dim \mathrm{SI}(Q,\beta)_{\langle \alpha, \cdot \rangle} \leqslant 1.$$

If $\mathrm{Rep}(Q,\alpha)$ has a dense $\mathrm{GL}(Q,\alpha)$-orbit then the proof is similar. □

Remark 4.3. — An exceptional sequence $\underline{\varepsilon} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_s)$ is a Schur sequence. The space $\mathrm{Rep}(Q,\varepsilon_j)$ has a dense $\mathrm{GL}(\varepsilon_j)$-orbit, so by Lemma 4.2

$$\varepsilon_i \circ \varepsilon_j \leqslant 1.$$

This shows that $\varepsilon_i \perp\!\!\!\perp \varepsilon_j$ for all $i < j$.

LEMMA 4.4. — *Suppose that $\alpha \perp \gamma$ and $\beta \perp \gamma$.*

(a) *If $(\alpha+\beta) \circ \gamma = 1$, then $\alpha \circ \gamma = 1$ and $\beta \circ \gamma = 1$;*
(b) *if $\mathrm{ext}(\alpha,\beta) = 0$, $\alpha \circ \gamma = 1$ and $\beta \circ \gamma = 1$ then $(\alpha+\beta) \circ \gamma = 1$.*

Proof. — (a) Since $\alpha \perp \gamma$ and $\beta \perp \gamma$ we have $\alpha \circ \gamma \geqslant 1$ and $\beta \circ \gamma \geqslant 1$. Choose $f \in \mathrm{SI}(Q,\gamma)_{\langle \alpha,\cdot\rangle}$. Then we have

$$f\,\mathrm{SI}(Q,\gamma)_{\langle \beta,\cdot\rangle} \subseteq \mathrm{SI}(Q,\gamma)_{\langle \alpha+\beta,\cdot\rangle}.$$

This shows that

$$\begin{aligned} 1 \leqslant \beta \circ \gamma = \dim \mathrm{SI}(Q,\gamma)_{\langle \beta,\cdot\rangle} &= \dim f\,\mathrm{SI}(Q,\gamma)_{\langle \beta,\cdot\rangle} \\ \leqslant \dim SI(Q,\gamma)_{\langle \alpha+\beta,\cdot\rangle} &= (\alpha+\beta)\circ\gamma \leqslant 1. \end{aligned}$$

Hence we get $\beta \circ \gamma = 1$. Similarly we obtain $\alpha \circ \gamma = 1$.

(b) Any $(\alpha+\beta)$-dimensional representation V has an α-dimensional subrepresentation V'. If $V'' = V/V'$ then $c^V = c^{V'} \cdot c^{V''}$ up to a scalar as functions on $\mathrm{Rep}(Q,\gamma)$ by Lemma 2.2. Since $\mathrm{SI}(Q,\gamma)_{\langle \alpha+\beta,\cdot\rangle}$ is spanned by semi-invariants of the form c^V (see Theorem 2.3), we have

$$\mathrm{SI}(Q,\gamma)_{\langle \alpha+\beta,\cdot\rangle} = \mathrm{SI}(Q,\gamma)_{\langle \alpha,\cdot\rangle}\,\mathrm{SI}(Q,\gamma)_{\langle \beta,\cdot\rangle}.$$

It follows that

$$1 \leqslant (\alpha+\beta)\circ\gamma \leqslant (\alpha\circ\gamma)(\beta\circ\gamma) = 1\cdot 1 = 1,$$

so $(\alpha+\beta)\circ\gamma = 1$. □

Corollary 4.5. — *If $\gamma_1,\gamma_2,\ldots,\gamma_s$ is a Schur sequence, and $p\gamma_i + q\gamma_{i+1}$ is a Schur root, then*

$$\gamma_1,\gamma_2,\ldots,\gamma_{i-1}, p\gamma_i + q\gamma_{i+1}, \gamma_{i+2},\ldots,\gamma_s$$

is a Schur sequence.

Proof. — This follows from Lemma 4.4(b) and Theorem 2.22. □

Remark 4.6. — Suppose that $\alpha = \alpha_1^{\oplus c_1} \oplus \alpha_2^{\oplus c_2} \oplus \cdots \oplus \alpha_s^{\oplus c_s}$ is the canonical decomposition with all α_i distinct. By Theorem 2.13 $\mathrm{ext}(\alpha_i,\alpha_j) = 0$ for all $i \neq j$. After reordering we may assume that $\hom(\alpha_i,\alpha_j) = 0$ for all $i < j$ by Lemma 2.14. So $\alpha_i \perp \alpha_j$ for all $i < j$. We claim that in fact $\alpha_1,\alpha_2,\ldots,\alpha_s$ is a Schur sequence. This follows from the algorithm in [12] for finding the canonical decomposition and Corollary 4.5.

Definition 4.7. — *A Schur sequence $\underline{\gamma} = (\gamma_1,\gamma_2,\ldots,\gamma_s)$ is called a quiver Schur sequence if $\langle \gamma_j,\gamma_i\rangle \leqslant 0$ for all $i < j$.*

Remark 4.8. — Suppose that $\alpha = c_1 \cdot \alpha_1 \dotplus c_2 \cdot \alpha_2 \dotplus \cdots \dotplus c_r \cdot \alpha_r$ is a σ-stable decomposition. By Proposition 3.15 (d) we may assume that $\mathrm{ext}(\alpha_i,\alpha_j) = 0$ for all $i < j$. By Proposition 3.15 (a),(c), $\alpha_1,\alpha_2,\ldots,\alpha_r$ is a quiver Schur sequence.

Remark 4.9. — Suppose that $E_1, \dots, E_n$ is a maximal exceptional sequence and $\varepsilon_1, \dots, \varepsilon_n$ is a quiver Schur sequence. Then $\mathcal{C}(E_1, \dots, E_n)$ is equal to $\mathrm{Rep}(Q)$ and $E_1, \dots, E_n$ are the simple objects in $\mathrm{Rep}(Q)$ by Lemma 2.35.

Definition 4.10. — *Let $\underline{\gamma} = (\gamma_1, \dots, \gamma_r)$, $\underline{\beta} = (\beta_1, \dots, \beta_s)$ be two sequences of dimension vectors. We say that $\underline{\beta}$ is a refinement of $\underline{\gamma}$ if there exists a sequence $0 = b_0 < b_1 < \dots < b_{r-1} < s = b_r$ such that for each $j = 1, \dots, r$ the dimension vector γ_j is a positive linear combination of $\beta_{b_{j-1}+1}, \dots, \beta_{b_j}$.*

Theorem 4.11 (Refinement Theorem). — *Let $\underline{\gamma} = (\gamma_1, \dots, \gamma_r)$ be a Schur sequence. Then there exists an exceptional sequence $\underline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_s)$ such that $\underline{\varepsilon}$ is a refinement of $\underline{\gamma}$.*

Proof. — For a dimension vector α we define $\tau(\alpha) = \sum_{x \in Q_0} \alpha(x)$. We will prove the theorem by induction on n, the number of vertices of the quiver Q, and by induction on $\tau(\gamma_1)$. If $n = 1$ there is nothing to prove. If $\tau(\gamma_1) = 0$ then there is nothing to prove either since this is impossible.

Let us assume that in a Schur sequence the first dimension vector γ_1 is a real Schur root. Let V be the unique indecomposable representation corresponding to the dense orbit in $\mathrm{Rep}(Q, \gamma_1)$. Then the dimension vectors $\gamma_2, \dots, \gamma_r$ are Schur roots in the right orthogonal category $V^{\perp}$. By Theorem 2.27 the category $V^{\perp}$ is equivalent to the category of representations of a quiver Q' with no oriented cycles and $n-1$ vertices. Let $I : \mathbb{N}^{n-1} \to \mathbb{N}^{Q_0}$ as in Section 2.6. We can write $\gamma_i = I(\delta_i)$. Then $\delta_2, \dots, \delta_n$ is a Schur sequence for Q'. We can refine $\delta_2, \dots, \delta_n$ to an exceptional sequence $\underline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_t)$ by induction on n. Then the sequence $\underline{\varepsilon} = (\gamma_1, I(\varepsilon_2), \dots, I(\varepsilon_t))$ is clearly an exceptional sequence for Q which refines $\underline{\gamma}$.

We now assume that γ_1 is an imaginary Schur root. Since $\gamma_1 \circ \gamma_j = 1$ for all $j \geqslant 2$, it follows by induction from Lemma 4.4 (b) that $\gamma_1 \circ \delta = 1$ where $\delta = \gamma_2 + \dots + \gamma_r$. Let $\gamma_1^{\perp}$ be the set of all dimension vectors α with $\gamma_1 \perp \alpha$. By Theorem 2 from [11], $\mathbb{R}_+ \gamma_1^{\perp}$ is a rational polyhedral cone in $\mathbb{R}^{n-1}$. Suppose that δ is in the interior of the cone. For each $\alpha \in \gamma_1^{\perp}$ there exists $\beta \in \gamma_1^{\perp}$ such that $\alpha + \beta = p\delta$ for some positive integer p. From Lemma 4.4 (a) follows that $\gamma_1 \circ \alpha = 1$. This shows that for all $\sigma \in \Gamma^{\star}$, $\dim \mathrm{SI}(Q, \gamma_1)_{\sigma} \leqslant 1$. Indeed, if $\mathrm{SI}(Q, \gamma_1)_{\sigma} \neq 0$ then $\sigma = -\langle \cdot, \alpha \rangle$ for some $\alpha \in \gamma_1^{\perp}$ and $\dim \mathrm{SI}(Q, \gamma_1)_{\sigma} = \gamma_1 \circ \alpha = 1$. This implies that all $\mathrm{GL}(\gamma_1)$-invariant rational functions on $\mathrm{Rep}(Q, \gamma_1)$ are constant. From this follows that $\mathrm{GL}(\gamma_1)$ has a dense orbit in $\mathrm{Rep}(Q, \gamma_1)$. Therefore, γ_1 must be a *real* Schur root. Contradiction, so it follows that δ is not in the interior of $\mathbb{R}_+ \gamma_1^{\perp}$.

Let $\sigma = -\langle \cdot, \delta\rangle$ and let us study the σ-stable decomposition of γ_1. By Theorem 2.3 there exists a $\beta \hookrightarrow \gamma_1$ such that $\sigma(\beta) = 0$. In particular, the σ-stable decomposition of γ_1 is nontrivial. Suppose that

$$\gamma_1 = c_1 \cdot \beta_1 \dotplus c_2 \cdot \beta_2 \dotplus \cdots \dotplus c_l \cdot \beta_l$$

is the σ-stable decomposition of γ_1. We may assume that $\beta_i \perp\!\!\!\perp \beta_j$ for $i < j$. From $\gamma_1 \perp\!\!\!\perp \delta$ and Lemma 4.4 follows that $\beta_i \perp\!\!\!\perp \gamma_j$ for all $j \geqslant 2$ and all i. Therefore

$$\underline{\gamma}' = (\beta_1, \beta_2, \ldots, \beta_l, \gamma_2, \ldots, \gamma_r) \tag{4.1}$$

is a Schur sequence using Lemma 4.4 (a). Notice that β_1 is smaller than γ_1. Now $\tau(\beta_1) < \tau(\gamma_1)$ so by induction there exists an exceptional sequence which is a refinement of $\underline{\gamma}'$. □

Corollary 4.12. — *Let $\underline{\gamma} = (\gamma_1, \ldots, \gamma_r)$ be a Schur sequence. Then the vectors $\gamma_1, \ldots, \gamma_r$ are linearly independent.*

Proof. — First assume that $\underline{\gamma}$ is an exceptional sequence. We have

$$\langle \gamma_i, \gamma_j\rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i < j. \end{cases}$$

The matrix

$$\big(\langle \gamma_i, \gamma_j\rangle\big)_{i,j=1}^{n}$$

is invertible, so $\gamma_1, \ldots, \gamma_r$ are linearly independent.

If $\underline{\gamma}$ is not an exceptional sequence then it has a refinement which is an exceptional sequence. So again, it follows that $\gamma_1, \ldots, \gamma_r$ are linearly independent.

□

5. The faces of the cone $\mathbb{R}_+\Sigma(Q, \alpha)$

As before, $\Sigma(Q, \alpha)$ denotes the set of all weights $\sigma \in \Gamma^\star$ such that α is σ-semistable. If α is a sincere dimension vector, (i.e., $\alpha(x) > 0$ for all $x \in Q_0$) then there is a bijection between $\alpha^\perp = \{\beta \in \mathbb{N}^{Q_0} \mid \alpha \perp \beta\}$ and $\Sigma(Q, \alpha)$ by

$$\beta \in \alpha^\perp \leftrightarrow \sigma := -\langle \cdot, \beta\rangle \in \Sigma(Q, \alpha).$$

Similarly there is also a bijection between $\Sigma(Q, \alpha)$ and ${}^\perp\alpha$.

Let $\mathbb{R}_+$ be the set of nonnegative real numbers. We consider the cone $\mathbb{R}_+\Sigma(Q, \alpha) \subseteq \mathbb{R} \otimes_{\mathbb{Z}} \Gamma^\star \cong (\mathbb{R}^n)^\star$. In this section, we will unravel the geometry of this cone.

The refinement Theorem (Theorem 4.11) allows us to obtain a beautiful description of the faces of $\mathbb{R}_+\Sigma(Q,\alpha)$. Let us denote by $\mathcal{W}_r(Q,\alpha)$ the set of all sets $\{\gamma_1,\ldots,\gamma_r\}$ such that $\underline{\gamma}=(\gamma_1,\ldots,\gamma_r)$ is a quiver Schur sequence of length r such that $\alpha=\sum_{i=1}^r a_i\gamma_i$ with

(1) a_i a positive integer for all i;
(2) if γ_i is imaginary and not isotropic, then $a_i=1$.

Let $\mathcal{F}_r(Q,\alpha)$ be the set of faces of dimension $n-r$ of $\mathbb{R}_+\Sigma(Q,\alpha)$.

Theorem 5.1. — *Let Q be a quiver and let α be a dimension vector. For each r, $1\leqslant r\leqslant n-1$ we can define a bijective map*

$$\psi(r):\mathcal{W}_r(Q,\alpha)\to\mathcal{F}_r(Q,\alpha)$$

which sends the quiver Schur sequence $\underline{\gamma}=(\gamma_1,\ldots,\gamma_r)$ to the face

$$\mathbb{R}_+\Sigma(Q,\gamma_1)\cap\ldots\cap\mathbb{R}_+\Sigma(Q,\gamma_r) = \mathbb{R}_+\Sigma(Q,\alpha)\cap\{\sigma\in(\mathbb{R}^n)^\star\mid\sigma(\gamma_1)=\cdots=\sigma(\gamma_r)=0\}.$$

The inverse of $\psi(r)$ is the map

$$\Theta(r):\mathcal{F}_r(Q,\alpha)\to\mathcal{W}_r(Q,\alpha)$$

defined as follows. For a given face F we take a weight $\sigma\in\Gamma^\star$ in the relative interior of F. Then $\Theta(r)$ is the quiver Schur sequence coming from the σ-stable decomposition of α.

