

EXPLICIT TENSORS OF BORDER RANK AT LEAST $2d - 2$ IN $K^d \otimes K^d \otimes K^d$ IN ARBITRARY CHARACTERISTIC

HARM DERKSEN AND VISU MAKAM

ABSTRACT. For tensors in $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$, Landsberg provides non-trivial equations for tensors of border rank $2d - 3$ for d even and $2d - 5$ for d odd in [1]. In [6], we observe that Landsberg's method can be interpreted in the language of tensor blow-ups of matrix spaces, and using concavity of blow-ups we improve the case for odd d from $2d - 5$ to $2d - 4$. The purpose of this paper is to show that the aforementioned results extend to tensors in $K^d \otimes K^d \otimes K^d$ for any field K .

1. INTRODUCTION

Over the last decade, tensors have received a lot of attention as a consequence of its wide ranging applications in mathematics as well as other scientific disciplines. We refer to [2] for several open conjectures in the subject, as well as a detailed introduction to the subject. The subject begins with the concept of tensor rank which is a generalization of matrix rank.

Definition 1.1. For a tensor $T \in K^{a_1} \otimes K^{a_2} \otimes \cdots \otimes K^{a_i}$, we define its tensor rank $\text{trk}(T)$ to be the smallest integer m such that T can be written as a sum of m pure tensors.

Let Z_m denote the set of tensors of rank $\leq m$. The set Z_m need not be Zariski closed, and we consider its Zariski closure \overline{Z}_m . This gives rise to the definition of border rank.

Definition 1.2. For a tensor T , we define its border rank $\text{brk}(T)$ to be the smallest integer m such that $T \in \overline{Z}_m$.

It is only natural to try and understand the polynomials that define the closed subset \overline{Z}_m . If f is a polynomial that vanishes on \overline{Z}_m (or even Z_m), then if $f(T) \neq 0$ for some tensor, we immediately know that $\text{brk}(T) > m$. In other words, f can be used a test to prove that a tensor has border rank $> m$.

1.1. Blow-ups of linear subspaces. Flattenings are a useful tool to find polynomial tests for the border rank of tensors in $K^a \otimes K^b \otimes K^c$. We present flattenings using the language of blow-ups of linear subspaces of matrices and the combinatorics of their ranks. We will recall these notions briefly.

Let $\text{Mat}_{r,s}$ denote the set of $r \times s$ matrices with entries in the field K . Let \mathcal{X} be a linear subspace of $\text{Mat}_{r,s}$. We define blow-ups of \mathcal{X} .

Definition 1.3. Let $\mathcal{X} \subseteq \text{Mat}_{r,s}$ be a linear subspace. We define its (p, q) tensor blow-up $\mathcal{X}^{\{p,q\}}$ to be

$$\mathcal{X} \otimes \text{Mat}_{p,q} = \left\{ \sum_i X_i \otimes T_i \mid X_i \in \mathcal{X}, T_i \in \text{Mat}_{p,q} \right\},$$

viewed as a subspace of $\text{Mat}_{rp, sq}$. We will write $\mathcal{X}^{\{d\}} = \mathcal{X}^{\{d,d\}}$.

The authors were supported by NSF grant DMS-1601229.

Definition 1.4. The rank of a linear subspace $\mathcal{X} \subseteq \text{Mat}_{r,s}$ is given by

$$\text{rk}(\mathcal{X}) = \max\{\text{rk}(X) \mid X \in \mathcal{X}\}.$$

We will now describe a method of finding lower bounds for border rank in tensor product spaces with three tensor factors. Given a tensor $T \in K^a \otimes K^b \otimes K^c$, we can write $T = \sum_i s_i \otimes X_i$, with $s_i \in K^a$ and $X_i \in K^b \otimes K^c$. Let $L : K^a \rightarrow \text{Mat}_{p,q}$ be a linear map, and denote the image by \mathcal{X}_L . We identify $K^b \otimes K^c$ with $\text{Mat}_{b,c}$, and identify $\text{Mat}_{p,q} \otimes \text{Mat}_{b,c}$ with $\text{Mat}_{pb,qc}$. This gives the following map.

$$\begin{aligned} \phi_L : K^a \otimes K^b \otimes K^c &\longrightarrow \text{Mat}_{pb,qc} \\ \sum_i s_i \otimes X_i &\longmapsto \sum_i L(s_i) \otimes X_i. \end{aligned}$$

In [6], we describe how this map can be used to prove lower bounds for tensors.

Lemma 1.5 ([6]). *We have $\text{brk}(T) \geq \frac{\text{rk}(\phi_L(T))}{\text{rk}(\mathcal{X}_L)}$.*

Corollary 1.6 ([6]). *Let $D = m \text{rk}(\mathcal{X}_L)$. Then the $(D+1) \times (D+1)$ minors of $\phi_L(T)$ are polynomials that vanish on \overline{Z}_m .*

The difficult part in using such a method to prove lower bounds for border rank of tensors is that the aforementioned polynomials coming from the minors might simply turn out to be the zero polynomial. In [6], we give a criterion for these minors to be nontrivial polynomials in terms of the ranks of blow-ups of the linear subspace \mathcal{X}_L .

Lemma 1.7 ([6]). *One of the $d \times d$ minors of ϕ_L is a nontrivial polynomial if and only if $\text{rk}(\mathcal{X}_L^{\{b,c\}}) \geq d$.*

In order to use this method effectively, one would require a linear subspace for which the ranks of the blow-ups are much larger than expected. For this phenomenon to happen, it is useful to pick linear subspaces with a large ratio of noncommutative rank to rank. While we do not recall the notion of noncommutative rank, we refer to [6] for a discussion on the extremal examples and the limitations of such methods.

