LOCALLY NILPOTENT DERIVATIONS

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We will assume that K is a field and R is a commutative K-algebra with $1 \neq 0$.

1. Derivations

Definition 1.1. A K-*derivation* D on a ring R is a K-linear map $D : R \to R$ that satisfies the Leibniz (or product) rule:

$$D(fg) = fD(g) + D(f)g.$$

The set of K-derivations on R is denoted by $\mathrm{Der}_K(R)$.

For the remainder of the section we will assume that D is a K-derivation on R. If $f \in R$ then fD is again a derivation. One can verify that this makes $\mathrm{Der}_K(R)$ into an R-module.

Exercise 1.2. If $f_1, \dots, f_r \in R$, show that

$$D(f_1 f_2 \cdots f_r) = D(f_1)f_2 \cdots f_r + f_1 D(f_2) f_3 \cdots f_r + \cdots + f_1 f_2 \cdots f_{r-1} D(f_r).$$

Exercise 1.3. For $f \in R$, show that $D(f^n) = nf^{n-1}D(f)$.

Exercise 1.4. Suppose that $f, g \in R$. Show that

$$D^n(fg) = \sum_{i=0}^{n} \binom{n}{i} D^i(f) D^{n-i}(g)$$

for all nonnegative integers n.

Exercise 1.5 (Chain Rule)**.** Suppose that $g(Y_1, \dots, Y_m) \in K[Y_1, \dots, Y_m]$ and $f_1, f_2, \dots, f_m \in R$. Then we have

$$D(g(f_1, \dots, f_m)) = \sum_{i=1}^{m} \frac{\partial g}{\partial Y_i}(f_1, \dots, f_m) D(f_i)$$

(Hint: first prove this in the case where g is a monomial.)

If E is another K-derivations, then the composition DE may not be a K-derivation. However, the *Lie bracket* of D and E defined by

$$[D, E] = DE - ED$$

is again a K-derivation. Indeed, for $f, g \in R$ we have

$$\begin{aligned}
[D, E](fg) &= D(E(fg)) - E(D(fg)) = D(E(f)g + fE(g)) - E(D(f)g + fD(g)) = \\
&= D(E(f))g + E(f)D(g) + D(f)E(g) + fD(E(g)) \\
&\quad - E(D(f))g - D(f)E(g) - E(f)D(g) - fE(D(g)) = [D, E](f) \cdot g + f \cdot [D, E](g).
\end{aligned}$$

This makes $\mathrm{Der}_K(R)$ into a Lie algebra.

Exercise 1.6. Suppose that σ is a K-algebra automorphism of R. Show that $\sigma D\sigma^{-1}$ is again a K-derivation on R.

The kernel of D is

$$R^D := \ker(D) := \{f \in R \mid D(f) = 0\}.$$

If $f, g \in \ker(D)$, then we have $D(fg) = fD(g) + gD(f) = 0 + 0 = 0$. So $fg \in \ker(D)$. This shows that $\ker(D)$ is a K-subalgebra of R.

The set of units in R is denoted by $R^\times$. If $f \in R^\times$, then we have

$$0 = D(1) = D(ff^{-1}) = D(f)f^{-1} + fD(f^{-1}),$$

so

$$D(f^{-1}) = \frac{-D(f)}{f^2}.$$

In particular, if $f \in R^D$ is a unit, then f^{-1} lies in R^D as well.

We now easily deduce the quotient rule

$$D\left(\frac{f}{g}\right) = D(fg^{-1}) = (Df)g^{-1} + fD(g^{-1}) = \frac{D(f)}{g} + \frac{-fD(g)}{g^2} = \frac{D(f)g - fD(g)}{g^2}.$$

Exercise 1.7. Show that D is an R^D-module homomorphism from R to itself.

An ideal I of R is called D-stable if $D(I) \subseteq I$.

Exercise 1.8. Suppose that I is a D-stable ideal. Show that D induces a derivation $\overline{D}$ on R/I by

$$\overline{D}(f + I) := D(f) + I.$$

Exercise 1.9. Suppose that R is a domain and $f \in R$ is nonzero. Show that D extends uniquely to the localized ring R_f. Also show that D extends uniquely to the quotient field $Q(R)$.

Exercise 1.10. Suppose that R is a domain and S is a domain containing R such that every element of S is algebraic over R (i.e., satisfies a non-constant polynomial with coefficients in R). Then D extends uniquely to a derivation on S.

Exercise 1.11. Suppose that R is a domain. Show that if $f \in R$ is algebraic over R^D, then $f \in R^D$.

Exercise 1.12. Suppose that R is a domain, $f_1, \dots, f_n \in R$, $D_1, \dots, D_n$ are derivations on R and consider the matrix

$$J = \begin{pmatrix} D_1(f_1) & D_2(f_1) & \cdots & D_n(f_1) \\ D_1(f_2) & D_2(f_2) & \cdots & D_n(f_2) \\ \vdots & \vdots & \ddots & \vdots \\ D_1(f_n) & D_2(f_n) & \cdots & D_n(f_n) \end{pmatrix}.$$

Show that if J is invertible, then $f_1, \dots, f_n$ are algebraically independent over K.

Example 1.13. Assume D is a derivation on $R = K[X_1, X_2, \dots, X_n]$, the polynomial ring in n variables. From Exercise 1.5 follows that for every $f \in R$ we have

$$D(f) = D(X_1)\frac{\partial f}{\partial X_1} + D(X_2)\frac{\partial f}{\partial X_2} + \cdots + D(X_n)\frac{\partial f}{\partial X_n}.$$

It follows that D on $R = K[X_1, \dots, X_n]$ is equal to

$$D(X_1)\frac{\partial}{\partial X_1} + \cdots + D(X_n)\frac{\partial}{\partial X_n}.$$

So $\mathrm{Der}_K(R)$ is a free R-module generated by $\frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_n}$.

For $f \in R$, we define an ideal

$$\mathfrak{a}(f) = \{D(f) \mid D \in \mathrm{Der}_K(R)\}.$$

Geoometrically, we can think of this ideal as the ideal of critical points of f.

Exercise 1.14. For a K-algebra automorphism σ of R, show that $\mathfrak{a}(\sigma(f)) = \sigma(\mathfrak{a}(f))$.

2. Locally nilpotent derivations

Definition 2.1. A derivation on a ring R is called *locally nilpotent* if for every $f \in R$ there exists a nonnegative integer n such that $D^n(f) = 0$.

Example 2.2. The derivation $\frac{\partial}{\partial X}$ on the polynomial ring $K[X]$ is locally nilpotent. Indeed, $\frac{\partial}{\partial X}$ decreases the degree by 1. If $f \in K[X]$ is a polynomial of degree d then $\left(\frac{\partial}{\partial X}\right)^{d+1} f = 0$.