Proof. — Suppose that $\underline{\gamma}=(\gamma_1,\ldots,\gamma_r)\in\mathcal{W}_r(Q,\alpha)$. Clearly,

$$\Sigma(Q,\gamma_1)\cap\ldots\cap\Sigma(Q,\gamma_r)\subseteq\Sigma(Q,\alpha)\cap\{\sigma\in\Gamma^\star\mid\sigma(\gamma_1)=\cdots=\sigma(\gamma_r)=0\}.$$

The converse inclusion

$$\Sigma(Q,\gamma_1)\cap\ldots\cap\Sigma(Q,\gamma_r)\supseteq\Sigma(Q,\alpha)\cap\{\sigma\in\Gamma^\star\mid\sigma(\gamma_1)=\cdots=\sigma(\gamma_r)=0\}.$$

also holds: If α is σ-semistable, and $\sigma(\gamma_1)=\cdots=\sigma(\gamma_r)=0$ then $\gamma_1,\ldots,\gamma_r$ are σ-semistable. We conclude that

$$\Sigma(Q,\gamma_1)\cap\ldots\cap\Sigma(Q,\gamma_r)=\Sigma(Q,\alpha)\cap\{\sigma\in\Gamma^\star\mid\sigma(\gamma_1)=\cdots=\sigma(\gamma_r)=0\}.$$

We get

$$\mathbb{R}_+\Sigma(Q,\gamma_1)\cap\ldots\cap\mathbb{R}_+\Sigma(Q,\gamma_r) = \mathbb{R}_+\Sigma(Q,\alpha)\cap\{\sigma\in(\mathbb{R}^n)^\star\mid\sigma(\gamma_1)=\cdots=\sigma(\gamma_r)=0\}.$$

We prove by induction on r that

$$\mathbb{R}_+\Sigma(Q,\gamma_1)\cap\ldots\cap\mathbb{R}_+\Sigma(Q,\gamma_r)$$

is a face of codimension r in the space of dimension vectors. Suppose that $r=1$. Then α is an integer multiple of γ_1. We have $\Sigma(Q,\alpha)=\Sigma(Q,\gamma_1)$

by the saturation property. The cone $\mathbb{R}_+\Sigma(Q,\alpha) = \mathbb{R}_+\Sigma(Q,\gamma_1)$ is given by one equality, $\sigma(\gamma_1) = 0$ and many inequalities of the form $\sigma(\beta) \leqslant 0$ for all $\beta \hookrightarrow \gamma_1$. By Proposition 3.11 there exists a weight τ such that γ_1 is τ-stable. This means that $\tau(\beta) < 0$ for all $\beta \hookrightarrow \gamma_1$ and $\beta \neq 0, \gamma_1$. If we view $\mathbb{R}_+\Sigma(Q,\alpha)$ inside the space

$$\{\sigma \in (\mathbb{R}^n)^\star \mid \sigma(\gamma_1) = 0\} \cong \mathbb{R}^{n-1}$$

then $\mathbb{R}_+\Sigma(Q,\alpha)$ contains an open neighborhood of τ. This shows that $\mathbb{R}_+\Sigma(Q,\alpha) = \mathbb{R}_+\Sigma(Q,\gamma_1)$ is a cone of dimension $n-1$. In particular $\mathbb{R}_+\Sigma(Q,\alpha)$ is a face of dimension $n-1$.

Suppose that $r > 1$. By the refinement Theorem (Theorem 4.11) there exists an exceptional sequence $\underline{\varepsilon} = (\varepsilon_1,\ldots,\varepsilon_s)$ which is a refinement of $\underline{\gamma}$, i.e., there exists a sequence $0 = b_0 < b_1 < \ldots < b_{r-1} < n = b_r$ such that for each $j = 1,\ldots,r$ the dimension vector γ_j is a positive linear combination of $\varepsilon_{b_{j-1}+1},\ldots,\varepsilon_{b_j}$. Let V_i be the unique representation corresponding to the dense orbit in $\operatorname{Rep}(Q,\varepsilon_i)$, for $i = 1,2,\ldots,b_1$. The representations $V_1,\ldots,V_{b_1}$ generate the full subcategory $\mathcal{C}(V_1,\ldots,V_{b_1})$ of $\operatorname{Rep}_K(Q)$ which is closed under extensions, direct sums, and taking kernels and cokernels. This category is equivalent to the category of representations of some quiver Q' with b_1 vertices and without oriented cycles. (See Lemma 2.34 and Theorem 2.27). The right orthogonal category

$$(V_1,\ldots,V_{b_1})^\perp = V_1^\perp \cap \cdots \cap V_{b_1}^\perp$$

is the category of representations of a quiver Q'' with $n - b_1$ vertices and without oriented cycles (Theorem 2.27). Define $I' : \mathbb{N}^{Q_0'} \to \mathbb{N}^{Q_0}$ and $I'' : \mathbb{N}^{Q_0''} \to \mathbb{N}^{Q_0}$ as in Theorem 2.38. The image of I' contains $\varepsilon_1,\ldots,\varepsilon_{b_1}$, and hence it contains γ_1 as well. The image of I'' contains $\varepsilon_{b_1+1},\ldots,\varepsilon_s$, so it also contains $\gamma_2,\ldots,\gamma_r$.

By the induction hypothesis, we can find linearly independent

$$\tau_1',\ldots,\tau_{b_1-1}' \in \Sigma(Q',(I')^{-1}(\gamma_1)).$$

and linearly independent

$$\tau_1'',\ldots,\tau_{n-b_1-r+1}'' \in \Sigma(Q'',(I'')^{-1}(\gamma_2)) \cap \cdots \cap \Sigma(Q'',(I'')^{-1}(\gamma_r)).$$

Define $\sigma_1',\ldots,\sigma_{b_1-1}' \in \Gamma^\star$ such that $\sigma_i'(I'(\cdot)) = \tau_i'$ and $\sigma_i'(I''(\cdot)) = 0$ for $i = 1,2,\ldots,b_1-1$. We can write $\tau_i' = \langle \beta_i,\cdot\rangle_{Q'}$ for some dimension vector $\beta_i \in \mathbb{N}^{Q_0'}$. Then we have $\sigma_i' = \langle I'(\beta_i),\cdot\rangle_Q$. It follows that $\gamma_1,\ldots,\gamma_r$ are σ_i'-semistable.

Define $\sigma_1'',\ldots,\sigma_{n-b_1-r-1}'' \in \Gamma^\star$ by $\sigma_i''(I''(\cdot)) = \tau_i''$ and $\sigma_i''(I'(\cdot)) = 0$ for $i = 1,2,\ldots,n-b_1-r-1$. By similar arguments as before, we have that

$\gamma_1, \ldots, \gamma_r$ are σ_i''-semistable. The weights

$$\sigma_1', \ldots, \sigma_{b_1-1}', \sigma_1'', \ldots, \sigma_{n-b_1-r+1}'' \in \mathbb{R}_+\Sigma(Q, \gamma_1) \cap \cdots \cap \mathbb{R}_+\Sigma(Q, \gamma_r)$$

are independent. This shows that the face

$$\mathbb{R}_+\Sigma(Q, \gamma_1) \cap \cdots \cap \mathbb{R}_+\Sigma(Q, \gamma_r)$$

has dimension $n - r$.

We have proven that the map $\psi(r)$ defined above is well defined. We now show that $\psi(r)$ is injective. Suppose that $\alpha = \sum_{i=1}^{r} a_i\gamma_i$ with $\underline{\gamma} = (\gamma_1, \ldots, \gamma_r)$ and let

$$F := \psi(r)(\underline{\gamma}) = \mathbb{R}_+\Sigma(Q, \gamma_1) \cap \cdots \cap \mathbb{R}_+\Sigma(Q, \gamma_r).$$

Choose any σ in the relative interior of F. Let

$$\alpha = b_1 \cdot \delta_1 \dotplus \cdots \dotplus b_s \cdot \delta_s$$

be the σ-stable decomposition of α. Now σ lies in $\psi(s)(\underline{\delta})$, which is a face of codimension s. It follows that $s \leqslant r$. A general representation of dimension α is σ-semistable and has a filtration with quotients of dimensions $\gamma_1, \ldots, \gamma_r$. Each γ_i is a nonnegative integral combination of the δ's. By Lemma 3.12, the σ-stable decomposition of γ_i is of the form

$$\gamma_i = c_{i,1} \cdot \delta_1 \dotplus \cdots \dotplus c_{i,s} \cdot \delta_s.$$

for some nonnegative integers $c_{i,j}$.
Since $\sum_j b_j\delta_j = \alpha = \sum_i a_i\gamma_j = \sum_{i,j} a_i c_{i,j}\delta_j$, we have $b_j = \sum_i a_i c_{i,j}$ for all j.

Claim. — *If δ_j is imaginary, then $c_{k,j} > 0$ for exactly one k.*

We prove the claim. Suppose that δ_j is imaginary and non-isotropic. Then $b_j = 1$, so $c_{k,j} > 0$ for exactly one k.

Suppose that δ_j is isotropic and $c_{k,j} > 0$, $c_{l,j} > 0$ and $k < l$. Define $\beta = \delta_j + \sum_{t<j} c_{k,t}\delta_t$. Then $\operatorname{ext}(\beta, \gamma_k - \beta) = 0$ and $\beta \hookrightarrow \gamma_k$. Choose nonnegative integers q_t, $t = 1, 2, \ldots, j-1$ such that $\beta' \hookrightarrow \gamma_k$ where $\beta' = \delta_j + \sum_{t<j} q_t\delta_t$ and $\sum_t q_t$ is minimal. Suppose that $s < j$ and $\langle \beta', \delta_s \rangle > 0$. Then $q_{k,s} > 0$ because $\langle \delta_i, \delta_s \rangle \leqslant 0$ for all $i \neq s$. We have that $\hom(\beta', \delta_s) > 0$, so there exists a nontrivial homomorphism from a general β'-dimensional representation to a δ_s-dimensional representation. Such a homomorphism is surjective, because β' is σ-semistable and δ_s is σ-stable. The kernel of the homomorphism is a subrepresentation of dimension $\beta' - \delta_s$. This shows that $\beta' - \delta_s \hookrightarrow \beta'$. Combined with $\beta' \hookrightarrow \gamma_k$ this gives $\beta' - \delta_s \hookrightarrow \gamma_k$. This contradicts the minimality of $\sum_t q_t$. So $\langle \beta', \delta_s \rangle \leqslant 0$ for all $s < j$. It is also clear that $\langle \beta', \delta_s \rangle = 0$ for all $s \geqslant j$. It follows that $\langle \beta', \gamma_l \rangle \leqslant 0$. Since γ_k

is $-\langle\cdot,\gamma_l\rangle$-stable, we also have $\langle\beta',\gamma_l\rangle \geqslant 0$. So $\langle\beta',\gamma_l\rangle = 0$. From $\gamma_k \perp\!\!\!\perp \gamma_l$ follows that $\beta' \perp\!\!\!\perp \gamma_l$. Define $\beta'' = \sum_{t\geqslant j} c_{l,t}\delta_t$. Then we have $\gamma_l \twoheadrightarrow \beta''$ and $\langle\beta',\beta''\rangle = 0$. From $\beta' \perp\!\!\!\perp \gamma_l$ follows that $\beta' \perp\!\!\!\perp \beta''$. Since $\beta' \twoheadrightarrow \delta_j$ and $\langle\delta_j,\beta''\rangle = 0$ we get $\delta_j \perp\!\!\!\perp \beta''$. And finally, since $\langle\delta_j,\delta_j\rangle = 0$ and $\delta_j \hookrightarrow \gamma_l$, we get $\delta_j \perp\!\!\!\perp \delta_j$. This clearly contradicts the Refinement Theorem, because the pair of vectors δ_j, δ_j is linearly dependent. So $c_{k,j} > 0$ for exactly one k. This concludes the proof of the claim.

Let D be set of all real Schur roots in the set $\{\delta_1,\dots,\delta_s\}$, and let C be the set of all elements of $\{\gamma_1,\dots,\gamma_r\}$ that are linear combinations of elements of D. There is a well-defined surjective map $f : \{\delta_1,\dots,\delta_s\}\setminus D \to \{\gamma_1,\dots,\gamma_r\}\setminus C$ defined by $f(\delta_j) = \gamma_k$ where k is unique index for which $c_{k,j} > 0$. This shows that $s - |D| \geqslant r - |C|$. The elements of C define a sequence in the quiver associated to D. In particular $|D| \geqslant |C|$. Combined with $s-|D| \geqslant r-|C|$ this gives $s \geqslant r$. Since $s \leqslant r$ we get $s = r$, and $|D| = |C|$. It follows that f is a bijection. So for every $\gamma_k \in \{\gamma_1,\dots,\gamma_r\}\setminus C$ there exists a unique j such that $c_{k,j} > 0$. So now C is a maximal quiver Schur sequence for the quiver associated to D. It follows that C consists of real Schur roots by the Refinement Theorem (Theorem 4.11). By Remark 4.9, we have $C = D$. Suppose that $\gamma_k \notin C$. There exists a unique imaginary root j such that $c_{k,j} > 0$. Suppose that $c_{k,i} > 0$ for some $i < j$. Assume that i is minimal with $c_{k,i} > 0$. Then δ_i is a real root, and $\delta_i \hookrightarrow \gamma_k$. Since $\delta_i = \gamma_l$ for some l, we have $\gamma_l \hookrightarrow \gamma_k$ and $\hom(\gamma_l,\gamma_k) \neq 0$. Contradiction. Similarly, $c_{k,i} > 0$ for $i > j$ leads to a contradiction. It follows that, after a permutation of the γ's, $(c_{i,j})$ is a diagonal matrix. So we have $\gamma_i = c_{i,i}\delta_i$ for all i. Also, if δ_i is non-isotropic, then $c_{i,i} = 1$. If δ_i is isotropic, then $c_{i,i} = 1$ since γ_i is a Schur root. We conclude that

$$\{\gamma_1,\dots,\gamma_r\} = \{\delta_1,\dots,\delta_s\}.$$

The injectivity of $\psi(r)$ follows.

Let us show that $\Theta(r)$ is well defined. Let F be a face of dimension $n-r$ of $\mathbb{R}_+\Sigma(Q,\alpha)$. Take a weight $\sigma \in \Gamma^\star$ in the relative interior of F. Let

$$\alpha = c_1\cdot\delta_1 \dotplus c_2\cdot\delta_2 \dotplus \cdot \dotplus c_l\cdot\delta_l$$

be the σ-stable decomposition of α. Define

$$F' = \mathbb{R}_+\Sigma(Q,\delta_1) \cap\cdots\cap \mathbb{R}_+\Sigma(Q,\delta_l).$$

Since $\sigma \in F'$ and σ is in the relative interior of F, we have $F \subseteq F'$.