1.2. Lower bounds for border rank of tensors in $K^d \otimes K^d \otimes K^d$. Let $m = 2p + 1$ be a positive integer. Let $L : K^m \rightarrow \text{Hom}(\bigwedge^p K^m, \bigwedge^{p+1} K^m)$ be given by $L(v) : w \mapsto v \wedge w$, and let \mathcal{X}_L be its image. We write L_v instead of $L(v)$, so L_v is a linear map from $\bigwedge^p K^m$ to $\bigwedge^{p+1} K^m$ for all v . Let e_1, e_2, \dots, e_m be the standard basis of K^m . We will write L_i instead of $L_{e_i} = L(e_i)$.

Proposition 1.8. *For $1 \leq r \leq 2p + 1$, let S_r be the $(p+1) \times (p+1)$ matrix such that*

$$S_r(j, k) = \begin{cases} 1 & \text{if } k - j = p + 1 - r \\ 0 & \text{otherwise} \end{cases}$$

Then $L := L_1 \otimes S_1 + L_2 \otimes S_2 + \dots + L_{2p+1} \otimes S_{2p+1}$ is invertible.

The S_i are the most obvious basis of the space of $(p+1) \times (p+1)$ Toeplitz matrices.

Example 1.9. For $p = 1$, we have

$$S_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, S_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, S_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The above proposition was proved by Landsberg in [1] for $K = \mathbb{C}$. Landsberg's approach is to interpret L as a certain multiplication map, which is shown to be surjective in [3]. However, the argument for surjectivity requires the underlying field to be characteristic 0. Our approach to this is far more elementary and we simply compute the determinant of L in a chosen basis.

Theorem 1.10. *We have:*

- (1) $\text{rk}(\mathcal{X}_L) = \binom{2p}{p}$;
- (2) $\text{rk}(\mathcal{X}_L^{\{m+1\}})$ is full;
- (3) $\text{rk}(\mathcal{X}_L^{\{m\}}) > \binom{2p}{p}(2m - 4)$.

When $K = \mathbb{C}$, (1) and (3) can be found in [6], and (2) can be found in [1], although the presentation in [1] is not quite the same as ours. The contribution of this paper is to show that these results are true for any field K . Note that the proof of (1) in [6] holds for any field K .

Applying Lemma 1.5, Corollary 1.6 and Lemma 1.7, we get equations for the variety of tensors in $K^d \otimes K^d \otimes K^d$ of border rank at most $2d - 3$ (resp. $2d - 4$) when d is even (resp. odd). This is done for the case $K = \mathbb{C}$ in [1, 6]. The contribution of this paper is to extend Theorem 1.10 to any field K , and hence the aforementioned results on the equations for border rank extend to any field K . We will formulate the precise statements in Section 6.

1.3. Tensor rank and border rank for 3×3 determinant and permanent. We illustrate the method described above to compute the border rank and tensor rank for the 3×3 determinant and permanent tensors. The 3×3 determinant tensor is

$$\det_3 = \sum_{\sigma \in \Sigma_3} \text{sgn}(\sigma) e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes e_{\sigma(3)},$$

where Σ_3 denotes the symmetric group in 3 letters. The 3×3 permanent tensor is

$$\text{perm}_3 = \sum_{\sigma \in \Sigma_3} e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes e_{\sigma(3)}.$$

Let $L : K^3 \rightarrow \text{Hom}(\bigwedge^1 K^3, \bigwedge^2 K^3) \cong \text{Mat}_{3,3}$ be the map defined in Section 1.2 for $p = 1$ (i.e., $m = 3$). Let ϕ_L denote the composite map $K^3 \otimes K^3 \otimes K^3 \rightarrow \text{Mat}_{3,3} \otimes \text{Mat}_{3,3} \rightarrow \text{Mat}_{9,9}$ as described in the introduction. We have $\text{rk}(\mathcal{X}_L) = 2$, by part (1) of Theorem 1.10.

Lemma 1.11. *We have $\text{brk}(\det_3) \geq 5$ if $\text{char } K \neq 2$.*

Proof. The matrix $\phi_L(\det_3)$ is an explicit 9×9 matrix, which can be checked to be invertible if $\text{char } K \neq 2$. We write the matrix explicitly. For this, we need to first choose ordered basis. Let (e_1, e_2, e_3) denote the standard ordered basis for $K^3 = \bigwedge^1 K^3$. We choose the ordered basis $(e_2 \wedge e_3, e_3 \wedge e_1, e_1 \wedge e_2)$. In the corresponding basis for $\text{Hom}(\bigwedge^1 K^3, \bigwedge^2 K^3)$, we compute the matrices $L_i = L_{e_i} = L(e_i)$. We have

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \text{ and } L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For $\text{Mat}_{3,3}$, let $E_{j,k}$ denote the 3×3 matrix whose $(j, k)^{\text{th}}$ entry is 1 and all other entries are 0. Now, let us identify $K^3 \otimes K^3$ with $\text{Mat}_{3,3}$ explicitly by identifying $e_j \otimes e_k$ with $E_{j,k}$.

Thus, we have $\phi_L(e_i \otimes e_j \otimes e_k) = L_i \otimes E_{j,k}$. With these choices of coordinates, we write out $\phi_L(\det_3)$. We get

$$\phi_L(\det_3) = \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ \hline 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ \hline 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{array} \right)$$

This matrix contains only 12 nonzero entries of the form ± 1 . Six of these entries (marked red) are in a column or a row with no other nonzero entry, reducing our computation to a 3×3 minor. It is easy to see that this minor is of full rank if $\text{char } K \neq 2$ (and drops rank by 1 if $\text{char } K = 2$).