Example 2.3. The derivations $\frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_n}$ on $K[X_1, \dots, X_n]$ are locally nilpotent.

Exercise 2.4. Suppose that D is a derivation on a ring R, $f, g \in R$ and $D^{n+1}(f) = D^{m+1}(g)$. Then we have $D^{n+m+1}(fg) = 0$. Show that the set

$$S = \{f \in R \mid \text{there exists a nonnegative integer } n \text{ with } D^n(f) = 0\}.$$

is a subring of R.

Lemma 2.5. *Suppose that D is a derivation on a K-algebra R and R is generated as a K-algebra by $f_1, f_2, \dots, f_r$. Then D is locally nilpotent if and only if there exists a positive integer N such that $D^N(f_1) = D^N(f_2) = \cdots = D^N(f_r) = 0$.*

Proof. Suppose that $D^N(f_1) = \cdots = D^N(f_r) = 0$. Define S as in Exercise 2.4. Then S contains K and $f_1, \dots, f_r$. It follows that $S = R$ and D is locally nilpotent. The implication in the other direction is trivial. $\square$

Corollary 2.6. *A K-derivation D on $K[X_1, \dots, X_n]$ is locally nilpotent if and only if there exists an N such that $D^N(X_1) = \cdots = D^N(X_n) = 0$.*

Example 2.7. If $g(X) \in K[X]$ is a polynomial then $\frac{\partial}{\partial X} + g(X)\frac{\partial}{\partial Y}$ is locally nilpotent: we have $D^2(X) = D(1) = 0$ and if g has degree r then $D^{r+2}(Y) = D^{r+1}(g(X)) = 0$. By Corollary 2.6 we have that D is locally nilpotent (take $N = r + 2$).

Exercise 2.8. Suppose that $f_i \in K[X_1, \dots, X_{i-1}]$ for $i = 1, 2, \dots, n$ (so $f_1 \in K$ is a constant). Show that the *triangular* K-derivation

$$D = f_1\frac{\partial}{\partial X_1} + f_2(X_1)\frac{\partial}{\partial X_2} + \cdots + f_n(X_1, \dots, X_{n-1})\frac{\partial}{\partial X_n}$$

is locally nilpotent.

Exercise 2.9. Suppose that $f \in K[X_1, \dots, X_n]$ and D is a derivation on $K[X_1, \dots, X_n]$ such that fD is locally nilpotent. Show that D is triangular, and $f \in K[X_1, \dots, X_{n-1}]$.

Exercise 2.10. Suppose that D is a locally nilpotent derivation on R and $f \in \ker(D)$. Then fD is also locally nilpotent.

Example 2.11. The derivation $D = (X_1+X_2)(\frac{\partial}{\partial X_1} - \frac{\partial}{\partial X_2})$ on $K[X_1, X_2]$ is locally nilpotent, because $D^2(X_1) = D(X_1 + X_2) = 0$ and $D^2(X_2) = -D(X_1 + X_2) = 0$. The derivation D is not triangular. However, we can choose different coordinates. Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1$. Then $K[X_1, X_2] = K[Y_1, Y_2]$ and

$$D = Y_1 \frac{\partial}{\partial Y_2}.$$

So D is triangular with respect to these new coordinates. If σ is the K-automorphism sending Y_1 to X_1 and Y_2 to X_2, then we have

$$\sigma D \sigma^{-1} = X_1 \frac{\partial}{\partial X_2}.$$

It is a natural question to ask whether every locally nilpotent derivation is *triangularizable*: Does there always exist for every locally nilpotent derivation an automorphism σ such that $\sigma D \sigma^{-1}$ is triangular? The answer is no:

Example 2.12. Consider the derivation $E = X\frac{\partial}{\partial Y} + Y\frac{\partial}{\partial Z}$ is locally nilpotent because it is triangular. The polynomial $Y^2 - 2XZ$ lies in the kernel of E, so the derivation

$$D = (Y^2 - 2XZ)E = (Y^2 - 2XZ)(X\frac{\partial}{\partial Y} + Y\frac{\partial}{\partial Z})$$

on the ring $\mathbb{C}[X, Y, Z]$ is also locally nilpotent. We will show later that D is not triangularizable.

However, in 2 variables every locally nilpotent derivation is triangularizable.

Theorem 2.13. *Suppose that K is a field of characteristic 0 and D is a locally nilpotent K-derivation on $K[X, Y]$. Then D is triangularizable.*

3. The exponential map

In this section we will assume that K is a field of characteristic 0 and that D is locally nilpotent. We define a map $\Psi : R \to R[T]$ by

$$\Psi = \exp(TD) = \sum_{i=0}^{\infty} \frac{T^i D^i}{i!}.$$

This definition makes sense, because for every $f \in R$, $D^i(f) = 0$ for $i \gg 0$, and

$$\Psi(f) = \sum_{n=0}^{\infty} \frac{T^n D^n(f)}{i!} \in R[T]$$

is well defined. We have

$$\Psi(fg) = \sum_{n=0}^{\infty} \frac{T^n D^n(fg)}{n!} = \sum_{n=0}^{\infty} \frac{T^i T^{n-i} \sum_{i=0}^{n} \binom{n}{i} D^i(f) D^{n-i}(g)}{i!(n-i)!} =$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{T^i D^i(f)}{i!} \frac{T^j D^j(g)}{j!} = \Psi(f)\Psi(g).$$

This shows that Ψ is an algebra homomorphism.

For $f \in R \setminus \{0\}$ we define its D-degree by

$$\deg_D(f) = \min\{n \mid D^{n+1}(f) = 0\}.$$

We also define $\deg_D(0) = -\infty$. Note that $\deg_D(f)$ is exactly the degree of $\Psi(f)$ as a polynomial in T. So the following lemma is clear:

Lemma 3.1. *We have* $\deg_D(fg) \leq \deg_D(f) + \deg_D(g)$ *with equality if* R *is an integral domain.*

Proof.

$$\begin{aligned}\deg_D(fg) = \deg_T(\Psi(fg)) = \deg_T(\Psi(f)\Psi(g)) \leq \\ \leq \deg_T(\Psi(f)) + \deg_T(\Psi(g)) = \deg_D(f) + \deg_D(g).\end{aligned}$$

If R is an integral domain, then so is $R[T]$ and we have equality. ☐

Lemma 3.2. *Suppose that* D *is a locally nilpotent derivation on* R*, and* R *is an integral domain. If* $f, g \in R \setminus \{0\}$ *and* $fg \in \ker(D)$*, then we have* $f, g \in \ker(D)$*.*

Proof. Since $0 = \deg_D(fg) = \deg_D(f) + \deg_D(g)$ and $\deg_D(f), \deg_D(g) \geq 0$, it follows that $\deg_D(f) = \deg_D(g) = 0$ and $D(f) = D(g) = 0$. ☐

Exercise 3.3. Show that it R is a *unique factorization domain* (UFD), then R^D is also a UFD.

