The Graph Isomorphism Problem and the Module Isomorphism Problem

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The graph isomorphism problem and approximate categories. (English summary)


Summary: "It is unknown whether two graphs can be tested for isomorphism in polynomial time. A classical approach to the Graph Isomorphism Problem is the $d$-dimensional Weisfeiler-Lehman algorithm. The $d$-dimensional WL-algorithm can distinguish many pairs of graphs, but the pairs of non-isomorphic graphs constructed by Cai, Fürer and Immerman it cannot distinguish. If $d$ is fixed, then the WL-algorithm runs in polynomial time. We will formulate the Graph Isomorphism Problem as an Orbit Problem: Given a representation $V$ of an algebraic group $G$ and two elements $v_1, v_2 \in V$, decide whether $v_1$ and $v_2$ lie in the same $G$-orbit. Then we attack the Orbit Problem by constructing certain approximate categories whose objects include the elements of $V$. We show that $v_1$ and $v_2$ are not in the same orbit by showing that they are not isomorphic in the category $C_d$. For every $d$, this gives us an algorithm for isomorphism testing. We will show that the WL-algorithms reduce to our algorithms, but that our algorithms cannot be reduced to the WL-algorithms. Unlike the Weisfeiler-Lehman algorithm, our algorithm can distinguish the Cai-Fürer-Immerman graphs in polynomial time."

References:

Orbit Problem

Given \( v, v' \in V \), does there exist \( g \in G \) with \( g \cdot v = v' \)?
Orbit Problems

Orbit Problem

$G$ group acting on $k$-vector space $V$
given $v, v' \in V$, does there exist $g \in G$ with $g \cdot v = v'$?

Graph Isomorphism Problem (Hard)

$\Gamma, \Gamma'$ graphs with $n$ vertices, $A, A' \in \text{Mat}_{n,n}$ adjacency matrices
$G = \{ \text{permutation matrices} \}$ acts on $\text{Mat}_{n,n}$ by conjugation
does there exists a permutation matrix $P$ with $PAP^{-1} = A'$?
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**Module Isomorphism Problem (Easy)**

$G = \text{GL}_n$ acts on $\text{Mat}^m_{n,n}$ by simultaneous conjugation
$A = (A_1, \ldots, A_m), A' = (A'_1, \ldots, A'_m) \in \text{Mat}^m_{n,n}$
is there a $P \in \text{GL}_n$ with $(PA_1P^{-1}, \ldots, PA_mP^{-1}) = (A'_1, \ldots, A'_m)$?
Module Isomorphism Problem (Easy)

\[ R = k \langle x_1, \ldots, x_m \rangle \text{ free associative algebra} \]
\[(A_1, \ldots, A_m) \in \text{Mat}_{n,n}^m \text{ corresponds to module } M = k^n, \]
where \( x_i \cdot \nu = A_i \nu \) for \( \nu \in M \)
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\((A_1, \ldots, A_m) \in \text{Mat}_{n,n}^m\) corresponds to module \(M = k^n\),

where \(x_i \cdot v = A_i v\) for \(v \in M\)

\((A'_1, \ldots, A'_m) \in \text{Mat}_{n,n}^m\) corresponds to module \(M' = k^n\)

\(\text{Hom}_R(M, M') = \{ P \in \text{Mat}_{n,n} | \forall i \ PA_i = A'_i P \}\)
Module Isomorphism Problem (Easy)

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\( (A'_1, \ldots, A'_m) \in \text{Mat}^m_{n,n} \) corresponds to module \( M' = k^n \)

\( \text{Hom}_R(M, M') = \{ P \in \text{Mat}^n_{n,n} \mid \forall i \: PA_i = A'_i P \} \)

Probabilistic Module Isomorphism Algorithm

choose \( P \in \text{Hom}_R(M, M') \subseteq \text{Mat}^n_{n,n} \) at random

if \( P \) invertible, then \( M \cong M' \)

if \( P \) not invertible, then \( M \not\cong M' \) with high probability

polynomial time de-randomized algorithms for module isom.:


(for arbitrary finitely generated associative \( k \)-algebras)
Γ, Γ’ graphs on $n$ vertices with adjacency matrices $A = (a_{i,j})$, $A' = (a'_{i,j})$
does there exists a permutation $n \times n$ matrix $X = (x_{i,j})$ with $XAX^{-1} = A'$?
Graph Isomorphism by Solving Polynomial Equations

Γ, Γ' graphs on \( n \) vertices with adjacency matrices
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We need to solve a system of polynomial equations:
Graph Isomorphism by Solving Polynomial Equations

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We need to solve a system of polynomial equations:

\( X \) is a permutation matrix, means:

(1) \( x_{i,j}x_{i,\ell} = 0 = x_{j,i}x_{\ell,i} \) for all \( i \) and all \( j \neq \ell \)

(2) \( \sum_{j=1}^{n} x_{i,j} - 1 = \sum_{j=1}^{n} x_{j,i} - 1 = 0 \) for all \( i \)
\( \Gamma, \Gamma' \) graphs on \( n \) vertices with adjacency matrices 
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\( XA = A'X \) gives us the linear equations:

(3) \( \sum_{j=1}^{n} x_{i,j}a_{j,\ell} - \sum_{j=1}^{n} a'_{i,j}x_{j,\ell} = 0 \) for all \( i, \ell \)

system of linear and quadratic equations in \( n^2 \) variables
Gröbner Basis?

\[ R = k[x_{1,1}, x_{1,2}, \ldots, x_{n,n}] \] polynomial ring in \( n^2 \) variables

define \( \text{Eq}(\Gamma, \Gamma') \subset R \) as the set of poly’s from our system of equations (1)-(3)

let \( I = (\text{Eq}(\Gamma, \Gamma')) \subseteq R \) be the ideal generated by \( \text{Eq}(\Gamma, \Gamma') \)

Hilbert’s Nullstellensatz

\[ 1 \in I \iff \Gamma \cong \Gamma' \]
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\[ 1 \in I \iff \Gamma \not\sim \Gamma' \]

Algorithm 1, Gröbner basis (\textbf{GB})

compute Gröbner basis \( \mathcal{G} \) of \( I \) using Buchberger’s algorithm

then \( 1 \in \mathcal{G} \iff 1 \in I \iff \Gamma \not\sim \Gamma' \)
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Computing a Gröbner basis is known to be very slow

there is no reason to believe Algorithm 1 could be polynomial time

This is a stupid approach!

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Truncated Ideals

... or is it?

We restrict ourselves to computations in low degree. Fix a positive integer $d \geq 2$. $R_d = k[x_1, x_1, \ldots, x_n]$ is the space of polynomials of degree $\leq d$. The dimension of $R_d$ is polynomial in $n$ (for fixed $d$).

We construct subspaces $I[0] \subseteq I[1] \subseteq \cdots$ of $R_d$ as follows:

1. $I[0] \subseteq R_d$ is the $k$-span of $\text{Eq}(\Gamma, \Gamma')$ ($\text{Eq}(\Gamma, \Gamma')$ was the set of linear and quadratic equations).
2. $I[j+1] = \sum_{d \leq e \leq d} (I[j] \cap R_e)R_d - e$ for all $j$ for some $\ell \leq \text{dim} R_d$.

Let $I[\ell] = I[\ell+1] = I[\ell+2] = \cdots$ be this limit. This is the $d$-truncated ideal generated by $\text{Eq}(\Gamma, \Gamma')$. Harm Derksen

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$R_d = k[x_{1,1}, x_{1,2}, \ldots, x_{n,n}]_{\leq d}$ space of polynomials of degree $\leq d$

dim $R_d$ is polynomial in $n$ (for fixed $d$)

