Invariant Theory for Quivers

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Invariant Theory

$K = \overline{K}$ algebraically closed field
G **reductive** algebraic group (e.g., GL_n, semi-simple, finite,...)
V n-dimensional representation of G
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Theorem (Hilbert 1890, Nagata 1963/Haboush 1975)

$K[V]^G$ is a finitely generated K-algebra

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G acts on $K[V] = K[x_1, \dots, x_n]$ by permuting variables
$K[V]^{S_n} = K[e_1, e_2, \dots, e_n]$ where

$$e_k = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$$

is k-th elementary symmetric function

inclusion $K[V]^G \hookrightarrow K[V]$ corresponds to a **quotient**

$$\pi : V \to V /\!\!/ G$$

where $V /\!\!/ G = \operatorname{Spec} K[V]^G$

Geometric Invariant Theory

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Theorem

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(b) for $y \in V /\!\!/ G$, $\pi^{-1}(y)$ contains exactly 1 closed orbit, say $G \cdot z$
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Definition (Hilbert's Nullcone)

$\mathcal{N} = \pi^{-1}\pi(0)$ is **Hilbert's nullcone**
$\mathcal{N} = \{v \in V \mid f(v) = 0$ for all nonconst. homogen. $f \in K[V]^G\}$
$\mathcal{N} = \{v \in V \mid 0 \in \overline{G \cdot v}\}$

Examples

Example

$G = \mathrm{GL}_1 = K^\star$ acts on $V = K^3$:

$$t \cdot (x, y, z) = (tx, t^3 y, t^{-2} z)$$

$$K[V] = K[x, y, z]$$

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$$\mathcal{N} = \{x = y = 0\} \cup \{z = 0\}$$

Geometric Invariant Theory

Construction of quotient of $\mathbb{P}(V)$ with respect to G:

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(b) $v \in V$ is **stable** if and only if v is semistable and $\dim G \cdot v = \dim G$

V^{ss} semistable points
V^{s} stable points, open subset of V^{ss}
$\mathbb{P}(V^{\mathrm{s}}) \subseteq \mathbb{P}(V^{\mathrm{ss}}) \subseteq \mathbb{P}(V)$

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Theorem

(a) $\pi : \mathbb{P}(V^{\mathrm{ss}}) \to \mathbb{P}(V /\!\!/ G)$ *is a "good" quotient*

(b) *restriction to* $\mathbb{P}(V^{\mathrm{s}})$ *is geometric quotient (orbits=fibers)*

$\mathrm{char}(K) = 0$
$r = \dim K[V]^G \leq \dim V = n$

Theorem (Popov 1981)

if $\mathcal{N} \subseteq V$ is the zero set of $f_1, f_2, \ldots, f_s \in K[V]^G$ homogeneous of degree d, then $K[V]^G$ generated by invariants of degree $\leq rd$.

Constructive Invariant Theory

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Theorem (D. 2001)

if $\mathcal{N} \subseteq V$ is the zero set of $f_1, f_2, \ldots, f_s \in K[V]^G$ homogeneous of degree $\leq d$, then $K[V]^G$ generated by invariants of degree $\leq \max\{d, \frac{3}{8}rd^2\}$.

(improves bound $r\,\mathrm{lcm}(1, 2, \ldots, d)$ of Popov.)

Quiver Representations

Definition

A **quiver** is a 4-tuple $Q = (Q_0, Q_1, h, t)$, where
Q_0, finite set of vertices
Q_1, finite set of arrows
$h, t : Q_1 \to Q_0$
$h(a) = ha$ head of arrow a
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Fix a field K

Definition

A **quiver representation** V (over K) is
finite dimensional K-vector spaces $V(x)$, $x \in Q_0$, together with
K-linear maps $V(a) : V(ta) \to V(ha)$, $a \in Q_1$