Exercise 3.4. Show that if $D(f) \in (f)$, then we have $D(f) = 0$.

4. Slices and pre-slices

In this section we assume that D is a locally nilpotent derivation on R.

Definition 4.1. A *slice* is an element $s \in R$ such that $Ds = 1$.

If $s \in R$ is a slice, then we define an algebra homomorphism $\Phi : R \to R$ by

$$\Phi(f) = \Psi(f) \mid_{T=-s} .$$

Lemma 4.2. *If* s *is a slice for* D*, then the image of* Φ *is is* R^D*. In particular, if* $f_1, \dots, f_r \in R$ *generate the algebra* R*, then* $\Phi(f_1), \dots, \Phi(f_r)$ *generate the algebra* R^D*.*

Proof. Suppose that $f \in R$ and $D^{n+1}(f) = 0$. Then we have

$$\Phi(f) = \Psi(f) \mid_{T=-s} = \sum_{i=0}^{n} \frac{(-s)^i D^i(f)}{i!}.$$

We have

$$D(\Phi(f)) = -\sum_{i=1}^{n} \frac{(-s)^{i-1} D^i(f)}{(i-1)!} + \sum_{i=0}^{n-1} \frac{(-s)^i D^{i+1}(f)}{i!} = 0.$$

So the image of D is contained in R^D. Note that $\Psi(s) = s + T$ and $\Phi(s) = 0$. If $f \in R^D$, then $\Psi(f) = f$ and $\Phi(f) = f$. This proves that the image of Φ is exactly R^D. ☐

Lemma 4.3. *If $s \in R$ is a slice, then for every $f \in R$ we have*
$$f = \sum_{i=0}^{\infty} \frac{\Phi(D^i(f))s^i}{i!}$$

Proof. We prove the formula by induction on $\deg_D(f)$. The case that $\deg_D(f) = -\infty$ and $\deg_D(f) = 0$ are clear. Suppose that we have proven the formula for all $f \in R$ with $\deg_D(f) \leq k$. If $f \in R$ has D-degree $k+1$ then $D(f)$ has degree k and
$$D(f) = \sum_{i=0}^{\infty} \frac{\Phi(D^i(D(f)))s^i}{i!} = \sum_{i=1}^{\infty} \frac{\Phi(D^i(f))s^{i-1}}{(i-1)!} = D\Big(\sum_{i=0}^{\infty} \frac{\Phi(D^i(f))s^i}{i!}\Big).$$
So
$$f - \sum_{i=0}^{\infty} \frac{\Phi(D^i(f))s^i}{i!} \in R^D.$$
If we apply Φ, we get
$$f - \sum_{i=0}^{\infty} \frac{\Phi(D^i(f))s^i}{i!} = \Phi\Big(f - \sum_{i=0}^{\infty} \frac{\Phi(D^i(f))s^i}{i!}\Big) = \Phi(f) - \Phi(f) = 0.$$
□

Corollary 4.4. *If $s \in R$ is a slice, then we have $R^D[s] = R$.*

Note that $s \notin R^D$, so s is not algebraic over R^D. So we can view R as the polynomial ring in one variable over R^D.

Corollary 4.5. *If $s \in R$ is a slice, then we have $R^D \cong R/(s)$. In particular, R^D is a finitely generated algebra.*

If a locally nilpotent derivation does not have a slice, then the kernel may not be finitely generated as we will see later.

Proposition 4.6. *Suppose that $D_1, D_2, \dots, D_n$ are commuting locally nilpotent derivations on R, $f_1, \dots, f_n \in R$ such that $D_i f_j = \delta_{i,j}$ where $\delta_{i,j}$ is the Kronecker delta symbol ($\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$). Then we have $R = S[f_1, f_2, \dots, f_n]$ where*
$$S = \ker(D_1) \cap \ker(D_2) \cap \cdots \cap \ker(D_n).$$

Proof. The statement can be proven by induction on n. The case $n = 1$ is Corollary 4.4. Suppose that $n > 1$. We have $f_1, \dots, f_{n-1} \in R^{D_n}$. Restricting $D_1, \dots, D_{n-1}$ to R^{D_n} give commuting locally nilpotent derivations on R^{D_n} and we have $D_i(f_j) = \delta_{i,j}$ for $1 \leq i, j \leq n-1$. By the induction hypothesis, we have
$$R^{D_n} = S[f_1, \dots, f_{n-1}].$$
Since f_n is a slice for D_n, we have
$$R = R^{D_n}[f_n] = S[f_1, \dots, f_n]$$
by Corollary 4.4. □

Definition 4.7. An element $f \in R$ is called a *pre-slice* if $D^2(f) = 0$ and $D(f) \neq 0$.

If $f \in R$ is a pre-slice, then $g := D(f) \in R^D \setminus \{0\}$. The locally nilpotent derivation D extends to a derivation on R_g which is again locally nilpotent. The element $s := \frac{f}{g} \in R_g$ is a slice, and we can easily compute generators for the kernel $(R_g)^D$ using Lemma 4.2. Let $\Phi : R_g \to R_g$ be the map defined by

$$\Phi(f) = \Psi(f) \mid_{T=-s} .$$

If R is generated by $f_1, \dots, f_r$, then R_g is generated by $f_1, \dots, f_r, g^{-1}$ and R_g^D is generated by $\Phi(f_1), \dots, \Phi(f_r), \Phi(g^{-1}) = g^{-1}$. If $d_i = \deg_D(f_i)$, then let

$$h_i = g^{d_i}\Phi(f_i) = \sum_{j=0}^{d_i} \frac{D^j(f_i)(-f)^i g^{d_i-j}}{j!} \in R^D$$

for $i = 1, 2 \dots, r$ and

$$(R_g)^D = K[Phi(f_1), \dots, \Phi(f_r), g^{-1}] = K[h_1, \dots, h_r, g^{-1}]$$

Now we observe that

$$R^D = (R_g)^D \cap R = K[h_1, \dots, h_r, g^{-1}] \cap R.$$

Van den Essen's algorithm for finding generators of the kernel R^D is based on this idea. The intersection is computed using Gröbner basis techniques. (The algorithm will only terminate when the kernel R^D is a finitely generated algebra.)

5. Locally nilpotent derivations and the Jacobian Conjecture

Remark 5.1. We would like to view polynomials in $K[X_1, \dots, X_n]$ as functions on K^n. However, we have to be careful. If $K = \mathbb{F}_p$, the field with p elements, then the polynomial $f(X) = X^p - X \in \mathbb{F}_p[X]$ is a nonzero polynomial, but $f(\alpha) = 0$ for all $\alpha \in \mathbb{F}_p$.