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\[ \begin{align*}
  &\quad I^{[0]} \subseteq R_d \text{ is the } k\text{-span of } \text{Eq}(\Gamma, \Gamma') \\
  &\quad (\text{Eq}(\Gamma, \Gamma') \text{ was the set of linear and quadratic equations}) \\
  &\quad I^{[j+1]} = \sum_{e=0}^{d} (I^{[j]} \cap R_e) R_{d-e} \text{ for all } j
\end{align*} \]
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\begin{itemize}
  \item \( I^{[0]} \subseteq R_d \) is the \( k \)-span of \( \text{Eq}(\Gamma, \Gamma') \)
    \( (\text{Eq}(\Gamma, \Gamma') \) was the set of linear and quadratic equations) \n  \item \( I^{[j+1]} = \sum_{e=0}^{d} (I^{[j]} \cap R_e) R_{d-e} \) for all \( j \)
\end{itemize}
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we construct subspaces \( I[0] \subseteq I[1] \subseteq \cdots \) of \( R_d \) as follows:

- \( I[0] \subseteq R_d \) is the \( k \)-span of Eq(\( \Gamma, \Gamma' \))
  (Eq(\( \Gamma, \Gamma' \)) was the set of linear and quadratic equations)
- \( I[j+1] = \sum_{e=0}^{d} (I[j] \cap R_e) R_{d-e} \) for all \( j \)

for some \( \ell \leq \dim R_d, I[\ell] = I[\ell+1] = I[\ell+2] = \ldots \)

Let (Eq(\( \Gamma, \Gamma' \)))\( d = I[\ell] \) be this limit
this is the \( d \)-truncated ideal generated by Eq(\( \Gamma, \Gamma' \))
Comparison to the Weisfeiler-Leman Algorithm

a basis of \((\text{Eq}(\Gamma, \Gamma'))_d\) (as a \(k\)-vector space) can be computed with a polynomial number of arithmetic operations in the field \(k\)

Algorithm 2, Truncated Ideals \((\text{T}I_d)\)

compute \((\text{Eq}(\Gamma, \Gamma'))_d\) and test whether \(1 \in (\text{Eq}(\Gamma, \Gamma'))_d\)

if \(1 \in (\text{Eq}(\Gamma, \Gamma'))_d\) then \(\Gamma \not\cong \Gamma'\)
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A basis of \((\text{Eq}(\Gamma, \Gamma'))_d\) (as a \(k\)-vector space) can be computed with a polynomial number of arithmetic operations in the field \(k\).

**Algorithm 2, Truncated Ideals \((\text{TI}_d)\)**

Compute \((\text{Eq}(\Gamma, \Gamma'))_d\) and test whether \(1 \in (\text{Eq}(\Gamma, \Gamma'))_d\).

If \(1 \in (\text{Eq}(\Gamma, \Gamma'))_d\) then \(\Gamma \not\sim \Gamma'\).

This algorithm is polynomial time if we work over a finite field \(k = \mathbb{F}_q\) and \(q = q(n) = 2^{O(\text{poly}(n))}\).

**Theorem**

If \(q\) is a prime \(> n\), \(k = \mathbb{F}_q\).

If \(\text{WL}_d\) distinguishes \(\Gamma\) and \(\Gamma'\), then \(\text{TI}_{2d+2}\) distinguishes \(\Gamma\) and \(\Gamma'\).

So \(\text{TI}\) is as powerful as \(\text{WL}\) (but perhaps not more powerful).

But there is more structure ...
recall $R = k[x_{1,1}, x_{1,2}, \ldots, x_{n,n}]$ and $R_d = k[x_{1,1}, x_{1,2}, \ldots, x_{n,n}]_{\leq d}$
matrix multiplication gives a ring homomorphism

$$\varphi : R = k[x_{1,1}, \ldots, x_{n,n}] \to k[y_{1,1}, \ldots, y_{n,n}, z_{1,1}, \ldots, z_{n,n}] \cong R \otimes R$$
defined by $\varphi(x_{i,j}) = \sum_{\ell=1}^{n} y_{i,\ell} z_{\ell,j}$
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this ring homomorphism restricts to a linear map $R_d \rightarrow R_d \otimes R_d$, 