Suppose that $\gamma \hookrightarrow \alpha$ and $\sigma(\gamma) = 0$. Then γ is a linear combination of the δ_i's by the definition of σ-stable decomposition. But the description of

$\Sigma(Q,\alpha)$ given in Theorem 2.9 implies that

$$F' = \mathbb{R}_+\Sigma(Q,\alpha) \cap \{\sigma \in (\mathbb{R}^n)^\star \mid \sigma(\delta_1) = \cdots = \sigma(\delta_l) = 0\} \subseteq$$

$$\subseteq F = \mathbb{R}_+\Sigma(Q,\alpha) \cap \bigcap_{\gamma,\gamma\hookrightarrow\alpha,\langle\gamma,\beta\rangle=0} \{\sigma \in (\mathbb{R}^n)^\star \mid \sigma(\gamma) = 0\},$$

so we have $F' \subseteq F$. This shows that $\Theta(r)$ is well defined and $\psi(r) \circ \Theta(r)$ is equal to the identity. Since $\psi(r)$ is injective, we conclude that $\Theta(r)$ and $\psi(r)$ are bijective. □

Let us state the meaning of Theorem 5.1 in two extreme cases: for the walls of maximal dimension of $\Sigma(Q,\alpha)$ and for extremal rays.

Corollary 5.2. — *Let Q be a quiver and let α be a Schur root. Then the walls of $\Sigma(Q,\alpha)$ (i.e., faces of dimension $n-2$) are in one to one correspondence with the ways of writing*

$$\alpha = c_1\gamma_1 + c_2\gamma_2$$

where (γ_1,γ_2) is a quiver Schur sequence, c_1, c_2 positive integers with $c_i = 1$ whenever γ_i is imaginary and nonisotropic.

Let us consider an extremal ray $\mathbb{R}^+\sigma$ in $\mathbb{R}^+\Sigma(Q,\alpha)$. It corresponds to the linear combination

$$\alpha = c_1\gamma_1 + \ldots + c_{n-1}\gamma_{n-1},$$

where $(\gamma_1,\ldots,\gamma_{n-1})$ is a quiver Schur sequence. Theorem 4.11 implies that at least $n-2$ of the roots $\gamma_1,\ldots,\gamma_{n-1}$ are real Schur roots. Consider the subring

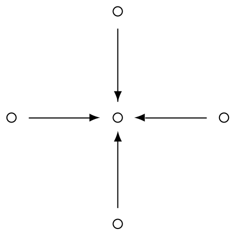
$$\mathrm{SI}(Q,\alpha,\sigma) = \bigoplus_{m\geqslant 0} \mathrm{SI}(Q,\alpha)_{m\sigma}.$$

By peeling off real roots from the left and from the right we can reduce the calculation of this ring to the ring of semi-invariants for a quiver $\theta(\ell)$ where $\theta(\ell)$ is the Kronecker quiver $\theta(\ell)$ with two vertices and ℓ equioriented arrows.

Corollary 5.3. — *Let Q be a quiver with no oriented cycles and let α be a Schur root. Suppose that $\sigma \in \Sigma(Q,\alpha)$ is indivisible and spans an extremal ray. Then there exists a positive integer ℓ and a Schur root β for the quiver $\theta(\ell)$ such that*

$$\mathrm{SI}(Q,\alpha,\sigma) \cong \mathrm{SI}(\theta(\ell),\beta).$$

Example 5.4. — Let Q be the quiver



and let α be the dimension vector

$$\begin{matrix} & 1 & \\ 1 & 2 & 1. \\ & 1 & \end{matrix}$$

There are 8 walls of $\Sigma(Q,\alpha)$. They are given by the Schur sequences

$$\begin{matrix} & 0 & \\ 1 & 2 & 1 \\ & 1 & \end{matrix}, \quad \begin{matrix} & 1 & \\ 0 & 0 & 0 \\ & 0 & \end{matrix} \qquad \text{(4 by symmetry)},$$

$$\begin{matrix} & 1 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix}, \quad \begin{matrix} & 0 & \\ 1 & 1 & 1 \\ & 1 & \end{matrix} \qquad \text{(4 by symmetry)}.$$

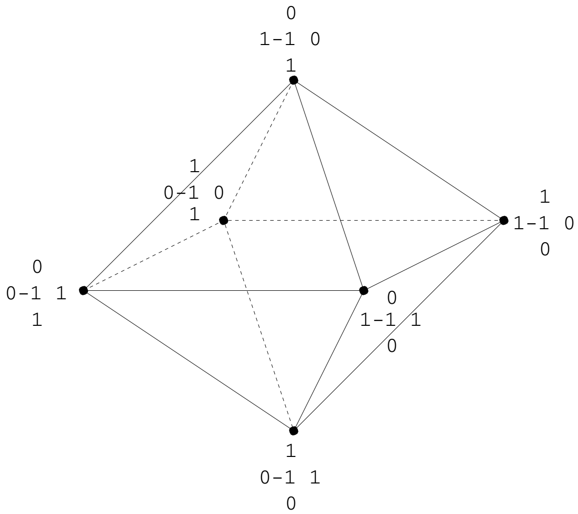
There are 12 two-dimensional faces of the cone given by the sequences

$$\begin{matrix} & 1 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix}, \quad \begin{matrix} & 0 & \\ 1 & 1 & 0 \\ & 1 & \end{matrix}, \quad \begin{matrix} & 0 & \\ 0 & 0 & 1 \\ & 0 & \end{matrix} \qquad \text{(12 by symmetry)}.$$

There are 6 extremal rays, given by the Schur sequences

$$\begin{matrix} & 1 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix}, \quad \begin{matrix} & 0 & \\ 1 & 1 & 0 \\ & 0 & \end{matrix}, \quad \begin{matrix} & 0 & \\ 0 & 0 & 0 \\ & 1 & \end{matrix}, \quad \begin{matrix} & 0 & \\ 0 & 0 & 1 \\ & 0 & \end{matrix} \qquad \text{(6 by symmetry)}.$$

The set $\Sigma(Q,\alpha)$ is a cone over a regular octahedron.

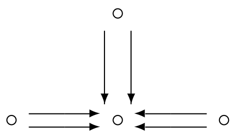


Each vertex of the octahedron corresponds to a weight and there is a unique semi-invariant of that weight (up to a scalar). If we multiply the semi-invariants corresponding to antipodal vertices, then we get 3 semi-invariants with weight

$$\begin{array}{ccc} & 1 & \\ 1 & -2 & 1. \\ & 1 & \end{array}$$

There is one linear dependence between them, the Plücker relation. The ratio of two such semi-invariants is a rational invariant, and it is the usual cross ratio.

Example 5.5. — Let Q be the quiver



and let α be the dimension vector

$$\begin{array}{ccc} & 1 & \\ 1 & 3 & 1 \end{array}.$$

Now $\Sigma(Q,\alpha)$ has 6 walls corresponding to the Schur sequences

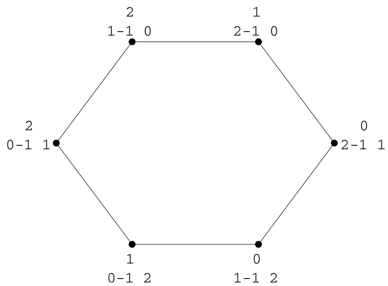
$$\begin{array}{ccc} & 0 & \\ 1 & 3 & 1 \end{array}, \begin{array}{ccc} & 1 & \\ 0 & 0 & 0 \end{array} \quad \text{(3 by symmetry)},$$

$$\begin{matrix} & 1 & & & 0 & \\ 0 & 2 & 0\text{'} & 1 & 1 & 1 \end{matrix} \quad \text{(3 by symmetry).}$$

The are also 6 extremal rays which correspond to the Schur sequences

$$\begin{matrix} & 1 & & & 0 & & & 0 & \\ 0 & 2 & 0\text{'} & 1 & 1 & 0\text{'} & 0 & 0 & 1 \end{matrix} \quad \text{(6 by symmetry).}$$

The cone $\Sigma(Q,\alpha)$ is a cone over a hexagon.



6. More on the σ-stable decompositions

6.1. The set of σ-stable dimension vectors

In the previous section we studied, how the σ-stable decomposition of α varies, when σ varies and α is fixed. This led to the description of the faces of $\Sigma(Q,\alpha)$. In this section, we will study how the σ-stable decomposition looks like for a fixed weight σ. Let us define $\overline{\Sigma}(Q,\sigma)$ as the set of all σ-semistable dimension vectors. Notice that

$$\alpha \in \overline{\Sigma}(Q,\sigma) \quad \Leftrightarrow \quad \sigma \in \Sigma(Q,\alpha).$$

Lemma 6.1. — *Suppose that $\alpha \in \overline{\Sigma}(Q,\sigma)$. There exists a sequence $\underline{\delta} = \delta_1,\dots,\delta_s$ of dimension vectors such that*

1. *$\alpha = \sum_{i=1}^{s} a_i\delta_i$ for some positive rational numbers $a_1,\dots,a_s$;*
2. *$\delta_1,\dots,\delta_s$ are linearly independent dimension vectors;*
3. *each δ_i generates an extremal ray, i.e., there exists an extremal ray $\mathcal{R}_i$ of $\mathbb{R}_+\overline{\Sigma}(Q,\sigma)$ such that $\mathcal{R}_i \cap \overline{\Sigma}(Q,\sigma) = \mathbb{N}\delta_i$.*

Proof. — This is trivial. □

Lemma 6.2. — *Suppose that $\alpha,\beta,\delta_1,\dots,\delta_s$ are σ-stable, and $\beta \hookrightarrow \alpha$.*

(a) *If α is a nonnegative integral combination of $\delta_1,\dots,\delta_s$ then so is β.*
(b) *If α is a nonnegative rational combination of $\delta_1,\dots,\delta_s$ then so is β.*

Proof. — Suppose that $\alpha = \sum_{i=1}^{s} a_i\delta_i$ for some integers $a_1, \dots, a_s$. Let V_i be σ-stable of dimension δ_i for all i. Consider the representation

$$V = V_1^{a_1} \oplus V_2^{a_2} \oplus \cdots \oplus V_s^{a_s}.$$

Now V has a semistable subrepresentation W of dimension β by Lemma 3.6. In a Jordan-Hölder filtration of W, only $V_1, \dots, V_s$ will appear, so β must be a nonnegative integral combination of $\delta_1, \dots, \delta_s$.

The second statement follows from the fact that for each positive integer m we have (see Theorem 2.7 and Corollary 2.19)

$$\beta \hookrightarrow \alpha \quad \Rightarrow \quad m\beta \hookrightarrow m\alpha.$$

□

Definition 6.3. — *For a sequence of roots $\underline{\alpha} = (\alpha_1, \dots, \alpha_s)$ (all α_i distinct) with $\langle \alpha_i, \alpha_j \rangle \leqslant 0$ for $i \neq j$, we define a quiver $Q(\underline{\alpha})$ as follows. The set of vertices of $Q(\underline{\alpha})_0$ is equal to $\{1, 2, \dots, s\}$. For $i \neq j$ there are $-\langle \alpha_i, \alpha_j \rangle$ arrows from i to j. There are $1 - \langle \alpha_i, \alpha_i \rangle$ arrows (loops) from i to i.*

Theorem 6.4. — *Under the assumptions of Lemma 6.1 above, α is σ-stable if and only if*

1. *either $\alpha = \delta_i$ and δ_i is a real Schur root for some i,*
2. *or $\langle \delta_i, \alpha \rangle \leqslant 0$ and $\langle \alpha, \delta_i \rangle \leqslant 0$ for all i, $Q(\underline{\delta})$ is path connected and α is indivisible if α is isotropic.*

Proof. — First we prove that the conditions are necessary. Assume that α is σ-stable. For $i \neq j$ we have $\hom(\delta_j, \delta_i) = 0$ because δ_i, δ_j are σ-stable (see Lemma 3.9). Suppose that $\langle \alpha, \delta_i \rangle > 0$. This is only possible when δ_i is a real Schur root because $\langle \delta_j, \delta_i \rangle \leqslant 0$ for all $j \neq i$. In that case we have $\hom(\alpha, \delta_i) \neq 0$, so $\alpha = \delta_i$ because α and δ_i are σ-stable (see Lemma 3.9). Similarly, if $\langle \delta_i, \alpha \rangle > 0$ then we have $\alpha = \delta_i$.

Consider the quiver $Q(\underline{\delta})$. Let S_1 be the set of all k such that there is a path from 1 to k and let $S_2 = \{1, 2, \dots, s\} \setminus S_1$. Define

$$\alpha_1 = \sum_{i \in S_1} a_i\delta_i, \qquad \alpha_2 = \sum_{i \in S_2} a_i\delta_i.$$

There are no arrows from S_1 to S_2. This shows that $\langle \alpha_1, \alpha_2 \rangle = 0$. Choose an integer m such that ma_i is a positive integer for all i. We have $m\alpha_1 \hookrightarrow m\alpha$. Since α is σ-stable, the σ-stable decomposition of $m\alpha$ is either $m\alpha$ or $\alpha \dotplus \cdots \dotplus \alpha$ by Proposition 3.13. Because $m\alpha_1$ is σ-semistable, $m\alpha_1$ must be proportional to α. This can only happen if $S_2 = \emptyset$. We have shown that the quiver $Q(\underline{\delta})$ is path connected.

If α is isotropic, then α must be indivisible because of Proposition 3.14.

Clearly, if condition (1) is satisfied, then α is σ-stable. Suppose that (2) is satisfied. Suppose that $\beta \hookrightarrow \alpha$, β is σ-stable and $\beta \neq 0, \alpha$. We will derive a contradiction. By Lemma 6.2, $\beta = \sum_{i=1}^{s} b_i\delta_i$ such that the b_i's are nonnegative rational numbers. Define

$$T_1 = \mathrm{Supp}(\beta) := \{i \mid 1 \leqslant i \leqslant s,\ b_i \neq 0\}.$$

Let

$$T_2 = \{1, 2, \ldots, s\} \setminus T_1 = \{i \mid 1 \leqslant i \leqslant s,\ b_i = 0\}.$$

Assume that $T_2 \neq \emptyset$. Define

$$\alpha_1 = \sum_{i \in T_1} a_i\delta_i, \quad \alpha_2 = \sum_{i \in T_2} a_i\delta_i.$$

Now $\alpha = \alpha_1 + \alpha_2$. Because $\langle \delta_i, \delta_j\rangle \leqslant 0$ for all $i \neq j$, and because there must be at least one arrow from T_1 to T_2, we get $\langle \beta, \alpha_2\rangle < 0$. Since $\mathrm{ext}(\beta, \alpha-\beta) = 0$, we get $\langle \beta, \alpha - \beta\rangle \geqslant 0$, so $\langle \beta, \gamma\rangle > 0$ with

$$\gamma = (\alpha - \beta) - \alpha_2 = \alpha_1 - \beta = \sum_{\alpha_i \in T_1} (a_i - b_i)\delta_i.$$

If

$$\gamma = \gamma_1 \dotplus \gamma_2 \dotplus \cdots \dotplus \gamma_r$$

is the σ-stable decomposition of γ, then $\langle \beta, \gamma_j\rangle > 0$ for some j. We get that $\beta = \gamma_j$ by Lemma 3.9. But then

$$\langle \beta, \gamma_j\rangle = \langle \beta, \beta\rangle = \langle \beta, \alpha\rangle - \langle \beta, \alpha - \beta\rangle \leqslant 0.$$

Contradiction.