Lemma 5.2. *Suppose that K is infinite. Then $f \in K[X_1, \dots, X_n]$ is nonzero, then there exists $x = (x_1, \dots, x_n) \in K^n$ with $f(x_1, \dots, x_n) \neq 0$.*

For the remainder of this section we will assume that K is infinite, so that every polynomial functions on K^n is represented by a unique formal polynomial in $K[X_1, \dots, X_n]$. Polynomial maps $K^n \to K^m$ will be identified with elements of R^m.

Definition 5.3. For a polynomial map $F = (F_1, \dots, F_m) \in R^m$ we define its Jacobi matrix by

$$J(F) = J_X(F) \begin{pmatrix} \frac{\partial F_1}{\partial X_1} & \frac{\partial F_1}{\partial X_2} & \cdots & \frac{\partial F_1}{\partial X_n} \\ \frac{\partial F_2}{\partial X_1} & \frac{\partial F_2}{\partial X_2} & \cdots & \frac{\partial F_2}{\partial X_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial X_1} & \frac{\partial F_m}{\partial X_2} & \cdots & \frac{\partial F_m}{\partial X_n} \end{pmatrix}.$$

Suppose that $F \in R^m$ is a polynomial map, and $G \in S^p$ is a polynomial map where $S = K[Y_1, \dots, Y_m]$. For the composition map $G \circ F$ we get, using the chain rule:

$$J_X(G \circ F) = \begin{pmatrix} \frac{\partial G_1}{\partial Y_1}(F) & \frac{\partial G_1}{\partial Y_2}(F) & \cdots & \frac{\partial G_1}{\partial Y_n}(F) \\ \frac{\partial G_2}{\partial Y_1}(F) & \frac{\partial G_2}{\partial Y_2}(F) & \cdots & \frac{\partial G_2}{\partial Y_n}(F) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial G_n}{\partial Y_1}(F) & \frac{\partial G_n}{\partial Y_2}(F) & \cdots & \frac{\partial G_n}{\partial Y_n}(F) \end{pmatrix} \begin{pmatrix} \frac{\partial F_1}{\partial X_1} & \frac{\partial F_1}{\partial X_2} & \cdots & \frac{\partial F_1}{\partial X_n} \\ \frac{\partial F_2}{\partial X_1} & \frac{\partial F_2}{\partial X_2} & \cdots & \frac{\partial F_2}{\partial X_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial X_1} & \frac{\partial F_n}{\partial X_2} & \cdots & \frac{\partial F_n}{\partial X_n} \end{pmatrix} = J_Y(G) \mid_{Y=F} J_X(F)$$

Definition 5.4. A polynomial map $F : K^n \to K^n$ is called *invertible* if there exists a polynomial map $G : K^n \to K^n$ with $G \circ F = F \circ G = \mathrm{id}$.

Lemma 5.5. *If $F = (F_1, \dots, F_n) \in R^n$ is an invertible polynomial map, then $J(F)$ is an invertible matrix (with entries in R) and $\det J(F) \in K^\times$.*

Proof. If F is invertible, then there exists a polynomial map $G \in K[Y_1, \dots, Y_n]^n$ such that $G \circ F = \mathrm{id}$. We have

$$I = J_X(F \circ G) = J_Y(G) \mid_{Y=F} J_X(F)$$

so $J(F) = J_X(F)$ is an invertible matrix. It follows that $\det(J(F)) \in R^\times = K^\times$. □

Lemma 5.6. *Suppose that $F = (F_1, \dots, F_n) \in R^n$. Then F is invertible if and only if $K[F_1, \dots, F_n] = R$.*

Proof. If F is invertible, then $G \circ F = \mathrm{id}$ for some polynomial map $G : K^n \to K^n$. Then we have $X_i = G_i(F_1, \dots, F_n) \in K[F_1, \dots, F_n]$ for all i, and $R = K[F_1, \dots, F_n]$.

Conversely, suppose that $K[F_1, \dots, F_n] = R$. Then we can find polynomials $G_1, \dots, G_n$ such that $X_i = G_i(F_1, \dots, F_n)$. If we set $G = (G_1, \dots, G_n)$ then we have $G \circ F = \mathrm{id}$. So we have $(F \circ G) \circ F = F \circ (G \circ F) = F$. By dimension arguments, $F_1, \dots, F_n$ must be algebraically independent. So $F \circ G = \mathrm{id}$. This proves that F is invertible. □

Exercise 5.7. Suppose that $F_1, \dots, F_{n-1} \in R$. And define $D : R \to R$ by

$$D(g) = \det(J(F_1, \dots, F_{n-1}g)).$$

Show that D is a derivation.

Conjecture 5.8 (Jacobian Conjecture)**.** *If K is a field of characteristic 0, and $F : K^n \to K^n$ is a polynomial map with $\det(J(F)) = 1$, then F is invertible.*

Suppose that $F = (F_1, \dots, F_n)$ and $\det(J(F)) = 1$. Define derivations $D_1, D_2, \dots, D_n$ on R by

$$D_i(g) = J(F_1, F_2, \cdots, F_{i-1}, g, F_{i+1}, \dots, F_n).$$

Theorem 5.9. *The derivations $D_1, \dots, D_n$ are locally nilpotent if and only if F is invertible.*

Proof. If F is invertible then we have $K[F_1, \dots, F_n] = K[X_1, \dots, X_n]$. Let $\frac{\partial}{\partial F_1}, \dots, \frac{\partial}{\partial F_n}$ be the partial derivatives with respect to the coordinate system $(F_1, \dots, F_n)$. These derivations are locally nilpotent. Note that $F_1, \dots, F_n$ lie in the kernel of $D_i - \frac{\partial}{\partial F_i}$ for all i. So the kernel of $D_i - \frac{\partial}{\partial F_i}$ contains $K[F_1, \dots, F_n] = R$ and $D_i = \frac{\partial}{\partial F_i}$ for all i. In particular, $D_1, \dots, D_n$ are locally nilpotent.

Suppose that $D_1, \dots, D_n$ are locally nilpotent. For all i and j, $F_1, \dots, F_n$ lie in the kernel of the derivation $[D_i, D_j]$. From $D_i F_j = \delta_{i,j}$ follows that $F_1, \dots, F_n$ are algebraically independent, and using dimension arguments we see that $X_1, \dots, X_n$ are algebraic over $K[F_1, \dots, F_n]$. So R is contained in the kernel of $[D_i, D_j]$ and $[D_i, D_j] = 0$. This shows that $D_1, \dots, D_n$ commute. By Proposition 4.6, $R = S[F_1, \dots, F_n]$ where

$$S = \ker(D_1) \cap \cdots \cap \ker(D_n).$$

Again by dimension arguments, $\dim(S) = 0$ and S is algebraic over K. It follows that $S = K$ and $R = K[F_1, \dots, F_n]$.