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An Associative Algebra

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defined by $\varphi(x_{i,j}) = \sum_{\ell=1}^{n} y_{i,\ell} z_{\ell,j}$

this ring homomorphism restricts to a linear map $R_d \to R_d \otimes R_d$, which dualizes to a linear map $R_d^* \otimes R_d^* \to R_d^*$.
An Associative Algebra

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which dualizes to a linear map $R_d^* \otimes R_d^* \to R_d^*$,

which defines a bilinear multiplication $R_d^* \times R_d^* \to R_d^*$,
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which defines a bilinear multiplication $R_d^* \times R_d^* \to R_d^*$,

which makes $R_d^*$ into an associative algebra
The Category $\mathcal{C}_{n,d}$

Definition (Approximate Category $\mathcal{C}_{n,d}$)

objects of $\mathcal{C}_{n,d}$ are graphs on $n$ vertices,
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Objects of $\mathcal{C}_{n,d}$ are graphs on $n$ vertices,

$$\text{Hom}_{\mathcal{C}_{n,d}}(\Gamma, \Gamma') = (R_d/(\text{Eq}(\Gamma, \Gamma'))_d)^* \subseteq R_d^*$$
The Category $C_{n,d}$

Definition (Approximate Category $C_{n,d}$)

Objects of $C_{n,d}$ are graphs on $n$ vertices,

$$\text{Hom}_{C_{n,d}}(\Gamma, \Gamma') = \left( \frac{R_d}{(\text{Eq}(\Gamma, \Gamma'))_d} \right)^* \subseteq R_d^*$$

Multiplication $R_d^* \times R_d^* \to R_d^*$ restricts to a bilinear map

$$\text{Hom}_{C_{n,d}}(\Gamma, \Gamma') \times \text{Hom}_{C_{n,d}}(\Gamma', \Gamma'') \to \text{Hom}_{C_{n,d}}(\Gamma, \Gamma'')$$
The Category $\mathcal{C}_{n,d}$

**Definition (Approximate Category $\mathcal{C}_{n,d}$)**

Objects of $\mathcal{C}_{n,d}$ are graphs on $n$ vertices,

$$\text{Hom}_{\mathcal{C}_{n,d}}(\Gamma, \Gamma') = \left( R_d/(\text{Eq}(\Gamma, \Gamma'))_d \right)^* \subseteq R_d^*$$

Multiplication $R_d^* \times R_d^* \to R_d^*$ restricts to a bilinear map

$$\text{Hom}_{\mathcal{C}_{n,d}}(\Gamma, \Gamma') \times \text{Hom}_{\mathcal{C}_{n,d}}(\Gamma', \Gamma'') \to \text{Hom}_{\mathcal{C}_{n,d}}(\Gamma, \Gamma'')$$

If $1 \in (\text{Eq}(\Gamma, \Gamma'))_d$ then $(\text{Eq}(\Gamma, \Gamma'))_d = R_d$ and $\text{Hom}_{\mathcal{C}_{n,d}}(\Gamma, \Gamma') = 0$
Properties of $\mathcal{C}_{n,d}$

suppose $\Gamma, \Gamma'$ graphs on $n$ vertices with adjacency matrices $A, A'$
if $\Gamma \cong \Gamma'$ then there is a permutation matrix $P$ with $PA = A'P$
Properties of $\mathcal{C}_{n,d}$

suppose $\Gamma, \Gamma'$ graphs on $n$ vertices with adjacency matrices $A, A'$
if $\Gamma \cong \Gamma'$ then there is a permutation matrix $P$ with $PA = A'P$
if $\text{ev}_P : R_d \to k$ is evaluation at $P$, then $\text{ev}_P \in \text{Hom}_{\mathcal{C}_{n,d}}(\Gamma, \Gamma') \subseteq R_d^*$
and $\text{ev}_P$ is an isomorphism in $\mathcal{C}_{n,d}$ (with inverse $\text{ev}_{P^{-1}}$)
Properties of $\mathcal{C}_{n,d}$