Hence, by Lemma 1.5, we have $\text{brk}(\det_3) \geq \frac{9}{2} = 4.5$. Since the border rank must be an integer, it must be at least 5. □

On the other hand, there is an explicit decomposition of \det_3 as a sum of 5 simple tensors if $\text{char } K \neq 2$, see [4].

Corollary 1.12. *Assume $\text{char } K \neq 2$. Then we have $\text{trk}(\det_3) = \text{brk}(\det_3) = 5$.*

Lemma 1.13. *We have $\text{brk}(\text{perm}_3) \geq 4$.*

Proof. A similar computation shows that rank of $\phi_L(\text{perm}_3)$ is 8. Hence we have $\text{brk}(\text{perm}_3) \geq \frac{8}{2} = 4$. □

Once again, if $\text{char } K \neq 2$, there is an explicit decomposition of perm_3 as a sum of 4 simple tensors due to Glynn, see [8].

Corollary 1.14. *Assume $\text{char } K \neq 2$. Then we have $\text{brk}(\text{perm}_3) = \text{trk}(\text{perm}_3) = 4$.*

In characteristic 0, the tensor rank of \det_3 and perm_3 were shown to be 5 and 4 respectively in [9]. While the arguments for bounding the tensor rank from above are still the same (i.e., explicit decompositions), the arguments for bounding the tensor rank from below are more complicated. Their approach is to analyze certain Fano schemes parametrizing linear subspaces contained in the hypersurfaces $\det_3 = 0$ and $\text{perm}_3 = 0$, and even involves a computation done with the help of a computer. The method we use for the lower bounds is far more elementary and holds in arbitrary characteristic.

1.4. Organization. In Sections 2 and 3, we develop the necessary linear algebra techniques. We prove Proposition 1.8 in an example in Section 4 and prove the main theorems in Section 5. Finally in Section 6, we give explicit equations for border rank.

2. PRELIMINARIES FROM LINEAR ALGEBRA

Let $\mathcal{B} = \{v_1, \dots, v_n\}$ denote an ordered basis for an n -dimensional vector space V . Consider the alternating power $\bigwedge^r V$. For a subset $I = \{i_1, \dots, i_r\} \subseteq [n]$ of size r , with $i_1 < i_2 < \dots < i_r$, we define $v_I = v_{i_1} \wedge v_{i_2} \wedge \dots \wedge v_{i_r}$. Here $[n]$ denotes the set $\{1, 2, \dots, n\}$. The following lemma is a well known fact.

Lemma 2.1. *For a given ordered basis $\mathcal{B} = (v_1, \dots, v_n)$ for K^n , define $\mathcal{B}(r)$ as the set $\{v_I \mid I \subseteq \{1, 2, \dots, n\}, \text{ with } |I| = r\}$ ordered lexicographically. Then $\mathcal{B}(r)$ is an ordered basis for $\bigwedge^r V$.*

Example 2.2. Let $n = 3$, and $r = 2$, then $\mathcal{B}(r)$ is the ordered basis $(v_{1,2}, v_{1,3}, v_{2,3})$.

Definition 2.3. Given an ordered basis $\mathcal{B} = (v_1, \dots, v_n)$ of V and an ordered basis $\mathcal{C} = \{w_1, \dots, w_m\}$ of W , we define $x_{i,j} = v_i \otimes w_j$. By $\mathcal{B} \otimes \mathcal{C}$, we mean the set $\{x_{i,j} \mid i \in [n], j \in [m]\}$ ordered lexicographically. This is a basis of $V \otimes W$.

Example 2.4. Let $n = 2, m = 2$, then $\mathcal{B} \otimes \mathcal{C} = (v_1 \otimes w_1, v_1 \otimes w_2, v_2 \otimes w_1, v_2 \otimes w_2) = (x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2})$.

Suppose that \mathcal{B} is a basis of V and \mathcal{C} is a basis of W and $L : V \rightarrow W$ is a linear map. Then $L_{\mathcal{C}, \mathcal{B}}$ denotes the matrix of the transformation L with respect to the bases \mathcal{B} and \mathcal{C} . If $M : W \rightarrow Z$ is a linear map and \mathcal{D} is a basis of Z , then we have $(ML)_{\mathcal{D}, \mathcal{B}} = M_{\mathcal{D}, \mathcal{C}} L_{\mathcal{C}, \mathcal{B}}$.

Let $\mathcal{B} = (b_1, b_2, \dots, b_n)$ and $\mathcal{B}' = (b'_1, b'_2, \dots, b'_n)$ be two ordered bases for V . Then denote by $X_{\mathcal{B}, \mathcal{B}'} = (\text{id}_V)_{\mathcal{B}, \mathcal{B}'}$ be the matrix of the identity with respect to \mathcal{B} and \mathcal{B}' . This is the base change matrix and its columns are the vectors b'_1, b'_2, \dots, b'_n expressed in the basis \mathcal{B} . Note that $X_{\mathcal{B}', \mathcal{B}} = X_{\mathcal{B}, \mathcal{B}'}^{-1}$. We recall the base change formula for linear transformations.