□

Definition 5.10. If D is a derivation on $K[X_1, \dots, X_n]$ then its *divergence* is

$$\operatorname{div}(D) = \sum_{i=1}^{n} \frac{\partial}{\partial X_i} D(X_i).$$

Lemma 5.11. *If D is a locally nilpotent derivation on $K[X_1, \dots, X_n]$, then* $\operatorname{div}(D) = 0$.

Proof. We can extend D to $\overline{K}[X_1, \dots, X_n]$ where $\overline{K}$ is the algebraic closure. So without loss of generality we may assume that K is algebraically closed. For $t \in K$, let

$$F_i(t) := \exp(tD)(X_i) = X_i + tD(X_i) + \frac{t^2 D^2(X_i)}{2} + \cdots$$

Then $F(t) = (F_1(t), \dots, F_n(t))$ is an automorphism of $K[X_1, \dots, X_n]$ for all $t \in K$. We have

$$J(F(t)) = \begin{pmatrix} \frac{\partial F_1(t)}{\partial X_1} & \cdots & \frac{\partial F_1(t)}{\partial X_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n(t)}{\partial X_1} & \cdots & \frac{\partial F_n(t)}{\partial X_n} \end{pmatrix} = I + t \begin{pmatrix} \frac{\partial}{\partial X_1} D(X_1) & \cdots & \frac{\partial}{\partial X_n} D(X_1) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial X_1} D(X_n) & \cdots & \frac{\partial}{\partial X_n} D(X_n) \end{pmatrix} + \cdots,$$

so

$$\det(J(F(t)) = 1 + t \operatorname{div}(D) + \cdots.$$

Now $\det(J(F(t))$ is a polynomial that is nonzero for all $t \in K$. It follows that $\det(J(F(t))) = 1$ for all t and $\operatorname{div}(D) = 0$. ☐

6. Affine algebraic geometry

From now on we will assume that K is an algebraically closed field of characteristic 0. We view $K[X_1, \dots, X_n]$ as the ring of polynomial functions on the affine space $\mathbb{A}^n := K^n$. An *affine variety* is a Zariski closed subset of $\mathbb{A}^n$ (for some n). The ring of polynomial functions on an affine variety X is denoted by $\mathcal{O}(X)$. This gives us a 1–1 correspondence between reduced finitely generated K-algebras and affine varieties. For a finitely generated reduced K-algebra R we denote the corresponding affine variety by $\operatorname{Spec}(R)$. For an ideal $J \subseteq \mathcal{O}(X)$ we denote its zero set by

$$\mathcal{V}(J) = \{x \in X \mid f(x) = 0 \text{ for all } f \in J\}.$$

For a subset $Y \subseteq X$ we define its vanishing ideal by

$$\mathcal{I}(Y) = \{f \in \mathcal{O}(X) \mid f(y) = 0 \text{ for all } y \in Y\}.$$

We have $\mathcal{V}(\mathcal{I}(Y)) = \overline{Y}$, the Zariski closure of Y in X and for an ideal $J \subseteq \mathcal{O}(X)$ we have

$$\mathcal{I}(\mathcal{V}(J)) = \sqrt{J} = \{f \in \mathcal{O}(X) \mid f^n \in J \text{ for some } n > 0\}.$$

A morphism $\varphi : X \to Y$ of affine varieties corresponds to a K-algebra homomorphism $\varphi^\star : \mathcal{O}(Y) \to \mathcal{O}(X)$ and every K-algebra homomorphism is obtained this way.

Definition 6.1. An affine algebraic group is a group G that also has the structure of an affine variety, such that the multiplication $G \times G \to G$ and the inverse map $\iota : G \to G$ defined by $\iota(g) = g^{-1}$ are morphisms of affine varieties.

Example 6.2. If we identify GL_n with the set of pairs $(A, B) \mid AB = I\} \in \mathrm{Mat}_{n,n}^2 \cong \mathbb{A}^{2n^2}$ then we can view it as an affine variety. The multiplication is given by

$$(A_1, B_1), (A_2, B_2) \mapsto (A_1A_2, B_2B_1)$$

and the inverse map is given by

$$(A, B) \mapsto (B, A).$$

These maps are polynomial, so GL_n is an affine algebraic group. The group $\mathbb{G}_m = \mathrm{GL}_1 = (K^\times, \cdot)$ is called the *multiplicative group*.

Example 6.3. A Zariski closed subgroup of GL_n is an affine algebraic group. Such a group is called a *linear algebraic group*. So linear algebraic groups are affine algebraic groups. The converse is also true: all affine algebraic groups are linear algebraic groups.

Example 6.4. Consider the additive group $\mathbb{G}_a = (K, +)$. The addition is given by $(\alpha, \beta) \mapsto \alpha + \beta$ and the inverse function is given by $\alpha \mapsto -\alpha$. These maps are clearly morphisms of affine varieties, so the additive group is an affine algebraic group. It can also be identified as the Zariski closed subgroup of all matrices of the form

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad t \in K.$$

Definition 6.5. If $G \times X \to X$ is a (left) action of an affine algebraic group G on an affine variety X, then the action is called regular if the map $G \times X \to X$ is a morphism of affine varieties.

If an affine algebraic group G acts regularly on an affine variety X, then G also acts on $\mathcal{O}(X)$ as follows: For $f \in \mathcal{O}(X)$, $x \in X$ and $g \in G$ we define

$$(g \cdot f)(x) = f(g \cdot x).$$

Note that $x \mapsto f(g \cdot x)$ is a polynomial function, so $g \cdot f \in \mathcal{O}(X)$. This action is a *right* action

$$(gh) \cdot f = h \cdot (g \cdot f)$$

for $g, h \in G$ and $f \in \mathcal{O}(X)$. If G is commutative then this is also a left action.

The *invariant ring* is defined by

$$\mathcal{O}(X)^G = \{f \in \mathcal{O}(X) \mid g \cdot f = f \text{ for all } g \in G\}.$$

A point $x \in X$ is called a *fixed point* for the action of G if $g \cdot x = x$ for all $x \in X$. The set of fixed points on X is denoted by $\mathrm{Fix}(G)$.