Representation Spaces

The **dimension vector** of a representation V is the function

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We can view V as an element in the **representation space**

$$V = (V(a), a \in Q_1) \in \mathrm{Rep}_\alpha(Q) := \prod_{a \in Q_1} \mathrm{Hom}(K^{\alpha(ta)}, K^{\alpha(ha)})$$

Rings of Invariants for Loop Quivers

Definition

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Special case Q quiver with 1 vertex, m-loops, $\alpha = (p)$
$\mathrm{Rep}_\alpha(Q) = \mathrm{Mat}_p(K)^m$, $\mathrm{GL}_\alpha = \mathrm{GL}_p$ acts by conjugation

Theorem (Procesi)

If $\mathrm{char}(K) = 0$, *then* $K[\mathrm{Mat}_{n,n}^m]^{\mathrm{GL}_n}$ *is generated by all*

$$(A_1, A_2, \dots, A_m) \mapsto \mathrm{Tr}(A_{i_1} A_{i_2} \cdots A_{i_d})$$

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Theorem (Razmyslov)

If $\mathrm{char}(K) = 0$, *then* $K[\mathrm{Mat}_{n,n}^m]^{\mathrm{GL}_n}$ *is generated by invariants of degree* $\leq n^2$

Rings of Invariants for Arbitrary Quivers

Q arbitary quiver
if V representation, and $p = a_k a_{k-1} \cdots a_1$ path, then
$V(p) := V(a_k)V(a_{k-1}) \cdots V(a_1)$
If $p_1, p_2, \dots, p_r$ paths (same head/tail), $\lambda_1, \dots, \lambda_r \in K$, then
$V(\sum_{i=1}^r \lambda_i p_i) := \sum_{i=1}^r \lambda_i V(p_i)$

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Theorem (LeBruyn-Procesi 1990)

if $\mathrm{char}(K) = 0$, *then* $\mathrm{I}(Q, \alpha)$ *is generated by invariants of the form* $V \mapsto \mathrm{Tr}(V(p))$ *with* p *a cyclic path*

so if Q has no oriented cycles, then $\mathrm{I}(Q, \alpha) = K$

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Theorem (Donkin 1994)

$\mathrm{I}(Q, \alpha)$ *is generated by the coefficients of the characteristic polynomial of all* $V(p)$ *with* p *a cyclic path*

Assume K is infinite
For $\sigma \in \mathbb{Z}^{Q_0}$ we define a **multiplicative character** $\chi_\sigma : \mathrm{GL}_\alpha \to K^\star$ by

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Semi-Invariants

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Definition

The space of semi-invariants of weight σ is

$$\mathrm{SI}(Q,\alpha)_\sigma = \{f \in K[\mathrm{Rep}_\alpha(Q)] \mid \forall A \in \mathrm{GL}_\alpha \; A \cdot f = \chi_\sigma(A)\}$$

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The ring of semi-invariants is

$$\mathrm{SI}(Q,\alpha) = K[\mathrm{Rep}_\alpha(Q)]^{\mathrm{SL}_\alpha} = \bigoplus_\sigma \mathrm{SI}(Q,\alpha)_\sigma$$

GIT for quivers

Definition

a representation V is σ-(semi)stable if $(V, 1) \in \mathrm{Rep}_\alpha(Q) \oplus \chi_\sigma$ is GL_α-(semi)stable

$\mathrm{Rep}_\alpha(Q)^{\mathrm{ss}}_\sigma$ (resp. $\mathrm{Rep}_\alpha(Q)^{\mathrm{s}}$) set of σ-semi-stable (resp. σ-stable) points

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Theorem (King 1994)

(a) *V is σ-semistable if and only if $\sigma(\alpha) = 0$ and $\sigma(\underline{\dim} W) \leq 0$ for every subrepresentation W of V*

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(d) *restriction to $\mathrm{Rep}_\alpha(Q)^{\mathrm{s}}$ is geometric quotient*

Schofield Semi-Invariants

Definition (Euler/Ringel Form)

for α, β dimension vectors

$$\langle \alpha, \beta \rangle = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha)$$