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Theorem

let $T = \text{Hom}_{\mathcal{C}_{n,d}}(\Gamma, \Gamma)$ (an associative $k$-algebra)
$\Gamma, \Gamma'$ are isomorphic in $\mathcal{C}_{n,d} \iff \text{Hom}_{\mathcal{C}_{n,d}}(\Gamma', \Gamma)$ and $\text{Hom}_{\mathcal{C}_{n,d}}(\Gamma, \Gamma)$
are isomorphic $T$-modules
Properties of $C_{n,d}$

suppose $\Gamma, \Gamma'$ graphs on $n$ vertices with adjacency matrices $A, A'$

if $\Gamma \cong \Gamma'$ then there is a permutation matrix $P$ with $PA = A'P$

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Theorem

let $T = \text{Hom}_{C_{n,d}}(\Gamma, \Gamma)$ (an associative $k$-algebra)

$\Gamma, \Gamma'$ are isomorphic in $C_{n,d}$ $\iff$ $\text{Hom}_{C_{n,d}}(\Gamma', \Gamma)$ and $\text{Hom}_{C_{n,d}}(\Gamma, \Gamma)$

are isomorphic $T$-modules

we can test whether $\Gamma, \Gamma'$ are isomorphic in $C_{n,d}$ in polynomial time
Properties of $C_{n,d}$

Suppose $\Gamma, \Gamma'$ graphs on $n$ vertices with adjacency matrices $A, A'$.

If $\Gamma \cong \Gamma'$, then there is a permutation matrix $P$ with $PA = A'P$.

If $ev_P : R_d \to k$ is evaluation at $P$, then $ev_P \in \text{Hom}_{C_{n,d}}(\Gamma, \Gamma') \subseteq R_d^*$ and $ev_P$ is an isomorphism in $C_{n,d}$ (with inverse $ev_{P^{-1}}$).

**Theorem**

Let $T = \text{Hom}_{C_{n,d}}(\Gamma, \Gamma)$ (an associative $k$-algebra).

$\Gamma, \Gamma'$ are isomorphic in $C_{n,d}$ $\iff$ $\text{Hom}_{C_{n,d}}(\Gamma', \Gamma)$ and $\text{Hom}_{C_{n,d}}(\Gamma, \Gamma)$ are isomorphic $T$-modules.

We can test whether $\Gamma, \Gamma'$ are isomorphic in $C_{n,d}$ in polynomial time.

**Algorithm 3 ($\text{AC}_d$)**

Test whether $\Gamma, \Gamma'$ are isomorphic in the category $C_{n,d}$ for all fields $k = \mathbb{F}_q$ with $q$ a prime $\leq 2n$.

If not isomorphic for some $k$, then $\Gamma$ and $\Gamma'$ are non-isomorphic graphs.
if $V = \{1, 2, \ldots, n\}$ is the set of vertices, then $\mathbf{WL}_{d-1}$ captures reasoning on subsets of $V^d$

it is as powerful as $d$-variable logic with counting (see Cai-Fürer-Immerman)

if $W = kV \cong k^n$ is the vector space whose basis is the set of vertices, then $\mathbf{AC}_{2d}$ captures reasoning with subspaces of $W \otimes^d = W \otimes \cdots \otimes W$ with operations such as tensor products, sums, intersections, projections and dimension count.
Cai-Fürer-Immerman constructed families of pairs of nonisomorphic graphs that cannot be distinguished by $\mathsf{WL}_d$ for any fixed $d$ so $\mathsf{WL}_d$ does not give a polynomial time algorithm for a pair of CFI graphs $(\Gamma, \Gamma')$, we can construct matrices $B$ and $B'$ from the adjacency matrices $A$ and $A'$ such that $B$ and $B'$ do not have the same rank if $k = \mathbb{F}_2$ 

$\mathsf{AC}_3$ can distinguish each pair of CFI-graphs $(\Gamma, \Gamma')$ if we work over $k = \mathbb{F}_2$