Lemma 6.6. *Suppose that D is a derivation on $R = K[X_1, \dots, X_n]$, $f \in R$ and fD is triangularizable. Then f has no isolated critical points.*

Proof. If $E = fD$ is triangularizable, then $\sigma E \sigma^{-1} = \sigma(f)\sigma D \sigma^{-1}$ is triangular for some automorphism σ. So $\sigma(f) \in K[X_1, \dots, X_{n-1}]$ by Exercise **??**. The critical points of $\sigma(f)$ is the zero set of

$$\mathfrak{a}(\sigma(f)) = \Big(\frac{\partial \sigma(f)}{\partial X_1}, \dots, \frac{\partial \sigma(f)}{\partial X_{n-1}}\Big).$$

It is clear that there are no isolated critical points, because this ideal is generated by $n-1$ polynomials in which the variable X_n does not appear. The critical points of f is the zero set of the ideal

$$\mathfrak{a}(f) = \sigma^{-1}(\mathfrak{a}(\sigma(f)).$$

So the set of critical points of f is isomorphic to the set of critical points of $\sigma(f)$. So f does not have isolated critical points. □

Corollary 6.7. *The derivation $E = (2XZ - Y^2)(X\frac{\partial}{\partial Y} + Y\frac{\partial}{\partial Z})$ is not triangularizable.*

Proof. The critical points of $f = 2XZ - Y^2$ is the zero set of the ideal

$$\left(\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}, \frac{\partial f}{\partial Z}\right) = (2Z, -2Y, 2X) = (X, Y, Z).$$

So there is an isolated critical point, namely $(0, 0, 0)$. By the lemma above, the derivation E cannot be linearizable. □

7. $\mathbb{G}_a$-actions and locally nilpotent derivations

Suppose that X is an affine variety over K. In this section we will show that there is a 1–1 correspondence between locally nilpotent derivations on $R = \mathcal{O}(X)$ and regular actions of $\mathbb{G}_a$ on X.

First we construct a $\mathbb{G}_a$ action to every locally nilpotent derivation on $\mathcal{O}(X)$. Let $R = \mathcal{O}(X)$. We have isomorphisms $\mathcal{O}(\mathbb{G}_a) \cong K[T]$ and

$$\mathcal{O}(\mathbb{G}_a \times X) \cong \mathcal{O}(\mathbb{G}_a) \otimes \mathcal{O}(X) \cong K[T] \otimes R \cong R[T].$$

Suppose that D is a locally nilpotent derivation on $R = \mathcal{O}(X)$. Recall the map $\Psi : R \to R[T]$ defined by

$$\Psi = \exp(TD) = \sum_{i=0}^{\infty} \frac{T^i D^i}{i!}.$$

The homomorphism Ψ corresponds to a morphism $\lambda : \mathbb{G}_a \times X \to X$.

Lemma 7.1. *The morphism λ is a regular action.*

Proof. We write $t \cdot x$ instead of $\lambda(t, x)$. If $s, t \in \mathbb{G}_a$, $x \in X$ and $f \in \mathcal{O}(X)$ then we have

$$f((s+t) \cdot x) = \exp((t+s)D)(f)(x) = \sum_{i=0}^{\infty} \frac{(t+s)^i D^i(f)(x)}{i!}.$$

On the other hand, we have

$$f(t \cdot x) = \sum_{i=0}^{\infty} \frac{t^i D^i(f)(x)}{i!}$$

and

$$f(s \cdot (t \cdot x)) = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \frac{s^j t^i D^{i+j}(f)(x)}{j!i!} = \sum_{n=0}^{\infty} \frac{\sum_{j=0}^{n}\binom{n}{j} s^j t^{n-j} D^n(f)}{n!} =$$

$$= \sum_{n=0}^{\infty} \frac{(t+s)^n D^n(f)}{n!} = f((s+t) \cdot x).$$

Since $f \in \mathcal{O}(X)$ was arbitrary, we have $(s+t)\cdot x = s\cdot(t\cdot x)$. So the map $\mathbb{G}_a \times X \to X$ is indeed an action. $\square$

So we have associated to every locally nilpotent derivation D on $\mathcal{O}(X)$ a regular action of $\mathbb{G}_a$ on X.

Proposition 7.2. *Every regular action of $\mathbb{G}_a$ on X comes from a uniquely determined locally nilpotent derivation D on $R = \mathcal{O}(X)$.*

Suppose we have a regular action $\lambda : \mathbb{G}_a \times X \to X$. This corresponds to a homomorphism

$$\lambda^\star : \mathcal{O}(X) \cong R \to \mathcal{O}(\mathbb{G}_a \times X) \cong R[T].$$

Define a linear map $D : R \to R$ by

$$D(f) = \Big(\frac{\partial}{\partial T}\lambda^\star(f)\Big) \mid_{T=0} .$$

So we have a commutative diagram

$$\begin{tikzcd} \mathcal{O}(X) \arrow[r, "\lambda^\star"] \arrow[d, "D"'] & \mathcal{O}(X)[T] \arrow[d, "\frac{\partial}{\partial T}"] \\ \mathcal{O}(X)) & \mathcal{O}(X)[T] \arrow[l, "T=0"] \end{tikzcd} .$$

For $t \in \mathbb{G}_a$ and $f \in \mathcal{O}(X)$ we have

$$(t \cdot f)(x) = f(t \cdot x) = \lambda^\star(f)(t, x) = \lambda^\star(f) \mid_{T=t} (x).$$

So we have

$$t \cdot f = \lambda^\star(f) \mid_{T=t}$$

for $t \in \mathbb{G}_a$ and $f \in R$. In particular, we have

$$\lambda^\star(f) \mid_{T=0} = 0 \cdot f = f$$

for all $f \in \mathcal{O}(X)$.

Lemma 7.3. *The map $D : R \to R$ is a K-derivation.*

Proof. We have

$$D(fg) = \Big(\frac{\partial}{\partial T}\lambda^\star(fg)\Big) \mid_{T=0} = \Big(\frac{\partial}{\partial T}\lambda^\star(f)\lambda^\star(g)\Big) \mid_{T=0} =$$
$$= \Big(\Big(\frac{\partial}{\partial T}\lambda^\star(f)\Big)\lambda^\star(g) + \Big(\frac{\partial}{\partial T}\lambda^\star(g)\Big)\lambda^\star(f)\Big) \mid_{T=0} = D(f)g + D(g)f.$$

and for $f \in K$ we have

$$D(f) = \Big(\frac{\partial}{\partial T}\lambda^\star(f)\Big) \mid_{T=0} = \Big(\frac{\partial}{\partial T} f\Big) \mid_{T=0} = 0.$$

