An Introduction to Invariant Theory

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Optimization, Complexity and Invariant Theory
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Plan of the Talk

- applications of invariants

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- a classical, motivating example : binary forms

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- polarization of invariants and Weyl's Theorem

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- polarization of invariants and Weyl's Theorem
- Invariant Theory for other fields

Applications of Invariants

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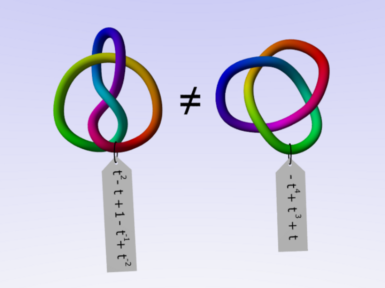
loop invariants can be used to prove the correctness of an algorithm although the number of iterations in a loop may vary, the loop invariant tell us to say something about the variables after the iterations

Knot invariants (such as the Jones polynomial) can be used to distinguish knots



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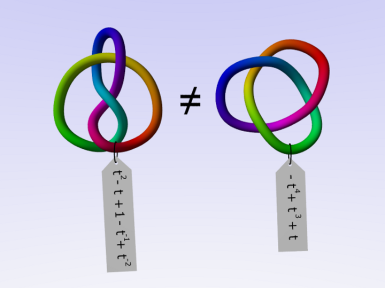

knot invariants remain unchanged under Reidemeister moves

(co-)homology groups are invariants of topological manifolds

Invariant Theory

in invariant theory we restrict ourselves to

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- invariants that remain unchanged under *group symmetries* such as rotations, permutations etc.

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we start with a motivating example from 19th century invariant theory

Classical Invariant Theory: Binary Forms

a *binary form* of degree 2 is a polynomial

$$p(z, w) = p_1 z^2 + p_2 zw + p_3 w^2$$

with $p_1, p_2, p_3 \in \mathbb{C}$

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$$\mathrm{SL}_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\}$$

is the group of 2×2 matrices with determinant one
a matrix $A \in \mathrm{SL}_2$ gives a linear change of coordinates in $\mathbb{C}^2$

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the group SL_2 acts on (the coefficients of) binary forms:
we make the substitution $(z,w) \mapsto (az + cw, bz + dw)$ and get another polynomial

$$p'(z,w) = p(az + cw, bz + dw) = p'_1 z^2 + p'_2 zw + p'_3 w^2$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2,$$

$$\begin{pmatrix} p_1' \\ p_2' \\ p_3' \end{pmatrix} = M_A \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

and

$$M_A = \begin{pmatrix} a^2 & ab & b^2 \\ ac & ad + bc & bd \\ c^2 & cd & d^2 \end{pmatrix}$$

Classical Invariant Theory: Binary Forms

the polynomial $f(x_1, x_2, x_3) = x_2^2 - 4x_1x_3 \in \mathbb{C}[x_1, x_2, x_3]$ (the discriminant) can be viewed as a function from $\mathbb{C}^3$ to $\mathbb{C}$ and an easy calculation shows that

$$f\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = p_2^2 - 4p_1p_3 = (p_2')^2 - 4p_1'p_3' = f\begin{pmatrix} p_1' \\ p_2' \\ p_3' \end{pmatrix}$$

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we say that $f(x_1, x_2, x_3)$ is an *invariant* under the action of SL_2

$f(x_1, x_2, x_3)$ is a *fundamental* invariant that generates all invariants: if $h(x_1, x_2, x_3)$ is another polynomial invariant, then there exists a polynomial $q(y)$ such that $h(x_1, x_2, x_3) = q(f(x_1, x_2, x_3))$

we may identify binary forms of degree n with vectors in $\mathbb{C}^{n+1}$:

$$p_1 z^n + p_2 z^{n-1} w + \cdots + p_{n+1} w^n \quad \leftrightarrow \quad \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_{n+1} \end{pmatrix}$$

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the vector space of binary forms of degree n is an $(n+1)$-dimensional representation of SL_2

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one of the main objectives was to find an explicit system of fundamental invariants for binary forms up to degree d

(currently known for $d \leq 10$)

The Polynomial Ring

$x_1, x_2, \ldots, x_n$ coordinate functions on $V = \mathbb{C}^n$
a polynomial $f(x_1, \ldots, x_n)$ can be viewed as function from V to $\mathbb{C}$
$\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \ldots, x_n]$ *graded ring* of polynomial functions

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Definition (Ideal)

a subset $I \subseteq \mathbb{C}[\mathbf{x}]$ is an *ideal* if

1. $0 \in I$;
2. $f(\mathbf{x}), g(\mathbf{x}) \in I \Rightarrow f(\mathbf{x}) + g(\mathbf{x}) \in I$;
3. $f(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]$, $g(\mathbf{x}) \in I \Rightarrow f(\mathbf{x})g(\mathbf{x}) \in I$.