This shows that $T_2 = \emptyset$ and $T_1 = \mathrm{Supp}(\beta) = \{1, 2, \ldots, s\}$. Let $\gamma = \alpha - \beta$. We have $\gamma \neq 0, \alpha$. We can find a σ-stable dimension vector γ' such that $\gamma \twoheadrightarrow \gamma'$ and therefore $\alpha \twoheadrightarrow \gamma'$. By a similar argument as before we obtain $\mathrm{Supp}(\gamma') = \mathrm{Supp}(\gamma) = \{1, 2, \ldots, s\}$. Write $\gamma = \sum_{i=1}^{s} c_i\delta_i$ with c_i a positive rational number for all i. We have

$$0 = \langle \beta, \gamma\rangle = \sum_{i=1}^{s} c_i\langle \beta, \delta_i\rangle. \tag{6.1}$$

Since β and δ_i are σ-stable, and $\beta \neq \delta_i$ we get that $\hom(\beta, \delta_i) = 0$ by Lemma 3.9. In particular, $\langle \beta, \delta_i\rangle \leqslant 0$ for all i. Combined with 6.1 we conclude that $\langle \beta, \delta_i\rangle = 0$ for all i.

Let $B = \max\{b_1, \ldots, b_s\}$. Let $S_1 = \{i \mid b_i = B\}$ and let $S_2 = \{1, 2, \ldots, s\} \setminus S_1$. Suppose $S_2 \neq \emptyset$. There must be an arrow from S_1 to S_2, say $j \to k$ with $j \in S_1$ and $k \in S_2$. Then

$$0 = \langle \beta, \delta_k\rangle \leqslant b_j\langle \delta_j, \delta_k\rangle + b_k\langle \delta_k, \delta_k\rangle$$

We know that $b_j > b_k$, $\langle \delta_j, \delta_k \rangle \leqslant -1$, and $\langle \delta_k, \delta_k \rangle \leqslant 1$. This leads to a contradiction.

So $S_2 = \emptyset$ and $b_i = B$ for all i. From $\langle \beta, \delta_i \rangle = 0$ follows that for every i there is exactly one arrow with tail i. Since $Q(\underline{\delta})$ is path connected, the quiver $Q(\underline{\delta})$ has to be a cycle. Now it easily follows from $\langle \alpha, \delta_i \rangle \leqslant 0$ that α must be proportional to β and $\langle \alpha, \alpha \rangle = 0$. Since α is indivisible in that case, $\alpha = \beta$. □

Remark 6.5. — The previous theorem also provides us an inductive way of finding the σ-stable decomposition, if α is σ-semistable, but not σ-stable. There are two cases.

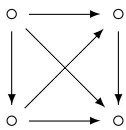
In the first case $\langle \delta_i, \alpha \rangle > 0$ or $\langle \alpha, \delta_i \rangle > 0$ for some extremal dimension vector $\delta_i \in \overline{\Sigma}(Q, \sigma)$. If we know the σ-stable decomposition of the smaller dimension vector $\alpha - \delta_i$, we know the σ-stable decomposition of α.

In the second case $Q(\underline{\delta})$ is not path-connected, say there is no path from i to j. Let S_1 be the set of all k such that there is a path from i to k and let S_2 be the complement. There are no arrows from S_1 to S_2. If we define

$$\alpha_1 = \sum_{i \in S_1} a_i \delta_i, \qquad \alpha_2 = \sum_{i \in S_2} a_i \delta_i.$$

For some m, $m\alpha_1, m\alpha_2$ are dimension vectors, and $m\alpha_1 \hookrightarrow m\alpha$. If we know the σ-stable decomposition of $m\alpha_1$ and $m\alpha_2$ then we know the σ-stable decomposition of $m\alpha$ and α. Now $m\alpha_1$ and $m\alpha_2$ have smaller support than α.

Example 6.6. — Let Q be the quiver



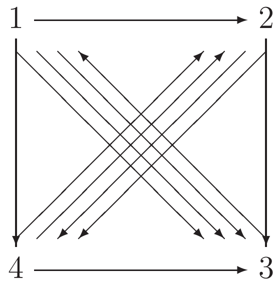
and let σ be the weight

$$\begin{matrix} 1 & -1 \\ 1 & -1 \end{matrix}.$$

The extremal rays of the cone $\overline{\Sigma}(Q, \sigma)$ are given by the dimension vectors

$$\delta_1 = \begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix}, \quad \delta_2 = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}, \quad \delta_3 = \begin{matrix} 0 & 0 \\ 1 & 1 \end{matrix}, \quad \delta_4 = \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}.$$

The quiver $Q(\underline{\delta})$ is

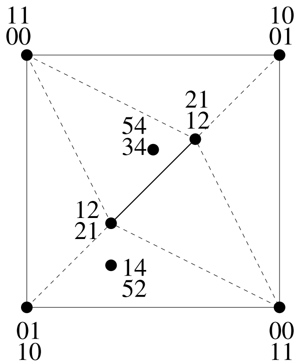


Any σ-stable dimension vector is a nonnegative rational combination of $\delta_1, \delta_2, \delta_4$ or a nonnegative rational combination of $\delta_2, \delta_3, \delta_4$. Suppose α is σ-stable and not equal to $\delta_1, \delta_2, \delta_3, \delta_4$. If α is a nonnegative rational combination $\delta_1, \delta_2, \delta_4$, then because the support has to be path connected, it is actually a nonnegative rational combination of δ_2, δ_4. Similarly, if α is a nonnegative rational combination of $\delta_2, \delta_3, \delta_4$, then it must be in fact a nonnegative rational combination of δ_2 and δ_4.

Now it easily follows that the set of σ-stable dimension vectors is

$$\delta_1, \quad \delta_2, \quad \delta_3, \quad \delta_4, \quad a\delta_2 + b\delta_4 \ (a, b > 0, a, b \in \mathbb{Z}, a \leqslant 2b, b \leqslant 2a).$$

The cone $\overline{\Sigma}(Q, \alpha)$ is a cone over a square. In the diagram below, the coordinates should be interpreted as projective coordinates.



The fat line in the middle of the square corresponds to the imaginary σ-stable dimension vectors. The dashed lines distinguish the regions where the σ-stable decomposition looks different.

Some examples of σ stable decomposition are:

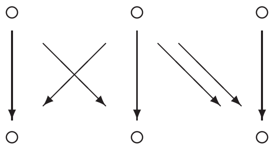
$$\begin{matrix} 5 & 4 \\ 3 & 4 \end{matrix} = \begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix} \dotplus \begin{matrix} 4 & 3 \\ 3 & 4 \end{matrix}$$

$$\begin{matrix}1 & 4\\ 5 & 2\end{matrix} = \begin{matrix}0 & 0\\ 1 & 1\end{matrix} + \begin{matrix}1 & 2\\ 2 & 1\end{matrix} + 2\cdot\begin{matrix}0 & 1\\ 1 & 0\end{matrix}.$$

6.2. σ-stable decomposition for quivers with oriented cycles

Doubling of the quiver, reduces the σ-stable decomposition for quivers with oriented cycles, to the case of quivers without oriented cycles. Suppose that Q is a quiver with oriented cycles. We define a new quiver $\widehat{Q}$ by $\widehat{Q}_0 = Q_0 \times \{0,1\}$. For every $a \in Q_1$ we define an arrow $\widehat{a} \in \widehat{Q}_1$ with $t\widehat{a} = (ta, 0)$ and $h\widehat{a} = (ha, 1)$ and for every $x \in Q_0$ we define an arrow $\widehat{x} \in \widehat{Q}_1$ with $t\widehat{x} = (x, 0)$ and $h\widehat{x} = (hx, 1)$. For example, if Q is the quiver

then $\widehat{Q}$ is the quiver



For a Q-dimension vector α, we define a dimension vector $\widehat{\alpha}$ of $\widehat{Q}$ by $\widehat{\alpha}(x,0) = \widehat{\alpha}(x,1) = \alpha(x)$ for all $x \in Q_0$. Similarly, if σ is a weight of Q, we define a weight $\widehat{\sigma}$ of $\widehat{Q}$ by $\widehat{\sigma}(x,0) = \widehat{\sigma}(x,1) = \sigma(x)$. We define the weight τ of $\widehat{Q}$ by $\tau(x,0) = 1$ and $\tau(x,1) = -1$ for all $x \in Q_0$. Note that for any $\alpha \in \mathbb{N}^{Q_0}$, $\widehat{\alpha}$ is τ-stable.

Proposition 6.7. — *Suppose that α is a dimension vector and σ is a weight for Q. Then α is σ-semistable (stable) if and only if for some large positive integer m, $\widehat{\alpha}$ is $\widehat{\sigma} + m\tau$-semistable (stable).*

Proof. — Suppose that α is σ-semistable. Let γ be a $\widehat{Q}$-dimension vector such that $\gamma \hookrightarrow \widehat{\alpha}$. Note that $\gamma(x,0) \leqslant \gamma(x,1)$ for all $x \in Q_0$ because $\widehat{\alpha}(x,0) = \widehat{\alpha}(x,1)$, and for a general representation V of dimension $\widehat{\alpha}$ the map $V(\widehat{x}) : V(x,0) \to V(x,1)$ is injective. If $\gamma(x,0) = \gamma(x,1)$ for all $x \in Q_0$ then γ is of the form $\widehat{\beta}$ and $\beta \hookrightarrow \alpha$. Then we have $\widehat{\sigma}(\gamma) = 2\sigma(\beta) \leqslant 0$. Also we have $(\widehat{\sigma} + m\tau)(\gamma) = \widehat{\sigma}(\gamma) \leqslant 0$.

Suppose that $\gamma(x,0) < \gamma(x,1)$ for some $x \in Q_0$. Then $\tau(\gamma) < 0$ so in particular for m large enough we will have $(\widehat{\sigma} + m\tau)(\gamma) < 0$.

Since there are only finitely many subdimension vectors γ, we can choose m large enough such that $(\widehat{\sigma} + m\tau)(\gamma) \leqslant 0$ for all $\gamma \hookrightarrow \widehat{\alpha}$. This shows that $\widehat{\alpha}$ is $(\widehat{\sigma} + m\tau)$-semistable.

Conversely, assume that $\widehat{\alpha}$ is $(\widehat{\sigma}+m\tau)$-semistable for some m and $\beta \hookrightarrow \alpha$. Then $\widehat{\beta} \hookrightarrow \widehat{\alpha}$, so $0 \geqslant (\widehat{\sigma}+m\tau)(\widehat{\beta}) = \widehat{\sigma}(\widehat{\beta}) = \sigma(\beta)$. This shows that α is σ-semistable.

A similar statement with stable instead of semistable is easy to prove. □

Suppose now that Q is quiver, possibly with oriented cycles. Let us consider the 0-stable decomposition. Clearly, every representation of Q is 0-semistable in the sense of Theorem 2.40. A representation V is 0-stable if the only subrepresentations are 0 and V itself. In other words, 0-stable representations are exactly simple representations. Notice that if there exists an α-dimensional simple representation, then the general representation of dimension α is simple. We will call such dimension vectors simple.

Corollary 6.8. — *Suppose that Q is an arbitrary quiver. For each $x \in Q_0$ we define a dimension vector δ_x by $\delta_x(y) = 0$ for $y \neq x$ and $\delta_x(x) = 1$. A dimension vector α is simple if*

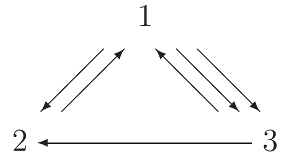
(1) *either $\alpha = \delta_x$ and δ_x is real (i.e., $\langle \delta_x, \delta_x \rangle = 1$);*
(2) *or $\langle \delta_x, \alpha \rangle \leqslant 0$ and $\langle \alpha, \delta_x \rangle \leqslant 0$ for all $x \in Q_0$, the full subquiver of Q with vertices*

$$\mathrm{Supp}(\alpha) := \{x \in Q_0 \mid \alpha(x) \neq 0\}$$

is path connected, and if α is isotropic, then α is indivisible.

Remark 6.9. — A statement similar to Corollary 6.8 was proven in [28]. Theorem 6.4 is a generalization of this result.

Example 6.10. — Consider the quiver



Suppose that α is the dimension vector (a_1, a_2, a_3). We will find necessary and sufficient conditions for α to be a simple dimension vector. Of course α can be equal to δ_1, δ_2, δ_3. The conditions $\langle \delta_i, \alpha \rangle \leqslant 0$ and $\langle \alpha, \delta_i \rangle \leqslant 0$ give the inequalities

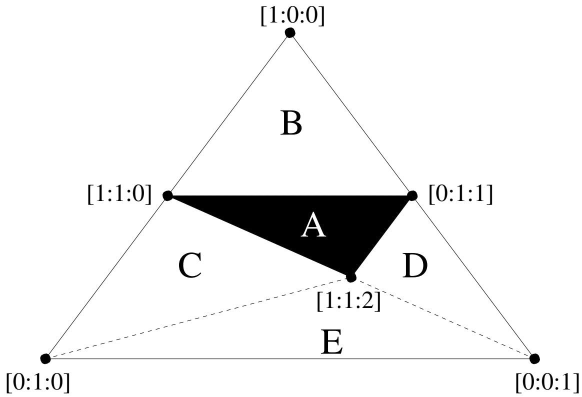
$$a_1 \leqslant a_2 + a_3,\ a_2 \leqslant a_1,\ a_3 \leqslant a_1 + a_2$$

(other inequalities turn out to be redundant). If $a_3 = 0$ and $a_1 = a_2$, then α is isotropic, so we must have that $a_1 = a_2 = 1$ in that case. The only support of α which is not possible (because it is not path connected) is $\{2, 3\}$, but this is already excluded by the inequalities.

From the inequalities and the remarks above it is easy to deduce that set of simple dimension vectors is given by

$$(1,0,0),\ (0,1,0),\ (0,0,1),\ (1,0,0),\ (1,1,0),\ \text{and all}$$
$$\{(a_1,a_2,a_3)\in\mathbb{Z}^3 \mid a_1\leqslant a_2+a_3,\ a_2\leqslant a_1,\ a_3\leqslant a_1+a_2, a_3>0\}$$

The picture shows how the simple decomposition looks like. We use projective coordinates.