This shows that D is a K-derivation. $\square$

Lemma 7.4. *We have*

$$\lambda^\star(D(f)) = \frac{\partial}{\partial T}\lambda^\star(f).$$

Proof. Suppose that $f \in \mathcal{O}(X)$ and let us write

$$\lambda^\star(f) = \sum_{i=0}^{d} f_i T^i.$$

Note that $f_0 = \lambda^\star(f) \mid_{T=0} = f$, and $f_1 = (\frac{\partial}{\partial T}\lambda^\star(f))_{T=0} = D(f)$. We have

$$t \cdot f = \lambda^\star(f) \mid_{T=t} = \sum_{i=0}^{\infty} f_i t^i.$$

We have

$$\sum_{i=0}^{d} (s \cdot f_i) t^i = s \cdot \sum_{i=0}^{d} f_i t^i = s \cdot (t \cdot f) = (t+s) \cdot f = \sum_{i=0}^{d} f_i (s+t)^i$$

Comparing the coefficient of t^1 yields

$$\lambda^\star(D(f)) \mid_{T=s} = s \cdot D(f) = s \cdot f_1 = \sum_{i=0}^{d} i f_i s^{i-1} = \Big(\frac{\partial}{\partial T} \sum_{i=0}^{d} f_i T^i\Big) \mid_{T=s}$$

□

By induction on can show that:

Corollary 7.5.

$$\lambda^\star(D^n(f)) = \Big(\frac{\partial}{\partial T}\Big)^n \lambda^\star(f).$$

Setting $T = 0$ yields

$$D^n(f) = \lambda^\star(D^n(f)) \mid_{T=0} = \Big(\frac{\partial}{\partial T}\Big)^n \lambda^\star(f) \mid_{T=0} .$$

This gives:

Corollary 7.6.

$$\lambda^\star(f) = \sum_{i=0}^{\infty} \frac{T^i D^i(f)}{i!}.$$

Corollary 7.7. *The derivation D is locally nilpotent.*

Proof. For large n, $\lambda^\star(D^n f) = 0$ and $D^n f = \lambda^\star(D^n f) \mid_{T=0} = 0$. □

So given a locally nilpotent derivation D, we can exponentiate it to obtain a regular $\mathbb{G}_a$-action. Moreover, every regular $\mathbb{G}_a$-action can be obtained in this way. The $\mathbb{G}_a$-action determines the locally nilpotent derivation D uniquely, because

$$\Big(\frac{\partial}{\partial T}\lambda^\star(f)\Big) \mid_{T=0} = \Big(\frac{\partial}{\partial T} \sum_{i=0}^{\infty} \frac{T^i D^i(f)}{i!}\Big) \mid_{T=0} = \Big(\sum_{i=1}^{\infty} \frac{T^{i-1} D^i(f)}{(i-1)!}\Big) \mid_{T=0} = D(f).$$

8. Quotients of $\mathbb{G}_a$-actions

Lemma 8.1. *If D is a locally nilpotent derivation on $\mathcal{O}(X)$, and $\lambda : \mathbb{G}_a \times X \to X$ is the corresponding $\mathbb{G}_a$-action, then we have*

$$R^D = \mathcal{O}(X)^{\mathbb{G}_a}.$$

Proof. If $D(f) = 0$, then we have

$$t \cdot f = \sum_{i=0}^{\infty} \frac{t^i D^i(f)}{i!} = f$$

for all $t \in \mathbb{G}_a$ and $f \in \mathcal{O}(X)^{\mathbb{G}_a}$.

Conversely, if $f \in \mathcal{O}(X)^{\mathbb{G}_a}$, then

$$f = t \cdot f = \Psi(f) \mid_{T=t} = \sum_{i=0}^{\infty} \frac{t^i D^i(f)}{i!}$$

for all t. So we have

$$f = \Psi(f) = \sum_{i=0}^{\infty} \frac{T^i D^i(f)}{i!}$$

and $D(f) = 0$. This shows that $f \in R^D$. □

A classical question of Hilbert is the following:

Question 8.2. *If $G \subseteq \mathrm{GL}_n$ is a Zariski closed subgroup, is the invariant ring $K[X_1, \dots, X_n]^G$ always finitely generated as a K-algebra?*

Nagata showed in 1958 that the answer to this question is no. Nagata's counterexample is a bit complicated. Weitzenböck showed that if $G = \mathbb{G}_a$, then this invariant ring is always finitely generated. However, if we consider nonlinear $\mathbb{G}_a$-actions, and actions of $\mathbb{G}_a$ on arbitrary affine varieties, then the answer is still no.

Example 8.3 (Daigle-Freudenburg)**.** Consider the locally nilpotent derivation

$$D = A^2 \frac{\partial}{\partial X} + (AX + B) \frac{\partial}{\partial Y} + Y \frac{\partial}{\partial Z}$$

on the ring $K[A, B, X, Y, Z]$. Then $\ker(D) = K[A, B, X, Y, Z]^{\mathbb{G}_a}$ is not finitely generated.

Exercise 8.4. Suppose that D is a locally nilpotent derivation on $\mathcal{O}(X)$, and $W \subseteq \mathcal{O}(X)$ is a subspace. The space W is called $\mathbb{G}_a$ stable if $g \cdot W \subseteq W$ for all $g \in G$. Show that W is $\mathbb{G}_a$-stable if and only if $D(W) \subseteq W$.

Generally, orbits of algebraic group actions on affine varieties may not be closed. However, the additive group behaves nicely:

Theorem 8.5. *If $\mathbb{G}_a$ acts regularly on an affine variety X, then all the orbits are Zariski closed.*

Proof. Suppose that $\mathbb{G}_a$ acts regularly on an affine variety X, and $x \in X$. We will show that the orbit $\mathbb{G}_a \cdot x$ is closed. Without loss of generality we may assume that $X = \overline{\mathbb{G}_a \cdot x}$. The orbit map $\rho : \mathbb{G}_a \to X$ given by $\rho(t) = t \cdot x$ corresponds to an injective ring homomorphism

$\rho : \mathcal{O}(X) \to \mathcal{O}(\mathbb{G}_a) \cong K[T]$. The group $\mathbb{G}_a$ also acts on itself by $s \cdot t = s + t$. This action corresponds to the locally nilpotent derivation $\frac{\partial}{\partial T}$. We have

$$\rho(s+t) = (s+t) \cdot x = s \cdot (t \cdot x) = s \cdot \rho(t)$$

It follows that for $f \in \mathcal{O}(X)$ we have

$$\rho^\star(s \cdot f)(x) = (s \cdot f)(\rho(x)) = f(s \cdot \rho(x)) = f(\rho(s \cdot x)) = \rho^\star(f)(s \cdot x) = (s \cdot \rho^\star(f))(x).$$

So it follows that $\rho^\star(s \cdot f) = s \cdot \rho^\star(f)$. From this it is clear that the image of $\rho^\star$ is $\mathbb{G}_a$-stable.