Hilbert's Basis Theorem

the *ideal* (S) generated by a subset $S \subseteq \mathbb{C}[\mathbf{x}]$ is

$$\{a_1(\mathbf{x})f_1(\mathbf{x})+\cdots+a_r(\mathbf{x})f_r(\mathbf{x}) \mid r \in \mathbb{N}, \forall i\, a_i(\mathbf{x}) \in \mathbb{C}[\mathbf{x}], f_i(\mathbf{x}) \in S\}$$

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every ideal $I \subseteq \mathbb{C}[\mathbf{x}]$ is generated by a finite set ($\mathbb{C}[\mathbf{x}]$ is noetherian) the *ideal* (S) generated by a subset $S \subseteq \mathbb{C}[\mathbf{x}]$ is

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Hilbert used this theorem to prove a his Finiteness Theorem in Invariant Theory (discussed later)

Action of a Group G

suppose $V = \mathbb{C}^n$ is a representation of a group G
this means that every $g \in G$ acts my some $n \times n$ matrix $M_g : V \to V$ (so $g \cdot v = M_g v$)

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if $f(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]$ and $M = (m_{i,j})$ is $n \times n$ matrix, then $v \mapsto f(Mv)$ is a polynomial function given by the formula

$$f\Big(\sum_{j=1}^{n} m_{1,j} x_j, \dots, \sum_{j=1}^{n} m_{n,j} x_j\Big)$$

Action of a Group G

G acts on $\mathbb{C}[\mathbf{x}]$ as follows:

if $g \in G$ and $f(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]$ then define $(g \cdot f)(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]$ by $(g \cdot f)(v) = f(M_{g^{-1}} v)$

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$\mathbb{C}[\mathbf{x}]$ is an ∞-dimensional $\mathbb{C}$-vector space
the monomials form a basis
G acts by linear transformations on $\mathbb{C}[\mathbf{x}]$
$\mathbb{C}[\mathbf{x}]$ is an ∞-dimensional representation of G

The Invariant Ring

$f(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]$ is G-invariant if $(g \cdot f)(\mathbf{x}) = f(\mathbf{x})$ for all $g \in G$
$f(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]$ is G-invariant if and only if it is constant on all G-orbits in V

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Definition

$\mathbb{C}[\mathbf{x}]^G$ is the set of all G-invariant polynomials in $\mathbb{C}[\mathbf{x}]$

$\mathbb{C}[\mathbf{x}]^G$ is a *subalgebra*, i.e., contains $\mathbb{C}$ and is closed under addition, subtraction and multiplication

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if $f_1(\mathbf{x}), \dots, f_r(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]$ then

$$\mathbb{C}[f_1(\mathbf{x}), \dots, f_r(\mathbf{x})] := \{p(f_1(\mathbf{x}), \dots, f_r(\mathbf{x})) \mid p(y_1, \dots, y_r) \in \mathbb{C}[y_1, \dots, y_r]\}$$

is the subalgebra of $\mathbb{C}[\mathbf{x}]$ generated by $f_1(\mathbf{x}), \dots, f_r(\mathbf{x})$.

The Symmetric Group

$G = S_n$ acts on $V = \mathbb{C}^n$ by permuting the coordinates
for $\sigma \in S_n$, M_σ is the corresponding permutation matrix
S_n acts on $\mathbb{C}[\mathbf{x}]$ as

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$$(\sigma \cdot f)(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

define the k-th elementary symmetric function as

$$e_k(\mathbf{x}) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}$$

for example $e_1 = x_1 + x_2 + \cdots + x_n$ and $e_n = x_1 x_2 \cdots x_n$

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Theorem

$\mathbb{C}[\mathbf{x}]^{S_n} = \mathbb{C}[e_1(\mathbf{x}), \ldots, e_n(\mathbf{x})]$

Hilbert's Finiteness Theorem

assume that G is (linearly) reductive, which means that every representation of G is a direct sum of irreducible representations
examples are GL_n, SL_n, O_n, finite groups

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Theorem (Hilbert 1890)

$\mathbb{C}[\mathbf{x}]^G$ *is a finitely generated algebra, i.e.,*
$\mathbb{C}[\mathbf{x}]^G = \mathbb{C}[f_1(\mathbf{x}), \dots, f_r(\mathbf{x})]$ *for some* $r < \infty$ *and*
$f_1(\mathbf{x}), \dots, f_r(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]^G$

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proof sketch:
$J \subseteq \mathbb{C}[\mathbf{x}]$ ideal generated by all homogeneous, non-constant $f(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]^G$ (∞ many!)