Lemma 2.5 (Base change formula). *We have $L_{\mathcal{C}', \mathcal{B}'} = X_{\mathcal{C}', \mathcal{C}} L_{\mathcal{C}, \mathcal{B}} X_{\mathcal{B}, \mathcal{B}'} = X_{\mathcal{C}, \mathcal{C}'}^{-1} L_{\mathcal{C}, \mathcal{B}} X_{\mathcal{B}, \mathcal{B}'}$.*

Let $\mathcal{B} = (b_1, b_2, \dots, b_n)$ be an ordered basis of V and we multiply the i^{th} basis vector by some scalar $\lambda \neq 0$ to obtain the basis $\mathcal{B}' = (b_1, \dots, b_{i-1}, \lambda b_i, b_{i+1}, \dots, b_n)$. Then $X_{\mathcal{B}, \mathcal{B}'}$ is a diagonal matrix. The i^{th} diagonal entry of $X_{\mathcal{B}, \mathcal{B}'}$ is λ and all other diagonal entries are 1. In particular, we have $\det(X_{\mathcal{B}, \mathcal{B}'}) = \lambda$. For our purposes we need to understand a more interesting base change matrix.

Proposition 2.6. *With \mathcal{B} and \mathcal{B}' as above, we have we have $\det(X_{\mathcal{B}(r), \mathcal{B}'(r)}) = \lambda^{\binom{n-1}{r-1}}$.*

Proof. It is easy to see that the basis $\mathcal{B}'(r)$ is gotten from $\mathcal{B}(r)$ by scaling some of its basis vectors. More precisely, if a subset I contains i , then the basis vector b_I is scaled by λ . All other basis vectors remain unchanged. The number of subsets containing i is given by $\binom{n-1}{r-1}$. Hence $X_{\mathcal{B}(r), \mathcal{B}'(r)}$ is a diagonal matrix in which $\binom{n-1}{r-1}$ diagonal entries are λ and all other diagonal entries are 1. The proposition follows since the determinant of a diagonal matrix is the product of the diagonal entries. \square

We also need to understand what happens to a linear transformation $L \in \text{Hom}(\bigwedge^r V, \bigwedge^{r+1} V)$ when we change basis. For a basis \mathcal{B} of V , let $L_{\mathcal{B}} = L_{\mathcal{B}(r+1), \mathcal{B}(r)}$ denote the matrix of L in the basis $\mathcal{B}(r)$ and $\mathcal{B}(r+1)$ for the domain and codomain respectively.

Corollary 2.7. *Let \mathcal{B} and \mathcal{B}' be as in Proposition 2.6. Then for $L \in \text{Hom}(\bigwedge^r V, \bigwedge^{r+1} V)$, we have $\det(L_{\mathcal{B}'}) = \lambda^{\binom{n-1}{r-1} - \binom{n-1}{r}} \det(L_{\mathcal{B}})$.*

Proof. This follows from applying Proposition 2.6 to the base change formula

$$L_{\mathcal{B}'} = X_{\mathcal{B}(r+1), \mathcal{B}'(r+1)}^{-1} L_{\mathcal{B}} X_{\mathcal{B}(r), \mathcal{B}'(r)}.$$

□

In fact, we need slightly more general results. An argument along the lines of the proof of Proposition 2.6 gives the following lemma.

Lemma 2.8. *Let \mathcal{B} and \mathcal{B}' be as in Proposition 2.6. Let W be a c -dimensional vector space with ordered basis \mathcal{C} . Then we have $\det(X_{\mathcal{B}(r) \otimes \mathcal{C}, \mathcal{B}'(r) \otimes \mathcal{C}}) = \lambda^{c \binom{n-1}{r-1}}$.*

For a linear transformation $L \in \text{Hom}((\bigwedge^r V) \otimes W, (\bigwedge^{r+1} V) \otimes W)$, let $L_{\mathcal{B} \otimes \mathcal{C}}$ denote the matrix for the linear transformation of L in the bases $\mathcal{B}(r) \otimes \mathcal{C}$ and $\mathcal{B}(r+1) \otimes \mathcal{C}$ for the domain and codomain respectively. Following the same idea as Corollary 2.7, we get the following:

Corollary 2.9. *Let \mathcal{B} and \mathcal{B}' be as in Proposition 2.6. Then for a linear transformation $L \in \text{Hom}((\bigwedge^r V) \otimes W, (\bigwedge^{r+1} V) \otimes W)$, we have $\det(L_{\mathcal{B}' \otimes \mathcal{C}}) = \lambda^{c \left(\binom{n-1}{r-1} - \binom{n-1}{r} \right)} \det(L_{\mathcal{B} \otimes \mathcal{C}})$.*

3. EFFECTS OF SCALING BASIS VECTORS ON THE MATRICES OF L_i 'S

Let $m = 2p + 1$ be a positive integer. Let $\mathcal{E} = (e_1, \dots, e_m)$ denote the standard ordered basis of K^m . Recall that for a $v \in K^m$, $L_v \in \text{Hom}(\bigwedge^p K^m, \bigwedge^{p+1} K^m)$ is the linear map that sends w to $v \wedge w$. Let \mathcal{E}' be the ordered basis obtained from \mathcal{E} by scaling the i^{th} basis vector by λ , i.e., $\mathcal{E}' = (e_1, \dots, e_{i-1}, \lambda e_i, e_{i+1}, \dots, e_m)$. It is easy to understand the effect of this base change on the matrices of L_i .