Region A is defined by $a_1\leqslant a_2+a_3,\ a_2\leqslant a_1,\ a_3\leqslant a_1+a_2$. This will always define a simple dimension vector except when $a_3=0$ (and $a_1=a_2$). In that case, the simple decomposition is

$$(a,a,0)=a\cdot(1,1,0).$$

Region B is defined by $a_2+a_3\leqslant a_1$. The simple decomposition in this region is

$$(a_1,a_2,a_3)=(a_1-a_2-a_3)\cdot(1,0,0)\dotplus(a_2+a_3,a_2,a_3) \qquad \text{if } a_3>0 \text{ and}$$
$$(a_1,a_2,0)=(a_1-a_2)\cdot(1,0,0)\dotplus a_2\cdot(1,1,0).$$

Region C is defined by $a_2\geqslant a_1$, $2a_1\geqslant a_3$. The simple decomposition is

$$(a_1,a_2,a_3)=(a_2-a_1)\cdot(0,1,0)\dotplus(a_1,a_1,a_3) \qquad \text{if } a_3>0 \text{ and}$$
$$(a_1,a_2,0)=(a_2-a_1)\cdot(0,1,0)\dotplus a_1\cdot(1,1,0).$$

Region D is defined by $a_1\geqslant a_2$ and $a_3\geqslant a_1+a_2$. The simple decomposition here is

$$(a_1,a_2,a_3)=(a_3-a_1-a_2)\cdot(0,0,1)\dotplus(a_1,a_2,a_1+a_2).$$

Region E is defined by $a_2\geqslant a_1,\ a_3\geqslant 2a_1$. The simple decomposition in this region is

$$(a_1,a_2,a_3)=(a_2-a_1)\cdot(1,0,0)\dotplus(a_3-2a_1)\cdot(0,1,0)\dotplus(a_1,a_1,2a_1).$$

Example 6.11. — Let Q be the quiver with 3 vertices (labeled 1, 2 and 3), with a loop at each vertex and with arrows $1 \to 2$, $2 \to 3$ and $3 \to 1$. The set of simple dimension vectors is

$$(1,0,0),\ (0,1,0),\ (0,0,1),\quad \text{and all } (a,b,c) \text{ with } a,b,c>0.$$

Notice that for example a dimension vector of the form $(a,b,0)$ $(a,b>0)$ is not simple because its support is not path connected.

7. Littlewood-Richardson coefficients

7.1. The Klyachko cone

Irreducible polynomial representations of GL_n are parameterized by nonincreasing integer sequences of length n. If $\lambda = (\lambda_1, \ldots, \lambda_n)$ is such a sequence, then we denote the corresponding representation by V_λ. We define

$$|\lambda| := \lambda_1 + \cdots + \lambda_n.$$

The *Littlewood-Richardson coefficient* $c^{\nu}_{\lambda,\mu}$ is defined by

$$\dim(V_\lambda \otimes V_\mu \otimes V_\nu^\star)^{\mathrm{GL}_n}.$$

We would like to study the set

$$\mathcal{K}_n = \{(\lambda,\mu,\nu) \in (\mathbb{Z}^n)^3 \mid \lambda,\mu,\nu \text{ are nonincreasing and } c^{\nu}_{\lambda,\mu} \neq 0\}.$$

The cone $\mathbb{R}_+\mathcal{K}_n$ is the *Klyachko cone*. The Klyachko cone has dimension $3n-1$. Let $T_{p,q,r}$ be the quiver with $p+q+r-2$ vertices:

$$\begin{array}{ccccccccc}
x_1 & \longrightarrow & x_2 & \cdots & x_{p-2} & \longrightarrow & x_{p-1} & & \\
 & & & & & & & \searrow & \\
y_1 & \longrightarrow & y_2 & \cdots & y_{q-2} & \longrightarrow & y_{q-1} & \longrightarrow & x_p \\
 & & & & & & & \nearrow & \\
z_1 & \longrightarrow & z_2 & \cdots & z_{r-2} & \longrightarrow & z_{r-1} & &
\end{array}$$

We use the convention $y_q = z_r = x_p$. In [11] we have seen that if we take the dimension vector

$$\begin{array}{ccccccc}
 & 1 & 2 & \cdots & n-1 & & \\
\beta = & 1 & 2 & \cdots & n-1 & n, & \\
 & 1 & 2 & \cdots & n-1 & &
\end{array}$$

for $T_{n,n,n}$, then we can view $\dim \mathrm{SI}(Q,\beta)_\sigma$ as a Littlewood-Richardson coefficient as follows. If σ is given by

$$\sigma = \begin{matrix} a_1 & a_2 & \cdots & a_{n-1} & \\ b_1 & b_2 & \cdots & b_{n-1} & c_n, \\ c_1 & c_2 & \cdots & c_{n-1} & \end{matrix}$$

then

$$\dim \mathrm{SI}(Q,\beta)_\sigma = c^{\nu}_{\lambda,\mu},$$

where

$$\begin{aligned} \lambda &= \lambda(\sigma) = (a_1 + \cdots + a_{n-1}, a_2 + \cdots + a_{n-1}, \cdots, a_{n-1}, 0), \\ \mu &= \mu(\sigma) = (b_1 + \cdots + b_{n-1}, b_2 + \cdots + b_{n-1}, \cdots, b_{n-1}, 0), \\ \nu &= \nu(\sigma) = (-c_n, -(c_n + c_{n-1}), \ldots, -(c_n + c_{n-1} + \cdots + c_1)). \end{aligned}$$

Conversely, if $\lambda, \mu, \nu \in \mathbb{Z}^n$, then

$$c^{\nu}_{\lambda,\mu} = \dim \mathrm{SI}(Q,\beta)_\sigma$$

where

$$\sigma = \sigma(\lambda,\mu,\nu) = \begin{matrix} \lambda_1 - \lambda_2 & \lambda_2 - \lambda_3, & \cdots & \lambda_{n-1} - \lambda_n & \\ \mu_1 - \mu_2 & \mu_2 - \mu_3 & \cdots & \mu_{n-1} - \mu_n & \lambda_n + \mu_n - \nu_1. \\ \nu_{n-1} - \nu_n & \nu_{n-2} - \nu_{n-1} & \cdots & \nu_1 - \nu_2 & \end{matrix}$$

The set $\Sigma(Q,\beta)$ is almost equal to $\mathcal{K}_n$. In fact, we define a bijection

$$\psi : \Sigma(Q,\beta) \times \mathbb{Z}^2 \to \mathcal{K}_n$$

by

$$\psi(\sigma, a, b) = (\lambda(\sigma) + a \cdot \mathbf{1}, \mu(\sigma) + b \cdot \mathbf{1}, \nu(\sigma) + (a+b) \cdot \mathbf{1}).$$

where $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{N}^n$. This bijection extends to an isomorphism of the cones $\mathbb{R}_+\Sigma(Q,\beta) \times \mathbb{R}^2$ and $\mathbb{R}_+\mathcal{K}_n$. The inverse of ψ is given by

$$(\lambda, \mu, \nu) \mapsto (\sigma(\lambda,\mu,\nu), \lambda_n, \mu_n).$$

Recall that if $\sigma = \langle \alpha, \cdot \rangle$, then $\dim \mathrm{SI}(Q,\beta)_\sigma = \alpha \circ \beta$. The numbers $\alpha \circ \beta$ can be interpreted as Littlewood-Richardson coefficients. See for example [11]. The calculations in [11] easily generalize to $T_{p,q,r}$ where p, q, r are arbitrary. This can be done as follows. Let us define

$$\widetilde{c}_{\lambda,\mu,\nu} = \widetilde{c}^{(n)}_{\lambda,\mu,\nu} = \dim(V_\lambda \otimes V_\mu \otimes V_\nu)^{\mathrm{SL}_n}.$$

If $\widetilde{c}_{\lambda,\mu,\nu} \neq 0$, then $|\lambda| + |\mu| + |\nu|$ must be a multiple of n, say mn. In that case

$$\widetilde{c}_{\lambda,\mu,\nu} = c^{\nu^\star}_{\lambda,\mu},$$

where

$$\nu^\star = (m - \nu_n, m - \nu_{n-1}, \ldots, m - \nu_1).$$

Definition 7.1. — *Let* $\underline{x} = (x_1, x_2, \ldots, x_n)$, $\underline{y} = (y_1, y_2, \ldots, y_n)$ *be two nondecreasing sequences of nonnegative integers. We define a partition* $P(\underline{x}, \underline{y})$ *by*

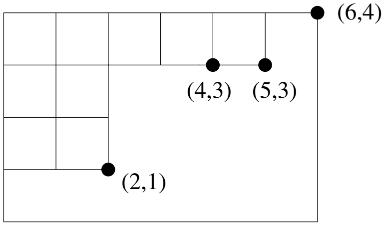
$$P(\underline{x}, \underline{y}) = (x_{n-1}^{y_n - y_{n-1}}, x_{n-2}^{y_{n-1} - y_{n-2}}, \ldots, x_1^{y_2 - y_1}, 0^{y_1}).$$

Graphically, this partition can be found as follows. In the Euclidian plane we draw a square with vertices $(0,0)$, $(x_n, 0)$, $(0, y_n)$, (x_n, y_n), *and we plot the points*

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n).$$

We take the region in the square above and left to those points. Viewed in the unit grid, the corresponding partition $P(\underline{x}, \underline{y})$ *can be read off.*

Example 7.2. — If $\underline{x} = (2,4,5,6)$ and $\underline{y} = (1,3,3,4)$ then $P(\underline{x}, \underline{y}) = (5,2,2,0)$ by the diagram below:



We consider the quiver $T_{p,q,r}$. Let α and β be dimension vectors. We write $\alpha(x) = (\alpha(x_1), \ldots, \alpha(x_p))$, $\alpha(y) = (\alpha(y_1), \ldots, \alpha(y_q))$ and in a similar way we define $\alpha(z)$, $\beta(x)$, $\beta(y)$, and $\beta(z)$.

Lemma 7.3. —

$$\alpha \circ \beta = \widetilde{c}_{\lambda,\mu,\nu}$$

where

$$\lambda = P(\alpha(x), \beta(x)), \ \mu = P(\alpha(y), \beta(y)), \ \nu = P(\alpha(z), \beta(z)).$$

Proof. — This is an easy computation, see [11]. □

Remark 7.4. — Note that if $1/p + 1/q + 1/r > 1$, then $T_{p,q,r}$ is a quiver of finite type and for all dimension vectors α, β we have $\alpha \circ \beta = 0$ or $\alpha \circ \beta = 1$. For a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ we define

$$j(\lambda) = \#\{i \mid \lambda_{i+1} \neq \lambda_i, 1 \leqslant i \leqslant n-1\} + 1.$$

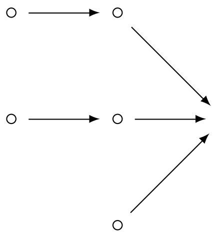
If λ is trivial then $j(\lambda) = 0$, if λ is a box then $j(\lambda) = 1$ and if λ is L-shaped (fat hook) then $j(\lambda) = 2$. A coefficient $\widetilde{c}_{\lambda,\mu,\nu}$ can be obtained from the quiver $T_{p,q,r}$ with $p = j(\lambda)+1, q = j(\mu)+1$ and $r = j(\nu)+1$. In particular, we get the corollary below:

Corollary 7.5. — *If*

$$\frac{1}{j(\lambda)+1}+\frac{1}{j(\mu)+1}+\frac{1}{j(\nu)+1}>1$$

then $\widetilde{c}_{\lambda,\mu,\nu}$ *equals 0 or 1.*

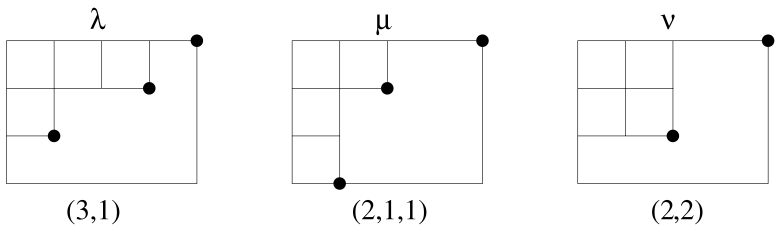
Example 7.6. — Let us consider the quiver $T_{3,3,2}$:



Let α, β be the dimension vectors

$$\begin{array}{cccc} & 1 & 3 & \\ \alpha = & 1 & 2 & 4, \\ & & 2 & \end{array} \qquad \begin{array}{cccc} & 1 & 2 & \\ \beta = & 0 & 2 & 3. \\ & & 1 & \end{array}$$

Now $\alpha \circ \beta$ is equal to the LR-coefficient $\widetilde{c}_{\lambda,\mu,\nu} = 1$ were $\lambda = (3,1)$, $\mu = (2,1,1)$ and $\nu = (2,2)$.



7.2. Walls of the Klyachko cone

Let us consider the quiver $Q = T_{n,n,n}$ and the dimension vector

$$\begin{array}{cccccc} & 1 & 2 & \cdots & n-1 & \\ \beta = & 1 & 2 & \cdots & n-1 & n. \\ & 1 & 2 & \cdots & n-1 & \end{array}$$

Lemma 7.7. — *The dimension vector* β *above is a Schur root.*

Proof. — For $n \leqslant 2$ this is an easy check that β is a real Schur root. For $n \geqslant 3$ the dimension vector β is indivisible and lies in the *fundamental set* (see [20, §1]). This implies that β is a Schur root by [20, Theorem B (d)]). □

We will study the cone $\mathbb{R}_+\Sigma(Q,\beta)$ which is essentially the Klyachko cone. The following Theorem is equivalent to a result of Knutson, Tao and Woodward (see [27]). It gives a precise description of the walls of the Klyachko cone.

Theorem 7.8. — *For every pair (β_1,β_2) with $\beta=\beta_1+\beta_2$, β_1,β_2 nondecreasing along arms, $\beta_1\circ\beta_2=1$ the inequality $\sigma(\beta_1)\leqslant 0$ defines a wall of $\mathbb{R}_+\Sigma(Q,\beta)$. All nontrivial walls can be uniquely obtained this way.*

Proof. — Clearly β_1 and β_2 have at most jumps 1 along the arms of the quiver Q. It follows from the proof of Lemma 7.7 that β_1,β_2 are Schur roots. Now either $\operatorname{ext}(\beta_2,\beta_1)=0$ or $\hom(\beta_2,\beta_1)=0$ by Theorem 2.37. The first would give a nontrivial decomposition of β, therefore $\hom(\beta_2,\beta_1)=0$ and β_1,β_2 is a quiver Schur sequence. This shows that $\sigma(\beta_1)\leqslant 0$ defines a wall by Theorem 5.1.