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by induction one shows that $\mathbb{C}[\mathbf{x}]^G = \mathbb{C}[f_1(\mathbf{x}), \ldots, f_r(\mathbf{x})]$

Degree Bounds

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Theorem (Emmy Noether 1916)

if G is finite then $\beta(\mathbb{C}[\mathbf{x}]^G) \leq |G|$

A Constructive Proof

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so Hilbert gave *another, more constructive* proof in 1893 of his Finiteness Theorem using his notion of the *null cone*

Hilbert's Null cone

for $v \in V$, $G \cdot v = \{g \cdot v \mid g \in G\}$ is orbit of v

$\overline{G \cdot v} \subseteq V$ closure of the orbit

Theorem

$\overline{G \cdot v} \cap \overline{G \cdot w} \neq \emptyset \Leftrightarrow f(v) = f(w)$ *for all* $f(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]^G$

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$\Rightarrow$: $f \in \mathbb{C}[\mathbf{x}]^G$ is constant on $\overline{G \cdot v}$ and $\overline{G \cdot w}$

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Definition

Hilbert's Null cone:

$$\mathcal{N} := \{v \in V \mid 0 \in \overline{G \cdot v}\} = \\ = \{v \in V \mid f(v) = f(0) \text{ for all } f(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]^G\}$$

if $\mathbb{C}[\mathbf{x}]^G = \mathbb{C}[f_1(\mathbf{x}), \ldots, f_r(\mathbf{x})]$ with $f_1(\mathbf{x}), \ldots, f_r(\mathbf{x})$ homogeneous, non-constant, then $\mathcal{N} = \{v \in V \mid f_1(v) = \cdots = f_r(v) = 0\}$

Example: Multiplicative Group

$G = \mathbb{C}^\star$, $V = \mathbb{C}^4$
for $t \in \mathbb{C}^\star$, define

$$M_t = \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & t^{-1} & 0 \\ 0 & 0 & 0 & t^{-1} \end{pmatrix}$$

$$t \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = M_t \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} tv_1 \\ tv_2 \\ t^{-1}v_3 \\ t^{-1}v_4 \end{pmatrix}$$

$$\mathcal{N} = \{v_1 = v_2 = 0\} \cup \{v_3 = v_4 = 0\}$$

Example: Multiplicative Group

$$\mathbb{C}[x_1, x_2, x_3, x_4]^{\mathbb{C}^\star} = \mathbb{C}[x_1x_3, x_1x_4, x_2x_3, x_2x_4]$$

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Note that in this case, there is an algebraic relation between the generators, namely

$$(x_1x_3)(x_2x_4) = (x_1x_4)(x_2x_3)$$

Hilbert-Mumford criterion

Definition

a one parameter subgroup (1-PSG) is a homomorphism of algebraic groups $\lambda : \mathbb{C}^{\star} \to G$

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Theorem (Hilbert-Mumford criterion)

if $v \in V = \mathbb{C}^n$*, then*
$v \in \mathcal{N} \Leftrightarrow$ *there exists a 1-PSG* $\lambda : \mathbb{C}^\star \to G$ *with* $\lim_{t \to 0} \lambda(t) \cdot v = 0$

Conjugation of $n \times n$ Matrices

$V = \mathrm{Mat}_{n,n}$, the space of $n \times n$ matrices
$G = \mathrm{GL}_n$ (the group of invertible $n \times n$ matrices) acts by conjugation: if $A = (a_{i,j}) \in V$ and $g \in G$ then $g \cdot A = gAg^{-1}$

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if

$$(\star) \quad \lambda(t) = \begin{pmatrix} t^{k_1} & & \\ & \ddots & \\ & & t^{k_n} \end{pmatrix}$$

with $k_1 \geq k_2 \geq \cdots \geq k_n$, then

$$\lambda(t) \cdot A = \lambda(t) A \lambda(t)^{-1} = (t^{k_i - k_j} a_{i,j}).$$

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every 1-PSG is of the form $(\star)$ after a base change, so