If the image of $\rho^\star$ is just K, then the map ρ is constant and the orbit is just a point which is clearly closed. Otherwise, the image of $\rho^\star$ contains a nonzero polynomial $f(T)$. But this image is stable under $\mathbb{G}_a$ and closed under $\frac{\partial}{\partial T}$. By differentiating enough times, we see that the image of $\rho^\star$ contains a linear polynomial of the form $T - x$. This implies that $\rho^\star$ is surjective. We conclude that $\rho^\star$ and ρ are isomorphisms. In particular, $\mathbb{G}_a \cdot x = X$ is closed. □

The image $\operatorname{im}(D)$ of $D : R \to R$ may not be an ideal, but it is an R^D module. Let $R \operatorname{im}(D)$ be the R-module generated by $\operatorname{im}(D)$.

Lemma 8.6. *The set* $\operatorname{Fix}(\mathbb{G}_a)$ *of fixed points is the zero set of the ideal* $R \operatorname{im}(D)$.

Proof. If x is a fixed point of $\mathbb{G}_a$, then the maximal ideal $\mathfrak{m}_x$ corresponding to x is $\mathbb{G}_a$-stable. It follows that $D(\mathfrak{m}_x) \subseteq \mathfrak{m}_x$. Since R is spanned by $\mathfrak{m}_x$ and $1 \in R$, we have $\operatorname{im}(D) \subseteq \mathfrak{m}_x$ and $R \operatorname{im}(D) \subseteq \mathfrak{m}_x$.

Let $Y \subseteq X$ be the zero set of the ideal $I = R \operatorname{im}(D)$. Clearly, $D(I) \subseteq I$ so I is $\mathbb{G}_a$-stable. If $t \in \mathbb{G}_a$, $f \in \mathcal{O}(X)$ and $y \in Y$, then we have

$$f(t \cdot y) = \sum_{i=0}^{\infty} \frac{t^i D^i(f)(y)}{i!} = f(y)$$

because $D^i(f) \in I$ for all $i \geq 1$. It follows that $t \cdot y = y$ for all $t \in \mathbb{G}_a$ and y is a fixed point. □

If D has a slice, then $1 \in R \operatorname{im}(D)$ and the action has no fixed points. However, an may have no fixed points but not have a slice.

Exercise 8.7. Suppose that D has a slice s. Show that the map

$$\mathbb{G}_a \times s^{-1}(0) \to X$$

defined by

$$(t, x) \mapsto t \cdot x$$

is an isomorphism of varieties. Note that $s = 0$ intersects each orbit exactly in one point.

Suppose that $\mathbb{G}_a$ acts regularly on an affine variety X. Assume that $\mathcal{O}(X)^{\mathbb{G}_a}$ is finitely generated. Then the algebra $\mathcal{O}(X)^{\mathbb{G}_a}$ corresponds to an affine variety $Y = \operatorname{Spec}(\mathcal{O}(X)^{\mathbb{G}_a})$, and the inclusion $\mathcal{O}(X)^{\mathbb{G}_a} \hookrightarrow \mathcal{O}(X)$ corresponds to a morphism $\pi : X \to Y$. The map π is constant on orbits. Ideally, π is surjective and the fibers of π are exactly all orbits. In that case π exactly parameterizes all orbits. But often, π will not be this nice.

Example 8.8. Consider the variety

$$\mathrm{SL}_2 = \left\{ \begin{pmatrix} w & x \\ y & z \end{pmatrix} \middle| wz - xy = 1 \right\}.$$

Then we have

$$R = \mathcal{O}(\mathrm{SL}_2) = K[W, X, Y, Z]/(WZ - XY - 1).$$

The additive group $\mathbb{G}_a$ acts on SL_2 by

$$t \cdot \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} w & x + tw \\ y & z + ty \end{pmatrix}.$$

This action has no fixed points. The locally nilpotent derivation on R corresponding to the $\mathbb{G}_a$ action is Let

$$D = W \frac{\partial}{\partial X} + Y \frac{\partial}{\partial Z}.$$

We have $R^{\mathbb{G}_a} = \ker(D) = K[W, Y]$ and $\mathrm{Spec}(\mathcal{O}(\mathrm{SL}_2)^{\mathbb{G}_a}) = \mathrm{Spec}(K[W, Y]) = \mathbb{A}^2$. The map π is given by

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix} \in \mathrm{SL}_2 \mapsto \begin{pmatrix} w \\ y \end{pmatrix}$$

This map is not surjective, because the image does not contain the origin.

Example 8.9. Consider the locally nilpotent derivation $D = X \frac{\partial}{\partial Y}$ on $K[X, Y]$. This derivation corresponds to the $\mathbb{G}_a$ action given by

$$t \cdot (x, y) = (x, y + tx).$$

We have

$$\ker(D) = K[X].$$

The quotient map is given by $(x, y) \mapsto x$. The fiber $\pi^{-1}(0) = \{0\} \times \mathbb{A}^1$ consists of infinitely may fixed points.

Example 8.10. Consider the locally nilpotent derivation $D = W \frac{\partial}{\partial X} + Y \frac{\partial}{\partial Z}$ on $K[W, X, Y, Z]$. Then we have

$$\ker(D) = K[W, X, Y, Z]^{\mathbb{G}_a} = K[W, Y, WZ - XY].$$

So the quotient map $\pi : \mathbb{A}^4 \to \mathbb{A}^3$ is given by

$$(w, x, y, z) \mapsto (w, y, zw - xy)$$

We have

$$\pi^{-1}(0, 0, 0) = \{(0, x, 0, z) \mid x, z \in K\} \cong \mathbb{A}^2.$$

This is a two dimensional fiber!

Suppose that R is a domain and f is a pre-slice. Let $g = D(f) \in R^D \setminus \{0\}$. Then the derivation D extends to the localization R_g. It is easy to verify that this extension is still locally nilpotent. Now f is a slice for the $\mathbb{G}_a$ action on R_g. Let $U = \mathrm{Spec}(R_g) = \{x \in X \mid g(x) \neq 0\}$. Then the action of $\mathbb{G}_a$ on U is trivial. We say that the action of $\mathbb{G}_a$ on X is locally trivial, if X can be covered with $\mathbb{G}_a$-stable open sets on which $\mathbb{G}_a$ acts trivially. If D is nonzero, then a pre-slice always exists: if $f \in R \setminus R^D$, then we can choose $k \geq 1$ such that $D^k(f) \neq 0$ and $D^{k+1}(f) \neq 0$. Then $D^{k-1}(f)$ is a pre-slice.

Exercise 8.11. Show that $\mathrm{im}(D) \cap R^D$ is an ideal of R^D. Show that if $R(R^D \cap \mathrm{im}(D)) = R$, then the $\mathbb{G}_a$-action is locally trivial.