For every wall, there exists a Schur sequence (β_1,β_2) such that that $\beta=c_1\beta_1+c_2\beta_2$ with c_i positive integers and the wall is defined by $\sigma(\beta_1)\leqslant 0$ by Theorem 5.1. Note that for $T_{n,n,n}$ a Schur root either has support on one arm (in which case it corresponds to a positive root of A_{n-1}), or it is nondecreasing along each arm. Because $\beta_1\hookrightarrow\beta$, it is easy to see that β_1 must also be nondecreasing. Indeed, for a general representation of dimension β, the linear maps along the arms are injective, so these maps are also injective for every subrepresentation. But β_2 could have support on one arm. In that case it follows from $\langle\beta_1,\beta_2\rangle=0$ that β_2 is simple. The inequalities following from such a quiver sequence are trivial, they say that the partitions λ, μ and ν must be weakly decreasing. If β_1 and β_2 are both nondecreasing along the arms, then $c_1=c_2=1$ because $\beta=\beta_1+\beta_2$ and β jumps only by steps of 1 along the arms. □

We need to find all β_1,β_2 such that $\beta=\beta_1+\beta_2$ and $\beta_1\circ\beta_2=1$. In that case $\sigma(\beta_1)\leqslant 0$ defines a wall of $\Sigma(Q,\beta)$. If we are dealing with a nontrivial wall, i.e., *not* $\lambda_i\geqslant\lambda_{i+1}$, $\mu_i\geqslant\mu_{i+1}$ or $\nu_i\geqslant\nu_{i+1}$ for some i, then both β_1 and β_2 are increasing along the arms, with jumps at most 1. The fact that β_1 has jumps of at most 1, gives these inequalities a special form, namely, if we take

$$I=\{i\mid\beta_1(x_{i-1})=\beta_1(x_i),1\leqslant i\leqslant n\},$$

$$J=\{i\mid\beta_1(y_{i-1})=\beta_1(y_i),1\leqslant i\leqslant n\},$$

$$K=\{i\mid\beta_1(z_{i-1})=\beta_1(z_i),1\leqslant i\leqslant n\},$$

(by convention $\beta_1(x_0) = \beta_1(y_0) = \beta_1(z_0) = 0$), then the inequality corresponding to β_1 is

$$\sum_{i\in I} \lambda_i + \sum_{i\in J} \mu_i \leqslant \sum_{i\in K} \nu_{n-i}. \tag{7.1}$$

Note that $\#I = \#J = \#K = \beta_1(x_n)$. Now (7.1) is a necessary inequality for the Klyachko cone if $\beta_1 \circ \beta_2 = 1$. If $\beta_1 \circ \beta_2 > 0$ then (7.1) still defines a true inequality. Now $\beta_1 \circ \beta_2$ is the value of a Littlewood-Richardson coefficient for $\mathrm{SL}_{\beta_2(x_n)}$. Since $\beta_2(x_n) < n$ we know necessary and sufficient inequalities for the corresponding LR-coefficient to be nonzero. *This explains the inductive nature of the Klyachko inequalities.*

Example 7.9. — Consider the quiver $T_{3,3,3}$ and

$$\beta = \begin{matrix} 1 & 2 & \\ 1 & 2 & 3. \\ 1 & 2 & \end{matrix}$$

The LR-coefficient $c_{\lambda,\mu}^{\nu}$ corresponds to $\dim \mathrm{SI}(Q,\beta)_\sigma$ where σ is given by

$$\sigma = \begin{matrix} \lambda_1-\lambda_2 & \lambda_2-\lambda_3 & \\ \mu_1-\mu_2 & \mu_2-\mu_3 & \lambda_3+\mu_3-\nu_1. \\ \nu_2-\nu_3 & \nu_1-\nu_2 & \end{matrix}$$

For example, the sequence

$$\begin{pmatrix} 1 & 2 & & 0 & 0 & \\ 0 & 1 & 2, & 1 & 1 & 1 \\ 0 & 1 & & 1 & 1 & \end{pmatrix}$$

is a Schur sequence, because $\widetilde{c}_{2,0,0} = c_{2,0}^2 = 1$. This Schur sequence corresponds to the wall

$$\lambda_1 + \lambda_2 + \mu_2 + \mu_3 \leqslant \nu_1 + \nu_2.$$

By permuting the arms, we get the inequalties

$$\begin{aligned} \lambda_2+\lambda_3+\mu_1+\mu_2 &\leqslant \nu_1+\nu_2, \\ \lambda_2+\lambda_3+\mu_2+\mu_3 &\leqslant \nu_2+\nu_3. \end{aligned}$$

Other walls are given by the Schur sequences

$$\begin{pmatrix} 1 & 1 & & 0 & 1 & \\ 1 & 1 & 2, & 0 & 1 & 1 \\ 0 & 1 & & 1 & 1 & \end{pmatrix} \quad (\widetilde{c}_{1,1,0}^{(1)} = c_{1,1}^2 = 1).$$

By permuting arms we get the inequalities

$$\begin{aligned}\lambda_1+\lambda_3+\mu_1+\mu_3 &\leqslant \nu_1+\nu_2,\\ \lambda_1+\lambda_3+\mu_2+\mu_3 &\leqslant \nu_1+\nu_3,\\ \lambda_2+\lambda_3+\mu_1+\mu_3 &\leqslant \nu_1+\nu_3.\end{aligned}$$

The Schur sequence

$$\begin{pmatrix}1 & 1 & & 0 & 1 & \\ 0 & 0 & 1, & 1 & 2 & 2\\ 0 & 0 & & 1 & 2 & \end{pmatrix} \qquad (\widetilde{c}^{(2)}_{11,0,0}=c^{11}_{11,0}=1)$$

gives the inequalities

$$\begin{aligned}\lambda_1+\mu_3 &\leqslant \nu_1,\\ \lambda_3+\mu_1 &\leqslant \nu_1,\\ \lambda_3+\mu_3 &\leqslant \nu_3.\end{aligned}$$

The Schur sequence

$$\begin{pmatrix}0 & 1 & & 1 & 1 & \\ 0 & 1 & 1, & 1 & 1 & 2\\ 0 & 0 & & 1 & 2 & \end{pmatrix} \qquad (\widetilde{c}^{(2)}_{1,1,0}=c^{11}_{1,1}=1)$$

gives the inequalities

$$\begin{aligned}\lambda_2+\mu_2 &\leqslant \nu_1,\\ \lambda_2+\mu_3 &\leqslant \nu_2,\\ \lambda_3+\mu_2 &\leqslant \nu_2.\end{aligned}$$

Besides these, there are 6 trivial walls corresponding to the inequalities $\lambda_1\geqslant\lambda_2\geqslant\lambda_3$, $\mu_1\geqslant\mu_2\geqslant\mu_3$ and $\nu_1\geqslant\nu_2\geqslant\nu_3$. The Schur sequence

$$\begin{pmatrix}0 & 2 & & 1 & 0 & \\ 1 & 2 & 3, & 0 & 0 & 0\\ 1 & 2 & & 0 & 0 & \end{pmatrix}$$

implies the inequalities

$$\begin{aligned}\lambda_1 &\geqslant \lambda_2,\\ \mu_1 &\geqslant \mu_2,\\ \nu_2 &\geqslant \nu_3.\end{aligned}$$

The Schur sequence

$$\begin{pmatrix} 1 & 1 & & 0 & 1 & \\ 1 & 2 & 3, & 0 & 0 & 0 \\ 1 & 2 & & 0 & 0 & \end{pmatrix}$$

leads to the inequalities

$$\begin{aligned} \lambda_3 &\geqslant \lambda_3, \\ \mu_2 &\geqslant \mu_3, \\ \nu_1 &\geqslant \nu_2. \end{aligned}$$

7.3. Faces of the Klyachko cone of arbitrary codimension

For the quiver $Q = T_{n,n,n}$, Theorem 5.1 translates to:

Corollary 7.10. — *There is a 1–1 correspondence between the faces of codimension l of $\mathcal{K}_n$ and Schur sequences $(\beta_1, \dots, \beta_{l+1})$ such that $\beta = c_1\beta_1 + \dots + c_{l+1}\beta_{l+1}$ for some positive integers $c_1, \dots, c_{l+1}$ and $\beta_i \circ \beta_j = 1$ for all $i < j$.*

However, if $l > 1$ then the c_i may be larger than 1. There is no easy criterion for a dimension vector to be a Schur root, but a fast algorithm for determining whether a dimension vector is a Schur root was given in [12]. This makes it more difficult to find the faces of higher codimension. Still, we obtain some interesting features.

Corollary 7.11. — *Suppose that (λ, μ, ν) lies in a face F of $\mathcal{K}_n$ of codimension l. Let $j(\lambda), j(\mu), j(\nu)$ be the number of jumps in λ, μ, ν respectively. Then,*

(a)

$$j(\lambda) + j(\mu) + j(\nu) \leqslant 4n - 4 - l,$$

(b) *if $\widetilde{c}_{\lambda,\mu,\nu} > 1$ then*

$$j(\lambda) + j(\mu) + j(\nu) \leqslant 4n - 6 - l.$$

Proof. — Let β be the usual dimension vector for $T_{n,n,n}$. Now $\alpha \circ \beta = \widetilde{c}_{\lambda,\mu,\nu}$ for some dimension vector α. Let $\sigma = \langle \alpha, \cdot \rangle$. Because σ is in a face of $\Sigma(Q, \beta)$ of codimension l, exactly $l+1$ distinct dimension vectors appear in the σ-stable decomposition of β. Suppose that the σ-stable decomposition of β is

$$\beta = c_1 \cdot \beta_1 \dotplus c_2 \cdot \beta_2 \cdots \dotplus c_{l+1} \cdot \beta_{l+1}.$$

Whenever $\beta_i(x_n) = 0$, then β_i has support on one arm and the equation $\sigma(\beta_i) = 0$ corresponds to the equation $\lambda_j = \lambda_k$, $\mu_j = \mu_k$ or $\nu_j = \nu_k$ for some $j \neq k$. Notice also that all β_j's correspond to linearly independent equations. It now follows that

$$j(\lambda) + j(\mu) + j(\nu) + \#\{i \mid \beta_i(x_n) = 0\} \leqslant 3n - 3.$$

Since $\sum_j c_j \beta_j(x_n) = n$, we have $\#\{j \mid \beta_j(x_n) > 0\} \leqslant n$ and $\#\{j \mid \beta_j(x_n) = 0\} \geqslant l + 1 - n$. Now (a) follows.

If $\widetilde{c}_{\lambda,\mu,\nu} > 1$, then at least one of the β_i's is an imaginary Schur root, so $\beta_i(x_n) \geqslant 3$. This shows that $\#\{j \mid \beta_j(x_n) > 0\} \leqslant n-2$ and (b) follows. □

The cone $\mathbb{R}_+\Sigma(Q,\beta)$ has one 0-dimensional face, namely $\{0\}$. This corresponds to the the 2-dimensional face of $\mathbb{R}_+\mathcal{K}_n$ consisting of all (λ,μ,ν), $\lambda = (a,\dots,a)$, $\mu = (b,\dots,b)$, $\nu = (a+b,\dots,a+b)$. It would be also interesting to study the extremal rays of the cone $\mathbb{R}_+\Sigma(Q,\beta)$ (or equivalently the 3-dimensional faces of $\mathbb{R}_+\mathcal{K}_n$). They span the cone $\Sigma(Q,\beta)$ (or the Klyachko cone). The codimension of the extremal rays is $3n-4$. One interesting question is, whether $c_{\lambda,\mu}^{\nu} = 1$ whenever (λ,μ,ν) is on an extremal ray of $\mathbb{R}_+\mathcal{K}_n$ (here by *extremal ray* we mean a 3-dimensional face of $\mathcal{K}_n$). We first give a positive result in this direction:

Corollary 7.12. — *If (λ,μ,ν) is in an extremal ray of $\mathbb{R}_+\mathcal{K}_n$, and $n \leqslant 7$, then $c_{\lambda,\mu}^{\nu} = 1$.*

Proof. — Suppose that $c_{\lambda,\mu}^{\nu} > 1$. Then $\widetilde{c}_{\lambda,\mu,\nu^\star} = c_{\lambda,\mu}^{\nu} > 1$ for some partition $\nu^\star$. Then $j(\lambda) + j(\mu) + j(\nu^\star) \geqslant 6$ (this follows from Remark 7.4). So

$$6 \leqslant 4n - 6 - (3n-4)$$

by Corollary 7.11 and we deduce that $n \geqslant 8$. Contradiction. □

Example 7.13. — We study $T_{8,8,8}$ with the weight

$$\sigma = \begin{matrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & -3. \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & \end{matrix}$$

The σ-stable decomposition of

$$\beta = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & \end{matrix}$$

is

$$\begin{matrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & \\ 0 & 0 & 1 & 1 & 1 & 2 & 2 & 3 \\ 0 & 0 & 1 & 1 & 1 & 2 & 2 & \end{matrix} \dotplus \begin{matrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \end{matrix} \dotplus$$

$$\begin{array}{cccccccccccccccccc}
0&0&0&0&1&1&1& & &0&0&0&0&0&0&0& & \\
0&0&0&0&0&0&0&1&\dotplus&0&0&0&0&0&1&1&1&\dotplus\\
0&0&1&1&1&1&1& & &0&0&1&1&1&1&1& &
\end{array}$$

$$\begin{array}{cccccccccccccccccc}
0&0&0&0&0&0&0& & &0&0&0&0&1&1&1& & \\
0&0&1&1&1&1&1&1&\dotplus&0&0&0&0&0&1&1&1&\dotplus\\
0&0&0&0&0&1&1& & &0&0&0&0&0&1&1& &
\end{array}$$

$$\delta_{x_2} \dotplus 2\cdot\delta_{x_3} \dotplus 3\cdot\delta_{x_4} + \delta_{x_6} + 2\cdot\delta_{x_7} \dotplus$$
$$\delta_{y_1} \dotplus 2\cdot\delta_{y_2} \dotplus \delta_{y_4} \dotplus 2\cdot\delta_{y_5} \dotplus \delta_{y_7} \dotplus$$
$$\delta_{z_1} \dotplus 2\cdot\delta_{z_2} \dotplus \delta_{z_4} \dotplus 2\cdot\delta_{z_5} \dotplus \delta_{z_7}.$$

Here for any vertex p of the quiver $Q = T_{8,8,8}$ δ_p denotes the dimension vector of the simple representation corresponding to the vertex p. In the σ-stable decomposition of β, there are 21 distinct Schur roots. The quiver $T_{8,8,8}$ has 22 vertices, so this proves that σ is in an extremal ray of $\Sigma(Q,\beta)$. The Littlewood-Richardson coefficient corresponding to β and σ is $\widetilde{c}_{\lambda,\mu,\nu}$ where

$$\lambda = (2,1,1,1,1,0,0,0),\ \ \mu = (2,2,2,1,1,1,0,0),\ \ \nu = (2,2,2,1,1,1,0,0).$$

The value of $\widetilde{c}_{\lambda,\mu,\nu}$ is 2. In fact, for any N we have $\widetilde{c}_{N\lambda,N\mu,N\nu} = N+1$.

7.4. A multiplicative formula for Littlewood-Richardson coefficients

Let β and $T_{n,n,n}$ as before.