$A \in \mathcal{N} \Leftrightarrow A$ conjugate to strict upper triang. mat. $\Leftrightarrow A$ is nilpotent

Conjugation of $n \times n$ Matrices

$X = (x_{i,j})$ where $x_{i,j}$ are indeterminates

$$\det(tI - X) = t^n - f_1(\mathbf{x})t^{n-1} + \cdots + (-1)^n f_n(\mathbf{x})$$

where $\mathbf{x} = x_{1,1}, x_{1,2}, \ldots, x_{n,n}$
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Theorem

$$\mathbb{C}[\mathbf{x}]^G = \mathbb{C}[f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})]$$

$A \in \mathcal{N} \Leftrightarrow f_1(A) = \cdots = f_n(A) = 0 \Leftrightarrow \det(tI - A) = t^n \Leftrightarrow A$ nilpotent

Theorem (Hilbert 1893)

suppose $f_1(\mathbf{x}), \dots, f_r(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]^G$ *are homogeneous and*
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then there exists finitely many homogenous invariants $h_1(\mathbf{x}), \dots, h_s(\mathbf{x})$ *such that every invariant* $p(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]^G$ *is of the form*

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Degree Bounds

Theorem (Popov 1980)

suppose $f_1(\mathbf{x}), \dots, f_r(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]^G$ *are homogeneous of the* **same** *degree* d *and* $\mathcal{N} = \{v \mid f_1(v) = \cdots = f_r(v) = 0\}$

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for some $a_1(\mathbf{x}), \dots, a_s(\mathbf{x}) \in \mathbb{C}[f_1(\mathbf{x}), \dots, f_r(\mathbf{x})]$

in particular, $\beta(\mathbb{C}[\mathbf{x}]^G) \leq \max\{d, n(d-1)\} \leq nd$
(because $\mathbb{C}[\mathbf{x}]^G = \mathbb{C}[f_1(\mathbf{x}), \dots, f_r(\mathbf{x}), h_1(\mathbf{x}), \dots, h_s(\mathbf{x})]$*)*

Polynomial Degree Bounds

Theorem (D.)

suppose $f_1(\mathbf{x}), \dots, f_r(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]^G$ *are homogeneous of the degree* **at most** d *and* $\mathcal{N} = \{v \mid f_1(v) = \cdots = f_r(v) = 0\}$

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for binary forms of degree n, the null cone is defined by homogeneous invariants of degree $\leq 2n^3$, and we get $\beta(\mathbb{C}[\mathbf{x}]^G) \leq \frac{3}{8}(n+1)(2n^3)^2$ (which is slightly worse than Jordan's bound)

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if G is fixed then the bound $\beta(\mathbb{C}[\mathbf{x}]^G)$ is polynomial in n (the dimension of V) and the largest euclidean length among the weights appearing in the representation

Polarization

V representation of G
$\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \ldots, x_n]$ ring of polynomial functions on $V = \mathbb{C}^n$
$\mathbb{C}[\mathbf{x}, \mathbf{y}]$ of polynomial functions on $V \oplus V = \mathbb{C}^{2n}$

for $f(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]^G$ we can write

$$f(x_1 + ty_1, \ldots, x_n + ty_n) = f_0(\mathbf{x}, \mathbf{y}) + f_1(\mathbf{x}, \mathbf{y})t + \cdots + f_d(\mathbf{x}, \mathbf{y})t^d$$

where $f_0(\mathbf{x}, \mathbf{y}), \ldots, f_d(\mathbf{x}, \mathbf{y}) \in \mathbb{C}[\mathbf{x}, \mathbf{y}]^G$

Polarization

$R[m] := \mathbb{C}[x_{1,1}, \dots, x_{n,1}, \dots, x_{1,m}, \dots, x_{n,m}]$ is the ring of polynomial functions on $V^m \cong \mathrm{Mat}_{n,m}$

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if $s > n = \dim V$ then polarizing generators from $R[n]^G$ give generators of $R[s]^G$.

in particular, $\beta(R[s]^G) = \beta(R[n]^G)$

Other Fields

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most statements are either false, or more difficult to prove in positive characteristic

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invariant rings of reductive groups are also finitely generated in positive characteristic, but the proof is harder (using theorems of Nagata and Haboush) and many of the geometric statements about the null cone etc. are still true