Lemma 3.1. *We have $(L_j)_{\mathcal{E}'} = \begin{cases} (L_j)_{\mathcal{E}} & \text{if } j \neq i, \\ \lambda^{-1} (L_i)_{\mathcal{E}} & \text{if } j = i. \end{cases}$*

Proof. It is easy to see that for any basis $\mathcal{B} = (b_1, \dots, b_m)$ of K^m , the matrix of L_{b_i} written in the basis $\mathcal{B}(r)$ and $\mathcal{B}(r+1)$ is the same, i.e., $(L_{b_i})_{\mathcal{B}} = (L_{c_i})_{\mathcal{C}}$ for any other basis $\mathcal{C} = (c_1, \dots, c_m)$. For $j \neq i$, we have $e_j = e'_j$, and hence

$$(L_j)_{\mathcal{E}'} := (L_{e_j})_{\mathcal{E}'} = (L_{e'_j})_{\mathcal{E}'} = (L_{e_j})_{\mathcal{E}} =: (L_j)_{\mathcal{E}}.$$

For $j = i$, we have $e_i = \lambda^{-1} e'_i$, and so

$$(L_i)_{\mathcal{E}'} := (L_{e_i})_{\mathcal{E}'} = (L_{\lambda^{-1} e'_i})_{\mathcal{E}'} = \lambda^{-1} (L_{e'_i})_{\mathcal{E}'} = \lambda^{-1} (L_{e_i})_{\mathcal{E}} =: \lambda^{-1} (L_i)_{\mathcal{E}}.$$

□

Let

$$L = L_1 \otimes S_1 + L_2 \otimes S_2 + \dots + L_{2p+1} \otimes S_{2p+1} \in \text{Hom}((\bigwedge^p K^m) \otimes K^{p+1}, (\bigwedge^{p+1} K^m) \otimes K^{p+1}),$$

where S_i is defined as in Proposition 1.8. Let \mathcal{F} denote the standard basis of K^{p+1} . Hence we have the bases $\mathcal{E}(p) \otimes \mathcal{F}$ and $\mathcal{E}'(p) \otimes \mathcal{F}$ for the domain and the bases $\mathcal{E}(p+1) \otimes \mathcal{F}$ and $\mathcal{E}'(p+1) \otimes \mathcal{F}$ for the codomain. Recall that for a linear transformation $L \in \text{Hom}((\bigwedge^r V) \otimes W, (\bigwedge^{r+1} V) \otimes W)$, $L_{\mathcal{B} \otimes \mathcal{C}}$ denotes the matrix for the linear transformation of L in the bases $\mathcal{B}(r) \otimes \mathcal{C}$ and $\mathcal{B}(r+1) \otimes \mathcal{C}$ for the domain and codomain respectively, where \mathcal{B} is a basis for V and \mathcal{C} is a basis for W .

Lemma 3.2. We have $\det(L_{\mathcal{E}' \otimes \mathcal{E}}) = \lambda^{-\binom{2p}{p}} \det(L_{\mathcal{E} \otimes \mathcal{E}})$

Proof. This follows from Corollary 2.9, since $(p+1)\binom{2p}{p-1} - \binom{2p}{p} = -\binom{2p}{p}$. \square

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m) \in K^m$ such that $\lambda_i \neq 0$ for $1 \leq i \leq m$. Given an ordered basis $\mathcal{E} = (e_1, \dots, e_m)$, we define another ordered basis $\lambda \cdot \mathcal{E} = (\lambda_1 e_1, \lambda_2 e_2, \dots, \lambda_m e_m)$. Applying the above lemma several times, we get:

Corollary 3.3. We have $\det(L_{(\lambda \cdot \mathcal{E}) \otimes \mathcal{E}}) = \left(\prod_{i=1}^m \lambda_i \right)^{-\binom{2p}{p}} \det(L_{\mathcal{E} \otimes \mathcal{E}})$.

Definition 3.4. Let M_i denote the matrix $(L_i)_{\mathcal{E}}$. We define

$$M(t_1, \dots, t_{2p+1}) := t_1 M_1 \otimes S_1 + t_2 M_2 \otimes S_2 + \dots + t_{2p+1} M_{2p+1} \otimes S_{2p+1}.$$

Define $p(t_1, \dots, t_{2p+1}) := \det(M(t_1, \dots, t_{2p+1}))$.

Corollary 3.5. We have $p(t_1, \dots, t_m) = \left(\prod_{i=1}^m t_i \right)^{\binom{2p}{p}} p(1, 1, \dots, 1)$.

Proof. This follows from applying Lemma 3.1 to Corollary 3.3, where $\lambda = (t_1^{-1}, t_2^{-1}, \dots, t_m^{-1})$. \square

4. EXAMPLES

Let us first recall that for an $m \times n$ matrix $A = (a_{i,j})$ and a $B = (b_{k,l})$, we define the Kronecker product $A \otimes B$ by

$$A \otimes B = \begin{pmatrix} a_{1,1}B & \dots & a_{1,n}B \\ \vdots & \ddots & \vdots \\ a_{m,1}B & \dots & a_{m,n}B \end{pmatrix}$$

If $A = (a_{i,j})$ is a square $n \times n$ matrix, then its determinant is equal to $\sum_{\sigma \in \Sigma_n} \text{sgn}(\sigma) r_\sigma$, where σ runs over all elements of the symmetric group Σ_n , $\text{sgn}(\sigma)$ is the sign of the permutation σ and $r_\sigma = \prod_{i=1}^n a_{i,\sigma(i)}$. To proceed further, we believe it is necessary to acquaint the reader with small examples.

Example 4.1 ($p = 1$). Suppose that $p = 1$ and $m = 3$. Let $\mathcal{E} = (e_1, e_2, e_3)$ be the standard basis of K^3 . Then the basis $\mathcal{E}(1)$ is \mathcal{E} itself, and the basis $\mathcal{E}(2) = (e_{1,2}, e_{1,3}, e_{2,3})$. In this basis $t_1 L_1 \otimes S_1 + t_2 L_2 \otimes S_2 + t_3 L_3 \otimes S_3$ is given by the block matrix

$$A := \begin{pmatrix} -t_2 S_2 & t_1 S_1 & 0 \\ -t_3 S_3 & 0 & t_1 S_1 \\ 0 & -t_3 S_3 & t_2 S_2 \end{pmatrix}$$

In other words $A = M(t_1, t_2, t_3)$. We also write out S_i . We have

$$S_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, S_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, S_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Observe that the matrix A is a 6×6 matrix with entries in $\mathbb{Z}[t_1, t_2, t_3]$. We will try to compute $\det A$ as an element of this ring. In fact, this has been computed by Domokos in [7] already in the context of understanding semi-invariants for Kronecker quivers. We will

however analyze the situation thoroughly as it will be useful in handling the general case. We know $\det A = k(t_1 t_2 t_3)^2$ by Corollary 3.5 and we want to establish that $k = \pm 1$.