THEOREM 7.14. — *Suppose $\beta = \beta_1 + \beta_2$ and $\beta_1 \circ \beta_2 = 1$. Let α be another dimension vector with $\alpha\circ\beta = \widetilde{c}_{\lambda,\mu,\nu}$. Put $\sigma = \langle \alpha, \cdot\rangle$. The inequality $\sigma(\beta_1) \leqslant 0$ translates to*

$$\sum_{i\in I}\lambda_i + \sum_{i\in J}\mu_i \leqslant \sum_{i\in K}\nu_i, \tag{7.2}$$

where I, J, K are subsets of $S = \{1,2,\ldots,n\}$ of the same cardinality. Suppose that equality in (7.2) holds for $(\lambda,\mu,\nu) \in (\mathbb{Z}^n)^3$. Define

$$\lambda^* = (\lambda_{i_1},\cdots,\lambda_{i_r}), \quad I = \{i_1, i_2, \ldots, i_r\},$$
$$\lambda^\# = (\lambda_{\widetilde{ß}_1},\cdots,\lambda_{\widetilde{ß}_{n-r}}), \quad S\setminus I = \{\widetilde{ß}_1, \widetilde{ß}_2, \ldots, \widetilde{ß}_{n-r}\},$$
$$\mu^* = (\mu_{j_1},\cdots,\mu_{j_r}), \quad J = \{j_1, j_2, \ldots, j_r\},$$
$$\mu^\# = (\mu_{\widetilde{æ}_1},\cdots,\mu_{\widetilde{æ}_{n-r}}), \quad S\setminus J = \{\widetilde{æ}_1, \widetilde{æ}_2, \ldots, \widetilde{æ}_{n-r}\},$$
$$\nu^* = (\nu_{k_1},\cdots,\nu_{j_r}), \quad K = \{k_1, k_2, \ldots, k_r\},$$

$$\nu^{\#} = (\nu_{\widetilde{k}_1}, \cdots, \nu_{\widetilde{k}_{n-r}}), \quad S \setminus K = \{\widetilde{k}_1, \widetilde{k}_2, \ldots, \widetilde{k}_{n-r}\}.$$

Then we have

$$c^{\nu}_{\lambda,\mu} = c^{\nu^*}_{\lambda^*,\mu^*} c^{\nu^{\#}}_{\lambda^{\#},\mu^{\#}}.$$

Proof. — If σ is in the interior of the wall, then the σ-stable decomposition of β is $\beta_1 \dotplus \beta_2$ and by Theorem 3.16 we get

$$c^{\nu}_{\lambda,\mu} = \alpha \circ \beta = (\alpha \circ \beta_1)(\alpha \circ \beta_2) = c^{\nu^*}_{\lambda^*,\mu^*} c^{\nu^{\#}}_{\lambda^{\#},\mu^{\#}}.$$

Assume σ is not in the interior of the wall. Suppose that the σ-stable decomposition of β_1 is

$$c_1 \cdot \gamma_1 \dotplus \cdots \dotplus c_r \cdot \gamma_r$$

and that the σ-stable decomposition of β_2 is

$$d_1 \cdot \delta_1 \dotplus \cdots \dotplus d_s \cdot \delta_s.$$

Then the σ-stable decomposition of β is

$$c_1 \cdot \gamma_1 \dotplus \cdots \dotplus c_r \cdot \gamma_r \dotplus d_1 \cdot \delta_1 \dotplus \cdots \dotplus d_s \cdot \delta_s.$$

Note that $\{\gamma_1, \ldots, \gamma_r\}$ and $\{\delta_1, \ldots, \delta_s\}$ are disjoint because $\gamma_i \perp\!\!\!\perp \delta_j$ for all i, j. By Theorem 3.16 we get

$$c^{\nu}_{\lambda,\mu} = \alpha \circ \beta = \prod(\alpha \circ (c_i\gamma_i)) \prod(\alpha \circ (d_i\delta_i)) = (\alpha \circ \beta_1)(\alpha \circ \beta_2) = c^{\nu^*}_{\lambda^*,\mu^*} c^{\nu^{\#}}_{\lambda^{\#},\mu^{\#}}.$$

□

Example 7.15. — For $n = 8$, consider $\beta = \beta_1 + \beta_2$ with

$$\beta_1 = \begin{matrix} 1 & 1 & 2 & 2 & 3 & 3 & 4 & \\ 1 & 1 & 2 & 2 & 3 & 3 & 4 & 5, \\ 0 & 0 & 1 & 2 & 3 & 3 & 4 & \end{matrix} \quad \beta_2 = \begin{matrix} 0 & 1 & 1 & 2 & 2 & 3 & 3 & \\ 0 & 1 & 1 & 2 & 2 & 3 & 3 & 3. \\ 1 & 2 & 2 & 2 & 2 & 3 & 3 & \end{matrix}$$

Now $\beta_1 \circ \beta_2 = \widetilde{c}^{(3)}_{321,321,3} = c^{552}_{321,321} = 1$, so β_1, β_2 is a Schur sequence. The corresponding inequality for the Klyachko cone is

$$\lambda_1 + \lambda_3 + \lambda_5 + \lambda_7 + \lambda_8 + \mu_1 + \mu_3 + \mu_5 + \mu_7 + \mu_8 \leqslant \nu_1 + \nu_2 + \nu_4 + \nu_5 + \nu_6.$$

or equivalently

$$\lambda_2 + \lambda_4 + \lambda_6 + \mu_2 + \mu_4 + \mu_6 \geqslant \nu_3 + \nu_7 + \nu_8.$$

If now these inequalities are equalities, then

$$c^{\nu}_{\lambda,\mu} = c^{\nu^*}_{\lambda^*,\mu^*} c^{\nu^{\#}}_{\lambda^{\#},\mu^{\#}}.$$

where

$$\lambda^* = (\lambda_1, \lambda_3, \lambda_5, \lambda_7, \lambda_8),\ \mu^* = (\mu_1, \mu_3, \mu_5, \mu_7, \mu_8),\ \nu^* = (\nu_1, \nu_2, \nu_4, \nu_5, \nu_6)$$

and

$$\lambda^{\#} = (\lambda_2, \lambda_4, \lambda_6),\ \mu^{\#} = (\mu_2, \mu_4, \mu_6),\ \nu^{\#} = (\nu_3, \nu_7, \nu_8).$$

For example, take $\lambda = \mu = (8,4,4,2,2,0,0,0)$, $\nu = (10,8,7,4,3,3,3,2)$.

Then $c^{\nu}_{\lambda,\mu} = c^{\nu^*}_{\lambda^*,\mu^*} c^{\nu^\#}_{\lambda^\#,\mu^\#}$ where $\lambda^* = \mu^* = (8,4,2,0,0)$, $\nu^* = (10,8,4,3,3)$, $\lambda^\# = \mu^\# = (4,2,0)$, $\nu^* = (-7,3,2)$. Indeed, $c^{\nu^*}_{\lambda^*,\mu^*} = 5$, $c^{\nu^\#}_{\lambda^\#,\mu^\#} = 2$ and $c^{\nu}_{\lambda,\mu} = 10$.

Appendix: Belkale's proof of Fulton's conjecture

At the *AMS Summer Institute on Algebraic Geometry Meeting* in Seattle, Belkale explained his geometric proof of Theorem 1.6 to the first author and how it generalizes to the more general quiver setting. What follows below is a reconstruction of that proof. We are grateful to Prakash Belkale for letting us include his proof in our paper.

For a nonnegative integer d, the Grassmannian of d-dimensional subspaces of the n-dimensional vector space V is denoted by

$$\mathrm{Gr}\binom{V}{d}.$$

Suppose that α, γ are dimension vectors for a quiver Q. We define

$$\mathrm{Gr}\binom{\alpha}{\gamma} = \prod_{x \in Q_0} \mathrm{Gr}\binom{K^{\alpha(x)}}{\gamma(x)},$$

$$\mathrm{Hom}(K^\alpha, K^\beta) = \prod_{x \in Q_0} \mathrm{Hom}(K^{\alpha(x)}, K^{\beta(x)}).$$

Following Schofield, we need to introduce the notion of the *general rank* of a morphism. For a morphism $\phi : V \to W$ between two representations of Q, define the rank function $\underline{\mathrm{rk}}\,(\phi) \in \mathbb{N}^{Q_0}$ by

$$\underline{\mathrm{rk}}\,(\phi)(x) = \dim_K \phi(x)(V(x)), \quad x \in Q_0.$$

Define the variety

$$H(\alpha,\beta) = \{(V, W, \phi) \in \mathrm{Rep}(Q,\alpha) \times \mathrm{Rep}(Q,\beta) \times \mathrm{Hom}(K^\alpha, K^\beta) \mid \\ \forall a \in Q_1\ \phi(ha)V(a) = W(a)\phi(ta)\}.$$

There is a natural projection $p : H(\alpha,\beta) \to \mathrm{Rep}(Q,\alpha) \times \mathrm{Rep}(Q,\beta)$ such that the fiber of $(V,W) \in \mathrm{Rep}(Q,\alpha) \times \mathrm{Rep}(Q,\beta)$ can be identified with $\mathrm{Hom}_Q(V,W)$. Suppose that $Z \subseteq \mathrm{Rep}(Q,\alpha) \times \mathrm{Rep}(Q,\beta)$ is an irreducible closed subset. There exists an open dense subset $U \subseteq Z$ such that U is smooth and the fibers of $p^{-1}(U) \to U$ have constant dimension. Then $p^{-1}(U)$ is irreducible. The rank depends semi-continuously on $(V,W,\phi) \in p^{-1}(U)$. So there exists an open dense subset $U' \subseteq p^{-1}(U)$ such that $\underline{\mathrm{rk}}\,(\phi)$

is constant on U'. The value of this rank is called the *general rank* of a morphism $\phi : V \to W$ with $(V, W) \in Z$.

By symmetry, the Generalized Fulton Conjecture (Theorem 2.22) follows from

THEOREM 7.16. — *If $\alpha \circ \beta = 1$, then $\alpha \circ (n\beta) = 1$ for all $n \geqslant 0$.*

Proof. — Assume that $\alpha \circ \beta = 1$ and $\alpha \circ (n\beta) > 1$ for some $n \geqslant 0$. Let us put $\sigma = -\langle \cdot, \beta \rangle$. Then

$$\dim \mathrm{SI}(Q, \alpha)_{n\sigma} = \alpha \circ (n\beta).$$

We construct a quotient as in Section 2.8. Define the projective variety

$$Y = \mathrm{Proj} \bigoplus_{m \geqslant 0} \mathrm{SI}(Q, \alpha)_{m\sigma}.$$

We claim that its dimension is positive. Let

$$\pi : \mathrm{Rep}(Q, \alpha)^{\mathrm{ss}}_{\sigma} \to Y$$

be the GIT quotient with respect to σ, where $\mathrm{Rep}(Q, \alpha)^{\mathrm{ss}}_{\sigma} \subseteq \mathrm{Rep}(Q, \alpha)$ is the dense, open subset of σ-semistable representations. Let $W \in \mathrm{Rep}(Q, \beta)$ in general position. The Schofield semi-invariant $c_W \in \mathrm{SI}(Q, \alpha)_{\sigma}$ is nonzero because W is in general position. By assumption, $\dim_K \mathrm{SI}(Q, \alpha)_{\sigma} = \alpha \circ \beta = 1$. So $\mathrm{SI}(Q, \alpha)_{\sigma}$ is spanned by c_W. Also, if $W' \in \mathrm{Rep}(Q, \beta)$ is any other representation then either $c_{W'}$ is identically 0 or $c_{W'}$ is equal to c_W up to a scalar. By assumption, $\dim \mathrm{SI}(Q, \alpha)_{n\sigma} = \alpha \circ (n\beta) > 1$ for some n. If $f_1, f_2 \in \mathrm{SI}(Q, \alpha)_{n\sigma}$ are linearly independent, then they must be algebraically independent. This implies that the Krull dimension of the ring $\bigoplus_{n \geqslant 0} \mathrm{SI}(Q, \alpha)_{n\sigma}$ is at least 2, and the dimension of Y is at least 1. The equation $c_W = 0$ defines a nonnegative divisor on the projective variety Y. Clearly this divisor is nonzero, because c_W is not a constant. Let $y \in Y$ such that $c_W(y) = 0$. Since π is surjective, we can choose $V \in \pi^{-1}(y) \subseteq \mathrm{Rep}(Q, \alpha)^{\mathrm{ss}}_{\sigma}$. It follows that $c_W(V) = 0$. Let $D \subseteq \mathrm{Rep}(Q, \alpha)$ be an irreducible component of the divisor a

$$\{Z \in \mathrm{Rep}(Q, \alpha) \mid c_W(Z) = 0\}$$

which contains V. In conclusion, D is a divisor of $\mathrm{Rep}(Q, \alpha)$ containing σ-semistable representations, and $c_W(V) = 0$ for all $(V, W) \in D \times \mathrm{Rep}(Q, \beta)$.

Let γ be the general rank of a homomorphism $\phi : V \to W$ where $(V, W) \in D \times \mathrm{Rep}(Q, \beta)$.

Define

$$\begin{aligned}\widetilde{M} = \{(V,W,V_1,W_1,\phi) \in \\ \operatorname{Rep}(Q,\alpha)\times\operatorname{Rep}(Q,\beta)\times\operatorname{Gr}\tbinom{\alpha}{\alpha-\gamma}\times\operatorname{Gr}\tbinom{\beta}{\gamma}\times\operatorname{Hom}(K^\alpha,K^\beta) \mid \\ \phi : V\to W \text{ is a morphism}, \\ \forall\ x\in Q_0\ \phi(V_1(x))=0 \text{ and } \phi(K^{\alpha(x)})\subseteq W_1(x)\}\end{aligned}$$

and

$$M = \{(V,W,V_1,W_1,\phi)\in\widetilde{M} \mid \phi(K^{\alpha(x)}) = W_1(x)\}.$$

We have that $\widetilde{M}$ is a Zariski closed subset of

$$\operatorname{Rep}(Q,\alpha)\times\operatorname{Rep}(Q,\beta)\times\operatorname{Gr}\tbinom{\alpha}{\alpha-\gamma}\times\operatorname{Gr}\tbinom{\beta}{\gamma}\times\operatorname{Hom}(K^\alpha,K^\beta),$$

and M is open in $\widetilde{M}$ and therefore it is locally closed. So M is a variety. Suppose that $(V,W,V_1,W_1,\phi)\in M$. Then $\phi : V\to W$ has kernel V_1, image W_1 and rank $\gamma = \underline{\dim}\, W_1$.

Define

$$\begin{aligned}N := \{(V_1,W_1,\phi)\in\operatorname{Gr}\tbinom{\alpha}{\alpha-\gamma}\times\operatorname{Gr}\tbinom{\beta}{\gamma}\times\operatorname{Hom}(K^{\alpha(x)},K^{\beta(x)}) \mid \\ \forall x\in Q_0\ \phi(V_1(x))=0,\ \phi(K^\alpha(x)) = W_1(x)\}.\end{aligned}$$

Again N is locally closed, hence a variety. The projection $p : M \to \operatorname{Gr}\binom{\alpha}{\alpha-\gamma}\times\operatorname{Gr}\binom{\beta}{\gamma}$ factors through a morphism $q : M\to N$ and the projection $r : N\to\operatorname{Gr}\binom{\alpha}{\alpha-\gamma}\times\operatorname{Gr}\binom{\beta}{\gamma}$.