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Definition

if $V \in \mathrm{Rep}_\alpha(Q)$, $W \in \mathrm{Rep}_\beta(Q)$, define

$$d^V_W : \bigoplus_{x \in Q_0} \mathrm{Hom}_K(V(x), W(x)) \to \bigoplus_{a \in Q_1} \mathrm{Hom}_K(V(ta), W(ha))$$

by

$$(\phi(x), x \in Q_0) \mapsto (\phi(ha)V(a) - W(a)\phi(ta), a \in Q_1)$$

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suppose that $\langle \alpha, \beta \rangle = 0$
d_W^V is a square matrix

Definition (Schofield 1991)

$c(V, W) = c^V(W) = c_V(W) = \det d_W^V$

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$\ker d_W^V = \mathrm{Hom}_Q(V, W)$, $\mathrm{coker}\, d_W^V = \mathrm{Ext}^1_Q(V, W)$, so

$$c(V, W) = 0 \Leftrightarrow \mathrm{Hom}_Q(V, W) = 0 \Leftrightarrow \mathrm{Ext}^1_Q(V, W) = 0$$

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Theorem (Schofield 1991)

$c^V \in \mathrm{SI}(Q,\beta)_{\langle\alpha,\cdot\rangle}$
$c_W \in \mathrm{SI}(Q,\alpha)_{-\langle\cdot,\beta\rangle}$

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Theorem (D.-Weyman 2000)

$\mathrm{SI}(Q, \beta)$ *spanned by Schofield semi-invariants* c^V *where* $V \in \mathrm{Rep}_\alpha(Q)$ *and* α *a dimension vector with* $\langle \alpha, \beta \rangle = 0$

(a similar statement is true for c_W's)

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Theorem (D.-Weyman 2000)

$\mathrm{SI}(Q,\beta)_{\langle \alpha, \cdot \rangle}$ *and* $\mathrm{SI}(Q,\alpha)_{-\langle \cdot, \beta \rangle}$ *are dual and have same dimension*

Saturation for Semi-Invariants, etc.

$\mathrm{char}(K) = 0$

Theorem (D.-Weyman 2002)

$\dim \mathrm{SI}(Q, \alpha)_{n\sigma}$ *is a polynomial in n*

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Theorem (D.-Weyman 2000, Generalized Saturation Conjecture)

if $\dim \mathrm{SI}(Q, \alpha)_{\sigma} = 0$ *then* $\dim \mathrm{SI}(Q, \alpha)_{n\sigma} = 0$ *for all* $n \geq 1$

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Theorem (Sherman 2016, gen. King-Tollu-Toumazet conj.)

if $\dim \mathrm{SI}(Q, \alpha)_{\sigma} = 2$ *then* $\dim \mathrm{SI}(Q, \alpha)_{n\sigma} = n + 1$ *for all* $n \geq 1$

Application to Littlewood-Richardson Coefficients

irreducible representations of GL_p are V_λ where λ is a partition (Young diagram)

$$c^{\nu}_{\lambda,\mu} = \dim \mathrm{Hom}(V_\mu, V_\lambda \otimes V_\mu)^{\mathrm{GL}_p}$$

is multiplicity of V_ν in $V_\lambda \otimes V_\mu$.

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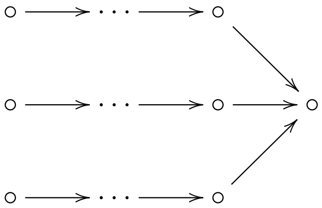

Then $c^{\nu}_{\lambda,\mu} = \dim \mathrm{SI}(Q,\alpha)_\sigma$ for some α, σ and $\dim \mathrm{SI}(Q,\alpha)_{n\sigma} = c^{\nu}_{n\lambda,n\mu}$

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Theorem (Knutson-Tao 1999, Klyachko Saturation Conjecture)

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if $c^{\nu}_{\lambda,\mu} = 1$ then $c^{n\nu}_{n\lambda,n\mu} = 1$ for all $n \geq 1$

Saturation for LR-Coefficients, etc.