Recall that $\det A = \sum_{\sigma \in \Sigma_6} \text{sgn}(\sigma) r_\sigma$, with $r_\sigma = \prod_{i=1}^6 a_{i, \sigma(i)}$. Now, observe that each entry of A is either 0 or $\pm t_i$. Hence each r_σ is either 0 or \pm monomial (in the t_i 's). We know that the final answer must be a multiple of the monomial $(t_1 t_2 t_3)^2$. So, it suffices to focus on the permutations σ such that $r_\sigma = \pm t_1^2 t_2^2 t_3^2$.

We claim that there is at most one permutation σ such that $r_\sigma = \pm t_1^2 t_2^2 t_3^2$. In other words, there is at most one choice of 6 entries, satisfying the condition that no two entries are in the same row and no two entries are in the same column such that the product of their entries is $\pm t_1^2 t_2^2 t_3^2$.

To see this, observe first there are only two entries of the form $\pm t_1$, since $t_1 S_1 = \begin{pmatrix} 0 & t_1 \\ 0 & 0 \end{pmatrix}$ and there are exactly two blocks which are $\pm t_1 S_1$. So, in order to get t_1^2 , we have no choice but to pick both entries.

Now, there are four entries of the form $\pm t_2$, two in each block of the form $\pm t_2 S_2$. Consider the northwest $-t_2 S_2$ block. This block occurs in the same block row as a $t_1 S_1$. We focus on these two blocks in the top block row.

$$(-t_2 S_2 \mid t_1 S_1) = \left(\begin{array}{cc|cc} -t_2 & 0 & 0 & t_1 \\ 0 & -t_2 & 0 & 0 \end{array} \right)$$

We have already argued that we must pick the blue t_1 in the $t_1 S_1$, since all $\pm t_1$'s must be picked. Hence we cannot pick any other entry from that row. This rules out the $-t_2$ that we have colored red. So only the $-t_2$ from the bottom row is available, which we have colored blue. A similar argument shows that you can only pick the t_2 in the left column of the southeast most block of the form $t_2 S_2$. Since there are only two $\pm t_2$'s available, we have no choice but to pick both of them.

Remark 4.2. We want to think of this in the following way. While considering the northwest block entry $-t_2 S_2$, we observe that there is exactly 1 block entry of the form $\pm t_i S_i$ in the same row with $i < 2$. This is the condition that rules out the top 1 rows. Similarly, there are 0 block entries of the form $\pm t_i S_i$ in the same column with $i < 2$. This is the condition that rules out the right 0 columns. This leaves precisely one non-zero entry in the northwest $t_2 S_2$ to choose from. A generalization of such an argument (see Proposition 5.11) will be the key to unlocking the general case.

Continuing with the example, observe that there are only two $\pm t_3$'s, and hence we must pick both of them. These $\pm t_3$'s could potentially be in the same row or column as the choices of t_1 's and t_2 's, which would be disastrous. However, this doesn't happen. In this case, one can check explicitly. In the general case, however, instead of an explicit check we will use the generalization of the argument mentioned in the above remark. Hence, there is exactly one permutation σ for which $r_\sigma = \pm (t_1 t_2 t_3)^2$. Thus we have that $\det A = \pm (t_1 t_2 t_3)^2$.

5. THE GENERAL CASE

We will prove Proposition 1.8 and consequently Theorem 1.10 in this section. Let $m = 2p + 1$ be a positive integer, and let $A := M(t_1, \dots, t_m)$. We will begin with some structural results on the matrix A . Let $M = t_1 M_1 + \dots + t_{2p+1} M_{2p+1}$.

Example 5.1. For $p = 1$, we have

$$M = \begin{pmatrix} -t_2 & t_1 & 0 \\ -t_3 & 0 & t_1 \\ 0 & -t_3 & t_2 \end{pmatrix},$$

and

$$A = \begin{pmatrix} -t_2 S_2 & t_1 S_1 & 0 \\ -t_3 S_3 & 0 & t_1 S_1 \\ 0 & -t_3 S_3 & t_2 S_2 \end{pmatrix}.$$

Lemma 5.2. *The matrix M is a $\binom{2p+1}{p} \times \binom{2p+1}{p}$ matrix, whose block entries are either 0 or $\pm t_i$.*

Proof. The positions of the nonzero entries of M_i 's are clearly distinct. \square

Lemma 5.3. *For each $i \in [2p + 1]$, there are $\binom{2p}{p}$ entries of the form $\pm t_i$ in M , and all other entries are 0.*

Proof. There are $\binom{2p}{p}$ subsets I of size p that do not contain i . For each such subset I , we have $L_i(e_I) = \pm e_{I \cup i}$. The corresponding entry in the matrix is ± 1 , and all other entries are 0. Thus $t_i M_i$ is a matrix with $\binom{2p}{p}$ entries of the form $\pm t_i$, and all other entries 0. Since the positions of the nonzero entries of the $t_i M_i$ are distinct from the positions of nonzero entries of $t_j M_j$ for $i \neq j$, we have the required conclusion. \square