Let $(V_1,W_1)\in\operatorname{Gr}\binom{\alpha}{\alpha-\gamma}\times\operatorname{Gr}\binom{\beta}{\gamma}$ and consider the fiber $r^{-1}(V_1,W_1)$. This can be seen as the set of all $\phi : K^\alpha/V_1\to W_1$ which induce an isomorphism at each vertex. So $r^{-1}(V_1,W_1)\cong\operatorname{GL}(Q,\gamma)$ has dimension

$$\sum_{x\in Q_0}\gamma(x)^2.$$

Note that r is actually a fiber bundle. We get

$$\begin{aligned}\dim N &= \dim\operatorname{Gr}\tbinom{\alpha}{\alpha-\gamma} + \dim\operatorname{Gr}\tbinom{\beta}{\gamma} + \sum_{x\in Q_0}\gamma(x)^2 = \\ &= \sum_{x\in Q_0}\big(\gamma(x)(\alpha-\gamma)(x)+\gamma(x)(\beta-\gamma)(x)+\gamma(x)^2\big) \\ &= \sum_{x\in Q_0}\big(\gamma(x)(\alpha-\gamma)(x)+\gamma(x)\beta(x)\big).\end{aligned}$$

The map $q : M\to N$ is a vector bundle. A fiber

$$q^{-1}(V_1,W_1,\phi)$$

is the set of all

$$(V, W) \in \operatorname{Rep}(Q, \alpha) \times \operatorname{Rep}(Q, \beta)$$

such that $V(a)(V_1(ta)) \subseteq V_2(ha)$ $((\alpha-\gamma)(ta)\gamma(ha)$ linear constraints) for all a and the restriction of $W(a)$ to $W_1(ta)$ is $\phi(ha)V(a)\phi(ta)^{-1}$ $(\gamma(ta)\beta(ha)$ linear constraints) for all a. The dimension of $q^{-1}(V_1, W_1, \phi)$ is

$$\dim \operatorname{Rep}(Q, \alpha) + \dim \operatorname{Rep}(Q, \beta) - \sum_{a \in Q_1} \big((\alpha-\gamma)(ta)\gamma(ha) - \gamma(ta)\beta(ha)\big).$$

Therefore,

$$\begin{aligned} \dim M = \dim \text{fiber of } q + \dim N = \dim \operatorname{Rep}(Q, \alpha) + \dim \operatorname{Rep}(Q, \beta) \\ - \sum_{a \in Q_1} \big((\alpha-\gamma)(ta)\gamma(ha) - \gamma(ta)\beta(ha)\big) + \sum_{x \in Q_0} \big(\gamma(x)(\alpha-\gamma)(x) + \gamma(x)\beta(x)\big) = \\ = \dim \operatorname{Rep}(Q, \alpha) + \dim \operatorname{Rep}(Q, \beta) - \langle \alpha-\gamma, \beta-\gamma \rangle. \end{aligned}$$

(Remember that $\langle \alpha, \beta \rangle = 0$). Since M is a fiber bundle over the smooth irreducible variety $\operatorname{Gr}\binom{\alpha}{\alpha-\gamma} \times \operatorname{Gr}\binom{\beta}{\gamma}$ with smooth irreducible fibers, we have that M is smooth and irreducible.

Consider now the projection $s : M \to \operatorname{Rep}(Q, \alpha) \times \operatorname{Rep}(Q, \beta)$. Since γ is the general rank of a homomorphism $\phi : V \to W$ for $(V, W) \in D \times \operatorname{Rep}(Q, \beta)$, we see that $\overline{s(M)}$ contains $D \times \operatorname{Rep}(Q, \beta)$. Since $\overline{s(M)}$ is irreducible and $D \times \operatorname{Rep}(Q, \beta)$ has codimension 1, we must have $\overline{s(M)} = D \times \operatorname{Rep}(Q, \beta)$ or $\overline{s(M)} = \operatorname{Rep}(Q, \alpha) \times \operatorname{Rep}(Q, \beta)$. The latter is impossible, because $\hom(\alpha, \beta) = 0$ and $\gamma \neq 0$. Therefore $\overline{s(M)} = D \times \operatorname{Rep}(Q, \beta)$. The dimension of a general fiber of s is

$$\begin{aligned} \dim M - \dim D \times \operatorname{Rep}(Q, \beta) \\ = \dim M - \dim \operatorname{Rep}(Q, \alpha) - \dim \operatorname{Rep}(Q, \beta) + 1 = \\ = -\langle \alpha-\gamma, \beta-\gamma \rangle + 1. \end{aligned}$$

Choose $(V, W, V_1, W_1, \phi) \in M$ in general position. Then $(V, W) \in D \times \operatorname{Rep}(Q, \beta)$ is in general position. Then a general element in $\operatorname{Hom}_Q(V, W)$ has rank γ. The fiber $s^{-1}(V, W)$ is the set of all $\psi \in \operatorname{Hom}_Q(V, W)$ of rank γ. Therefore,

$$\dim \operatorname{Ext}_Q(V, W) = \dim \operatorname{Hom}_Q(V, W) = \dim s^{-1}(V, W) = -\langle \alpha-\gamma, \beta-\gamma \rangle + 1.$$

Suppose that $V_1' \in \operatorname{Rep}(Q, \alpha-\gamma)$ and $W' \in \operatorname{Rep}(Q, \beta)$ are both in general position. We know that a general representation of dimension β has a γ-dimensional subrepresentation, because $\overline{s(M)} = D \times \operatorname{Rep}(Q, \beta)$. So let W_1' be a γ-dimensional subrepresentation of W'. Define $V' = V_1' \oplus W_1'$ and let $\phi' : V' \to W'$ be the projection $V' \twoheadrightarrow W_1' \subseteq W'$. Now $(V', W', V_1', W_1', \phi') \in$

M. Moreover $\dim \operatorname{Ext}(V_1', W')$ has the minimum possible value $\operatorname{ext}(\alpha - \gamma, \beta)$. The set of all $(V', W', V_1', W_1', \phi') \in M$ for which $\dim \operatorname{Ext}_Q(V_1', W')$ is minimal (or equivalently $\dim \operatorname{Hom}_Q(V_1', W')$ is minimal) is an open dense subset of M. Since (V, W, V_1, W_1, ϕ) was assumed to be in general position, we may assume $\dim \operatorname{Ext}_Q(V_1, W) = \operatorname{ext}(\alpha - \gamma, \beta)$.

From Theorem 2.17 follows that there exists a $\delta \hookrightarrow \alpha - \gamma$ such that

$$\operatorname{ext}(\alpha - \gamma, \beta) = -\langle \delta, \beta \rangle.$$

Since $V \in D$ is in general position, we know that V is σ-semistable. But V has a δ-dimensional subrepresentation $V_2 \subseteq V_1 \subseteq V$. Hence

$$\operatorname{ext}(\alpha - \gamma, \beta) = -\langle \delta, \beta \rangle = \sigma(\delta) \leqslant 0$$

by semi-stability. It follows that $\operatorname{ext}(\alpha - \gamma, \beta) = 0$ and $\operatorname{Ext}_Q(V_1, W) = 0$. From the long exact sequence of Ext's, the map $\operatorname{Ext}_Q(V_1, W) \to \operatorname{Ext}_Q(V_1, W/W_1)$ is surjective. So we also get $\operatorname{Ext}_Q(V_1, W/W_1) = 0$. Now

$$(7.3)\quad 0 = \dim \operatorname{Ext}_Q(V_1, W/W_1) = -\langle \alpha - \gamma, \beta - \gamma \rangle + \dim \operatorname{Hom}_Q(V_1, W/W_1)$$

and so

$$-\langle \alpha - \gamma, \beta - \gamma \rangle \leqslant 0.$$

On the other hand, $\operatorname{Hom}_Q(V, W) \neq 0$, so

$$1 \leqslant \dim \operatorname{Hom}_Q(V, W) = \dim \operatorname{Ext}_Q(V, W) = -\langle \alpha - \gamma, \beta - \gamma \rangle + 1.$$

We conclude that

$$-\langle \alpha - \gamma, \beta - \gamma \rangle = 0.$$

From this follows that $\operatorname{Hom}_Q(V_1, W/W_1) = 0$, by (7.3). We have an exact equence

$$\cdots \to \operatorname{Hom}_Q(V_1, W/W_1) \to \operatorname{Ext}_Q(V_1, W_1) \to \operatorname{Ext}_Q(V_1, W) \to \cdots$$

Since the outer two are equal to 0, we get $\operatorname{Ext}(V_1, W_1) = 0$. This implies that $\operatorname{ext}(\alpha - \gamma, \gamma) = 0$ and $(\alpha - \gamma) \hookrightarrow \alpha$. If $V' \in \operatorname{Rep}(Q, \alpha) \setminus D$ is general, then V' has an $(\alpha - \gamma)$-dimensional subrepresentation V_1'. Define $W_1' := V'/V_1'$ and let $W_2' \in \operatorname{Rep}(Q, \beta - \gamma)$ be any representation of dimension $\beta - \gamma$. Set $W' = W_1' \oplus W_2'$ and let $\phi' : V' \twoheadrightarrow V'/V_1' = W_1' \subseteq W'$ be the projection. We have $(V', W', V_1', W_1', \phi') \in M$, so $(V', W') \in s(M)$ and $V' \in D$. Contradiction!

□

BIBLIOGRAPHY

[1] P. Belkale, "Geometric proofs of Horn and saturation conjectures", *J. Algebraic Geom.* **15** (2006), no. 1, p. 133-176.

[2] ———, "Geometric proof of a conjecture of Fulton", *Advances Math.* **216** (2007), no. 1, p. 346-357.

[3] A. S. Buch, "The saturation conjecture (after A. Knutson and T. Tao)", With an appendix by William Fulton, *Enseign. Math. (2)* **46** (2000), no. 1-2, p. 43-60.

[4] C. Chindris, "Quivers, long exact sequences and Horn type inequalities", *J. Algebra* **320** (2008), no. 1, p. 128-157.

[5] ———, "Quivers, long exact sequences and Horn type inequalities II", *Glasg. Math. J.* **51** (2009), no. 2, p. 201-217.

[6] C. Chindris, H. Derksen & J. Weyman, "Non-log-concave Littlewood-Richardson coefficients", *Compos. Math.* **43** (2007), no. 6, p. 1545-1557.

[7] W. Crawley-Boevey, "Exceptional sequences of representations of quivers", *Canadian Math. Soc. Conf. Proceedings* **14** (1993), p. 117-124.

[8] ———, "Subrepresentations of general representations of quivers", *Bull. London Math. Soc.* **28** (1996), no. 4, p. 363-366.

[9] ———, "On matrices in prescribed conjugacy classes with no common invariant subspace and sum zero", *Duke Math. J.* **118** (2003), no. 2, p. 339-352.

[10] H. Derksen, A. Schofield & J. Weyman, "On the number of subrepresentations of a general quiver representation", *J. London Math. Soc. (2)* **76** (2007), no. 1, p. 135-147.

[11] H. Derksen & J. Weyman, "Semi-invariants of quivers and saturation for Littlewood-Richardson coefficients", *Journal of the AMS* **13** (2000), p. 467-579.

[12] ———, "On the canonical decomposition of quiver representations", *Compositio Math.* **133** (2002), p. 245-265.

[13] ———, "On the Littlewood-Richardson polynomials", *J. Algebra* **255** (2002), no. 2, p. 247-257.

[14] ———, "The combinatorics of quiver representations", arXiv:math/0608288.

[15] W. Fulton, "Eigenvalues, invariant factors, highest weights, and Schubert calculus", *Bull. Amer. Math. Soc.* **37** (2000), no. 3, p. 209-249.

[16] L. Hille, "Quivers, cones and polytopes", *Linear Algebra Appl.* **365** (2003), p. 215-237, special issue on linear algebra methods in representation theory.

[17] L. Hille & J. de la Peña, "Stable representations of quivers", *J. Pure Appl. Algebra* **172** (2002), no. 2-3, p. 205-224.

[18] A. Horn, "Eigenvalues of sums of Hermitian matrices", *Pacific J. Math.* **12** (1962), p. 620-630.

[19] V. Kac, "Infinite root systems, representations of graphs and Invariant Theory", *Invent. Math.* **56** (1980), p. 57-92.

[20] ———, "Infinite Root Systems, Representations of Graphs and Invariant Theory II", *J. Algebra* **78** (1982), p. 141-162.

[21] A. D. King, "Moduli of representations of finite dimensional algebras", *Quart. J. Math. Oxford (2)* **45** (1994), p. 515-530.

[22] R. C. King, C. Tollu & F. Toumazet, "The hive model and the factorisation of Kostka coefficients", *Sém. Lothar. Combin.* **54A** (2005/07), p. Art. B54Ah, 22 pp. (electronic).

[23] R. C. King, C. Tollu & F. Toumazet, "Factorisation of Littlewood-Richardson coefficients", *J. Combin. Theory Ser. A* **116** (2009), no. 2, p. 314-333.

[24] T. Klein, "The multiplication of Schur-functions and extensions of p-modules", *J. London Math. Soc.* **43** (1968), p. 280-284.

[25] A. Klyachko, “Stable bundles, representation theory and Hermitian operators”, *Selecta Math. (N.S.)* **4** (1988), no. 3, p. 419-445.

[26] A. Knutson & T. Tao, “The honeycomb model of $\mathrm{GL}_n(\mathbb{C})$ tensor products. I. Proof of the saturation conjecture”, *J. Amer. Math. Soc.* **12** (1999), no. 4, p. 1055-1090.

[27] A. Knutson, T. Tao & W. Woodward, “The honeycomb model of $\mathrm{GL}_n(\mathbb{C})$ tensor products. II. Puzzles determine facets of the Littlewood-Richardson cone”, *J. Amer. Math. Soc.* **17** (2004), no. 1, p. 19-48.

[28] L. Le Bruyn & C. Procesi, “Semisimple representations of quivers”, *Trans. Amer. Math. Soc.* **317** (1990), no. 2, p. 585-598.

[29] N. Ressayre, “Geometric invariant theory and generalized eigenvalue problem II”, arXiv:0903.1187.

[30] ———, “GIT cones for quivers”, arXiv: 0903.1202.

[31] ———, “Geometric invariant theory and the generalized eigenvalue problem”, *Invent. Math.* **180** (2010), no. 2, p. 389-441.

[32] C. M. Ringel, “Representations of K-species and bimodules”, *J. Algebra* **41** (1976), p. 269-302.

[33] ———, *Tame algebras and integral quadratic forms*, Lecture Notes in Math., vol. 1099, Springer, 1984.

[34] C. M. Ringel, “The braid group action on the set of exceptional sequences of a hereditary Artin algebra”, in *Abelian group theory and related topics (Oberwolfach, 1993)*, Contemp. Math., vol. 171, Amer. Math. Soc., Providence, RI, 1994, p. 339-352.

[35] A. Rudakov, “Stability for an abelian category”, *J. Algebra* **197** (1997), p. 231-245.

[36] A. Schofield, “Semi-invariants of Quivers”, *J. London Math. Soc.* **43** (1991), p. 383-395.

[37] ———, “General Representations of Quivers”, *Proc. London Math. Soc. (3)* **65** (1992), p. 46-64.

[38] ———, “Birational classification of moduli spaces of representations of quivers”, *Indag. Math., N.S.* **12** (3) (2001), p. 407-432.

[39] A. Schofield & A. van den Bergh, “Semi-invariants of quivers for arbitrary dimension vectors”, *Indag. Math., N.S.* **12** (1) (2001), p. 125-138.

[40] R. Vakil, “Schubert Induction”, *Annals of Math. (2)* **164** (2006), no. 2, p. 489-512.

[41] H. Weyl, “Das asymtotische Verteilungsgesetz der Eigenwerte lineare partieller Differentialgleichungen”, *Math. Ann.* **71** (1912), p. 441-479.

Manuscrit reçu le 1er août 2008,
accepté le 24 octobre 2008.

Harm DERKSEN
University of Michigan
Department of Mathematics
Ann Arbor MI 48109-1043 (USA)
hderksen@umich.edu

Jerzy WEYMAN
Northeastern University
Department of Mathematics
Boston MA 02115 (USA)
j.weyman@neu.edu