Theorem (D.-Weyman 2002)

$c^{\nu}_{n\lambda,n\mu}$ *is a polynomial in* n

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Theorem (Sherman 2015, King-Tollu-Toumazet conj.)

if $c^{\nu}_{\lambda,\mu} = 2$ *then* $c^{n\nu}_{n\lambda,n\mu} = n + 1$ *for all* $n \geq 1$

Semi-Invariants as determinants

suppose that $x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_s \in Q_0$ (possible repetition)
$p_{i,j}$ linear combination of paths from x_i to y_j, and
α is a dimension vector with $\sum_{i=1}^r \alpha(x_i) = \sum_{j=1}^s \alpha(y_j)$, then

$$V \in \mathrm{Rep}_\alpha(Q) \mapsto \det \begin{pmatrix} V(p_{1,1}) & \cdots & V(p_{s,r}) \\ \vdots & & \vdots \\ V(p_{s,1}) & \cdots & V(p_{s,r}) \end{pmatrix}$$

is a semi-invariant of weight $\sigma = \sum_{i=1}^r \mathbf{1}_{x_i} - \sum_{j=1}^s \mathbf{1}_{y_j}$

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Theorem (Domokos-Zubkov 2001)

$\mathrm{SI}(Q, \alpha)$ *is spanned by such semi-invariants*

Semi-Invariants of Generalized Kronecker Quiver

Q quiver with two vertices, x_1, x_2, and m arrows from x_1 to x_2
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Definition

for $T = (T_1, T_2, \ldots, T_m) \in \mathrm{Mat}_{d,d}^m$ define

$$f_T(A_1, \ldots, A_m) = \det(\textstyle\sum_{i=1}^m A_i \otimes T_i)$$

semi-invariant of weight $(d, -d)$ and degree dn

($\otimes$ is Kronecker product for matrices)

Domokos-Zubkov Thm: f_T's span $\mathrm{SI}(Q, \alpha) = K[\mathrm{Mat}_{n,n}^m]^{\mathrm{SL}_n \times \mathrm{SL}_n}$

Degree Bounds for Matrix Invariants

Theorem (Ivanyos-Qiao-Subrahmanyan)

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Theorem (D.-Makam 2017, Visu Makam thesis 2018)

$K[\mathrm{Mat}_{n,n}^m]^{\mathrm{SL}_n \times \mathrm{SL}_n}$ *generated by invariants of degree* $< mn^4$

(we may replace mn^4 by n^6)

About the Proof ...

King's criterion: $A = (A_1, A_2, \dots, A_m) \in \mathrm{Mat}_{n,n}^m$ lies on the nullcone $\mathcal{N}$ (i.e., is not σ-semistable) if and only if there exists a subspaces W_1, W_2 of K^n such that

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for all i, and $\dim W_2 < \dim W_1$

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f_T has degree $(n-1)n$, so together with Popov's bound (from earlier) we see that the invariant ring is generated in degree $\leq \dim \mathrm{Mat}^m_{n,n}((n-1)n) < mn^4$

Ivanyos, Qiao Subrahmanyan Regularity Lemma

Theorem (Ivanyos, Qiao, Subrahmanyan 2016)

For given $A_1, \ldots, A_m \in \mathrm{Mat}_{n,n}^m$ *and* generic $T_1, \ldots, T_m \in \mathrm{Mat}_{d,d}^m$ *the rank of* $\sum_{i=1}^m A_i \otimes T_i$ *is divisible by* d*.*

Example

Example: $n = m = 3$ and take:

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

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If $T = (t_1, t_2, t_3) \in \mathrm{Mat}_{1,1}^3$, then

$$f_T(A) = \det \begin{pmatrix} 0 & t_1 & t_2 \\ -t_1 & 0 & t_3 \\ -t_2 & -t_3 & 0 \end{pmatrix} = 0$$

because matrix is skew-symmetric of odd size

However, if $T = (T_1, T_2, T_3)$ then

$$T_1 \otimes A_1 + T_2 \otimes A_2 + T_3 \otimes A_3 = \begin{pmatrix} 0 & T_1 & T_2 \\ -T_1 & 0 & T_3 \\ -T_2 & -T_3 & 0 \end{pmatrix}$$

can be invertible, for example if we take $T_1 = I$, then

$$\begin{pmatrix} 0 & I & T_2 \\ -I & 0 & T_3 \\ -T_2 & -T_3 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & I & T_2 \\ -I & 0 & T_3 \\ -0 & 0 & T_3 T_2 - T_2 T_3 \end{pmatrix}$$

now take

$$T_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } T_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

so $A = (A_1, A_2, A_3)$ is semistable

Theorem (D.-Makam)

There exists $A = (A_1, A_2, \ldots, A_{d+1}) \in \mathrm{Mat}_{d^2-1,d^2-1}^{d+1}$ *with*

(a) *for all* $e \leq d$ *and all* $T \in \mathrm{Mat}_{e,e}^{d+1}$ $f_T(A) = 0$

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For $d = 3$, take $A = (A_1, A_2, A_3, A_4) \in \mathrm{Mat}^4_{8,8}$ such that

$$\sum_{i=1}^{4} A_i \otimes T_i = \left[\begin{array}{cc|cc|c|ccc}
T_1 & & & & & T_3 & & \\
-T_2 & T_1 & & & & & T_3 & \\
 & -T_2 & & & & & & T_3 \\
\hline
 & & T_1 & & & T_4 & & \\
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\hline
 & & & & T_1 & T_2 & & \\
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\end{array}\right]$$

Commutative Rank

K infinite field
if $A = (A_1, \ldots, A_m) \in \mathrm{Mat}_{n,n}^m$ then $A(t) = t_1 A_1 + \cdots + t_m A_m$
called **linear matrix**
$A(t)$ is $n \times n$ matrix whose entries are linear in $t_1, \ldots, t_m$

Definition

commutative rank of $A(t)$ is

$$\mathrm{cr}(A(t)) = \max\{\mathrm{rank}\, A(t_1, \ldots, t_m) \mid t_1, \ldots, t_m \in K\}$$

Non-Commutative Rank

$S = K \langle\!\langle t_1, \dots, t_m \rangle\!\rangle$ is free skew-field generated by $t_1, \dots, t_m$
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The **noncommutative rank** $\operatorname{ncrk}(A(t))$ of $A(t)$ is r, where r is the rank of the image of $A(t) : S^n \to S^n$ as a free S-module

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Theorem

$$\operatorname{ncrk} A(t) = \max\{\operatorname{rank}(A(T))/d \mid d \geq 1, T \in \operatorname{Mat}_{d,d}^m\}.$$

(this is an integer by regularity lemma)

Inequalities for Comm. and Non-Comm. Rank

Clearly, $\mathrm{crk}(A(t)) \leq \mathrm{ncrk}(A(t))$ We have seen that

$$A(t) = \begin{pmatrix} 0 & t_1 & t_2 \\ -t_1 & 0 & t_3 \\ -t_2 & -t_3 & 0 \end{pmatrix}$$

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Theorem (Flanders)

$$\operatorname{ncrk}(A(t)) \leq 2\operatorname{crk}(A(t)).$$

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$$\text{ncrk}(A(t)) \leq \text{ncrk}(B(t)\ \ C(t)) + \text{ncrk}(D(t)) \leq r + r = 2r.$$

($B(t)$ and $C(t)$ have r rows, $D(t)$ has r columns)

Theorem (D.-Makam 2016)

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then $\operatorname{crk}(A(t)) = \binom{2p}{p}$ and $\operatorname{ncrk}(A(t)) = \binom{2p+1}{p}$ and $\operatorname{ncrk}(A(t))/\operatorname{crk}(A(t)) = \frac{2p+1}{p+1} \to 2$ as $p \to \infty$