Lemma 5.4. *Fix an entry $\pm t_i$ in M . Then for each $j \neq i$, then the number of entries of the form $\pm t_j$ in the same row or column is exactly 1.*

Proof. The fixed entry $\pm t_i$ in M corresponds to the fact that $L_i(e_I) = \pm e_{I \cup \{i\}}$ for some I that does not contain i . Now, if $j \in I$, then let $J = I \cup \{i\} \setminus \{j\}$. Then we have $L_j(e_J) = \pm e_{J \cup \{j\}} = \pm e_{I \cup \{i\}}$. This corresponds to a $\pm t_j$ in the same row. On the other hand if $j \notin I$, then $L_j(e_I) = \pm e_{I \cup \{j\}}$ which corresponds to a $\pm t_j$ in the same column. \square

Remark 5.5. It follows from the definition of the tensor product of matrices that by replacing each t_i in M with the block matrix $t_i S_i$, we get the block matrix A . See Example 5.1.

The above remark applied to the above lemmas yield:

Corollary 5.6. *The matrix A is a $\binom{2p+1}{p}$ -block matrix, whose block entries are either 0 or $\pm t_i S_i$.*

Corollary 5.7. *For each $i \in [2p + 1]$, there are $\binom{2p}{p}$ block entries of the form $\pm t_i S_i$ in A , and all other block entries are 0.*

Corollary 5.8. *Fix a block entry $\pm t_i S_i$ in A . Then for each $j \neq i$, the number of block entries of the form $\pm t_j S_j$ in the same block row or same block column is exactly 1.*

Definition 5.9. Let $P = \pm t_i S_i$ be a block entry of A . Suppose there are x entries of the form $\pm t_j S_j$ with $j < i$ in the same block row and y entries of the form $\pm t_j S_j$ in the same block column. Then we call the $(x + 1, p - y)^{th}$ entry of P , the *elusive entry* of P .

Lemma 5.10. *The elusive entry of any block $P = \pm t_i S_i$ is a $\pm t_i$. Further, with x and y as defined in the previous definition, all other nonzero entries of P are in the top x rows or the right y columns.*

Proof. The equality $x+y = i-1$ follows from Corollary 5.8. Indeed, we have $S_i(x+1, p-y) = 1$ as $p-y = x+1-i+p+1$ follows from $x+y = i-1$. Thus there is a t_i in position $(x+1, p-y)$ in the block P . The second statement is obvious since the only nonzero entries are along the diagonal containing $(x+1, p-y)$. \square

Let us recall that a permutation $\sigma \in \Sigma_n$ is a choice of n entries subject to the condition that there are no two entries in the same row and no two entries in the same column. In order for $r_\sigma = \pm(t_1 t_2 \dots t_{2p+1})^{\binom{2p}{p}}$, we must make such a choice, where each entry chosen is of the form $\pm t_i$ and for each i , there are $\binom{2p}{p}$ entries chosen of the form $\pm t_i$.

Proposition 5.11. *In order for $r_\sigma = \pm(t_1 t_2 \dots t_{2p+1})^{\binom{2p}{p}}$, we must choose the elusive entry from each nonzero block entry.*

Proof. Let $P = \pm t_i S_i$ be a nonzero block entry of A . We proceed by induction on i .

- **Base Case:** $i = 1$.

In this case, observe that there is exactly one nonzero entry, which is $\pm t_1$, and that is precisely the elusive entry. There are $\binom{2p}{p}$ such block entries. In order for the power of t_1 in r_σ to be $\binom{2p}{p}$, we have no choice but to choose the elusive entries from each block entry of the form $\pm t_1 S_1$.

- **Induction Step:**

Suppose the claim is true for all $j < i$. Let the block entries in the same row of the form $\pm t_k S_k$ with $k < i$ be $Q_1 = \pm t_{j_1} S_{j_1}, Q_2 = \pm t_{j_2} S_{j_2}, \dots, Q_x = \pm t_{j_x} S_{j_x}$ with $1 \leq j_1 < j_2 < \dots < j_x < i$. Then clearly the block entry Q_k satisfies the hypothesis of the claim for $k-1$. Hence, by induction we would have picked the $\pm t_{j_k}$ from the k^{th} row. Hence, we cannot pick the $\pm t_i$'s in the first x rows of P .

By a similar argument, we cannot pick the t_i 's in the right y columns, where y is the number of the block entries of the form $\pm t_k S_k$ with $k < i$ in the same column. This leaves precisely one non-zero entry in P , which is the elusive entry. Now, once again we have precisely $\binom{2p}{p}$ blocks of the form $\pm t_i S_i$, and we can pick at most one $\pm t_i$ from each one. Since we want the power of t_i in r_σ to be $\binom{2p}{p}$, we have no choice but to pick all of them. \square

Corollary 5.12. *There is at most one permutation σ such that $r_\sigma = \pm(t_1 t_2 \dots t_{2p+1})^{\binom{2p}{p}}$.*

Proof of Proposition 1.8. We know that $p(t_1, \dots, t_m) = \det(M(t_1, \dots, t_m)) = k(t_1 t_2 \dots t_{2p+1})^{\binom{2p}{p}}$, where $k = p(1, \dots, 1) \in K$ by Corollary 3.5. We also know that each r_σ is \pm monomial. Further, by the above Proposition, there is exactly one r_σ which gives us $\pm(t_1 t_2 \dots t_{2p+1})^{\binom{2p}{p}}$, and hence we must have $k = \pm 1 \neq 0$. But $k = p(1, \dots, 1)$, and hence L is invertible, since $p(1, \dots, 1) = \det M(1, \dots, 1)$ and $M(1, \dots, 1)$ is the matrix for L in some coordinates. \square

Proof of Theorem 1.10. As remarked before, the proof of (1) as done in [6] works for any field K . For (2), observe that $(L_1 \otimes (S_1 \oplus S_1) + \dots + L_{2p+1} \otimes (S_{2p+1} \oplus S_{2p+1})) \in \mathcal{X}_L^{2p+2} = \mathcal{X}_L^{m+1}$ is invertible since we have

$$\begin{aligned} L_1 \otimes (S_1 \oplus S_1) + \cdots + L_m \otimes (S_m \oplus S_m) &= \\ &= (L_1 \otimes S_1 + \cdots + L_m \otimes S_m) \oplus (L_1 \otimes S_1 + \cdots + L_m \otimes S_m). \end{aligned}$$

For (3), we use the computation in [6, Theorem 6.1], which uses concavity of blow-ups proved in [5]. □

6. EXPLICIT EQUATIONS FOR BORDER RANK

In this section, we find non-trivial equations for the border rank of tensors in $K^d \otimes K^d \otimes K^d$. We first treat the case when d is odd.

6.1. The case d is odd. When d is odd, we set $d = m = 2p + 1$. Let $L : K^m \rightarrow \text{Hom}(\bigwedge^p K^m, \bigwedge^{p+1} K^m)$ be as in Section 1.2. Let $D = \dim \bigwedge^p K^m = \dim \bigwedge^{p+1} K^m = \binom{m}{p}$. We have the map

$$\begin{aligned} \phi_L : K^m \otimes K^m \otimes K^m &\longrightarrow \text{Mat}_{mD, mD} \\ \sum_i s_i \otimes X_i &\longmapsto \sum_i L(s_i) \otimes X_i. \end{aligned}$$

Corollary 6.1. *Let $N = (2m - 4) \cdot \binom{2p}{p} + 1$. Then the $N \times N$ minors of ϕ_L are polynomials that vanishes on tensors of border rank $\leq 2m - 4$. At least one of these is a non-trivial polynomial.*

Proof. By Theorem 1.10, we have $\text{rk}(\mathcal{X}_L) = \binom{2p}{p}$, and $\text{rk}(\mathcal{X}_L^m) \geq N$. Applying Corollary 1.6 and Lemma 1.7, we get the required result. □

Remark 6.2. In [6] under the assumption $K = \mathbb{C}$, an explicit tensor of border rank $\geq 2d - 3$ is given. Having extended results to any field K , it is clear that the same tensor has border rank $\geq 2d - 3$ in any field K .

6.2. The case d is even. In this case, we set $m = 2p + 1 = d - 1$. $L : K^m \rightarrow \text{Hom}(\bigwedge^p K^m, \bigwedge^{p+1} K^m)$ be as in Section 1.2. We have the map

$$\begin{aligned} \phi_L : K^m \otimes K^{m+1} \otimes K^{m+1} &\longrightarrow \text{Mat}_{(m+1)D, (m+1)D} \\ \sum_i s_i \otimes X_i &\longmapsto \sum_i L(s_i) \otimes X_i. \end{aligned}$$

$\det(\phi_L)$ is a polynomial on $K^{d-1} \otimes K^d \otimes K^d$. Take any projection $\pi : K^d \rightarrow K^{d-1}$, and let $\psi = \pi \otimes \text{id} \otimes \text{id} : K^d \otimes K^d \otimes K^d \rightarrow K^{d-1} \otimes K^d \otimes K^d$. Let $f = \psi^*(\det \phi_L)$ be the pull back of the polynomial $\det(\phi_L)$ under ψ .

Corollary 6.3. *The polynomial f is a non-trivial polynomial that vanishes on tensors of border rank $\leq 2d - 3$.*

Proof. If $T \in K^d \otimes K^d \otimes K^d$ has border rank $\leq 2d - 3$, then so does $\psi(T)$. A similar calculation as in Corollary 6.1 applied to $\psi(T)$ gives us the required result. □

Remark 6.4. Just as in [1], we have that the tensor $T = \sum_{i=1}^m e_i \otimes (S_i \oplus S_i)$ has border rank $\geq 2d - 2$ since the polynomial f does not vanish on it.

REFERENCES

- [1] J. M. Landsberg, *Non-triviality of equations and explicit tensors in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ of border rank at least $2m - 2$* , J. Pure Appl. Algebra **219** (2015), no. 8, 3677–3684.
- [2] J. M. Landsberg, *Tensors: Geometry and Applications*, Graduate Studies in Mathematics **128**, American Mathematical Society, Providence, RI, 2012.
- [3] J. M. Landsberg and G. Ottaviani, *New lower bounds for the border rank of matrix multiplication*, Theory of Computing **11** (2015), 285-298.
- [4] H. Derksen, *On the nuclear norm and singular value decomposition of tensors*, Found. Comput. Math. **16** (2016), no. 3, 779-811.
- [5] H. Derksen and V. Makam, *Polynomial degree bounds for matrix semi-invariants*, Adv. Math. **310** (2017), 44–63.
- [6] H. Derksen and V. Makam, *On non-commutative rank and tensor rank*, Linear and Multilinear Algebra, published online (2017).
- [7] M. Domokos, *Relative invariants of 3×3 matrix triples*, Linear and Multilinear algebra **47** (2000), no. 2, 175-190.
- [8] D. G. Glynn, *The permanent of a square matrix*, European J. Combin. **31** (2010), no. 7, 1887-1891.
- [9] N. Ilten and Z. Teitler, *Product ranks of 3×3 determinant and permanent*, Canad. Math. Bull. **59** (2016), no. 2, 311-319.