

COHEN-MACAULAY RINGS, COMBINATORICS,
AND SIMPLICIAL COMPLEXES

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§1. Introduction. Let Δ be an (abstract) finite simplicial complex with vertices $x_0, \dots, x_n$ (i.e., a family of subsets of $\{x_0, \dots, x_n\}$ such that if σ is in Δ and τ is a subset of σ then τ is in Δ , and such that $\{x_0\}, \dots, \{x_n\}$ are in Δ). Let K be a commutative ring, usually, a field (throughout, all rings are commutative, with identity). Let $S = K[x_0, \dots, x_n]$ be the polynomial ring generated by $x_0, \dots, x_n$: by abuse of notation, we are regarding the vertices x_i as indeterminates over K . Let I_Δ be the ideal generated by the monomials $x_{i_0} \dots x_{i_r}$, $i_0 < \dots < i_r$, such that $\{x_{i_0}, \dots, x_{i_r}\}$ is not in Δ . Let $K[\Delta] = S/I_\Delta$.

Recently G. Reisner [23] has characterized the Cohen-Macaulayness of $K[\Delta]$ in terms of purely topological properties of Δ and certain

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subcomplexes (the links) of Δ . Oddly enough, the proof depends on results of [13] which study the action of the Frobenius on local cohomology. The results are valid even when $\text{char}(K) = 0$, but the proof for that case depends on "reduction" to characteristic $p > 0$.

R. Stanley has used Reisner's result to prove the upper bound conjecture for triangulations of spheres (bounding the numbers of faces of various dimensions in a triangulation of an n -sphere in terms of n and the number of vertices: see (§2)).

We give here an expository account of Reisner's results (§2) (as well as certain background material about Cohen-Macaulay rings, Gorenstein rings, projective resolutions, local cohomology, etc. in (§3)), and we give a brief sketch of how Stanley uses the Cohen-Macaulayness of $K[\Delta]$ (§4). Moreover, we present several other applications of the ideas of Reisner's thesis:

- (a) determining the ranks of the free modules in a minimal (graded) free resolution of I_Δ , i.e., the "Betti numbers" of I_Δ (§5).
- (b) showing that the projective dimension of an ideal generated by monomials in a permutable R -sequence does not depend only on the "form" of the monomials, but on the specific R -sequence as well (§2).
- (c) characterizing when $K[\Delta]$ is Gorenstein - in particular, showing that any triangulation of a sphere is Gorenstein (§6).
- (d) determining when S/I is C-M (C-M $\equiv$ Cohen-Macaulay) for an arbitrary ideal I generated by monomials in $x_0, \dots, x_n$

(not necessarily square-free any more) (§7).

(e) recovering certain topological results from these ideas:

e.g., we show that Serre-Grothendieck duality on $\text{Proj}(K[\Delta])$ reduces to Poincaré duality on the geometric realization

$|\Delta|$ of Δ , if $|\Delta|$ is a manifold (this depends on (c) and on studying $\text{Pic}(\text{Proj}(K[\Delta]))$) (§6).

The results of (§5), while new, are obtained by a routine elaboration of Reisner's methods.

(§6) and (§7) are entirely new, however. We note particularly that if $\dim(\Delta) \geq 2$, then $K[\Delta]$ is Gorenstein for all Gorenstein K if and only if $Z[\Delta]$ is Cohen-Macaulay and the links of the $(\dim(\Delta) - 2)$ -simplices of Δ are either circles, or lines with at most 3 vertices. [See the remarks, "Added in Proof," at the end.]

§2. Reisner's Results: Cohen-Macaulayness of $K[\Delta]$. Let $\mathbb{R}$ be the real numbers and let Δ be a given abstract simplicial complex with vertices $x_0, \dots, x_n$. We identify $x_0, \dots, x_n$ with the standard basis for $\mathbb{R}^{n+1}$, and let

$$|\Delta| = \bigcup_{\sigma \text{ in } \Delta} \text{convex hull}(\sigma);$$

$|\Delta| \subset \mathbb{R}^{n+1}$ is a topological space. If $i_0 < \dots < i_r$, and $\sigma = \{x_{i_0}, \dots, x_{i_r}\}$ is in Δ , then σ is called an r-face of Δ .

$\dim(\Delta) = \max\{r : \Delta \text{ has an } r\text{-face}\} = \text{the topological dimension of } |\Delta|$.

If K is a ring, let $C.(\Delta)$ denote the simplicial chain complex with coefficients in K (we write $C.(\Delta; K)$ if we need to be specific

about what K is):

$C_i(\Delta)$ = the free K -module on the i -faces of Δ

(thus $C_i(\Delta) = 0$ if $i > \dim(\Delta)$ or if $i \leq -2$, but $C_{-1}(\Delta) \cong K$, since Δ has the unique (-1) -face $\emptyset$), while for each r -face

$\sigma = \{x_{i_0}, \dots, x_{i_r}\}$, $i_0 < \dots < i_r$,

$$d(\sigma) = \sum_{j=0}^r (-1)^j (\sigma \setminus \{x_{i_j}\}),$$

where $\sigma \setminus \{x_{i_j}\}$ is the $(r-1)$ -face obtained by deleting x_{i_j} from σ . There is a dual complex $C^*(\Delta; K)$, where

$$C^*(\Delta) = \text{Hom}_K(C_*(\Delta), K).$$

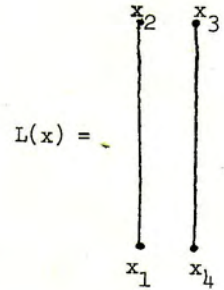
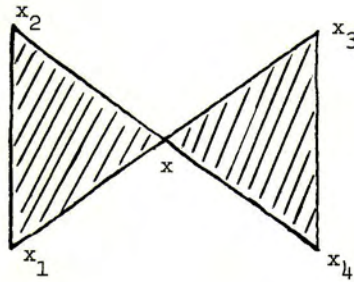
Note that if f is in $\text{Hom}_K(C_i(\Delta), K) = C^i(\Delta)$, and δ is the differentiation, we have that for each $\sigma = \{x_{i_0}, \dots, x_{i_{r+1}}\}$ in Δ , $i_0 < \dots < i_{r+1}$,

$$(\delta f)(\sigma) = \sum_{j=0}^{r+1} (-1)^j f(\sigma \setminus \{x_{i_j}\}).$$

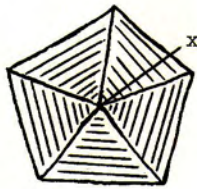
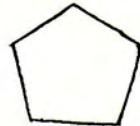
The homology of $C_*(\Delta)$ (resp., cohomology of $C^*(\Delta)$) is denoted $\tilde{H}_*(\Delta)$ (or $\tilde{H}_*(\Delta; K)$, etc.) (resp., $\tilde{H}^*(\Delta)$) and is called the reduced simplicial homology (cohomology) of Δ . If we omit C_{-1} (resp., C^{-1}), replacing it by 0, we get the (nonreduced) simplicial (co)homology, denoted $H_*(\Delta)$ ($H^*(\Delta)$). Note that $\tilde{H}_i = H_i$ for $i \geq 1$, and similarly for cohomology. H_i , H^i vanish for $i < 0$ and $i > \dim(r)$. If K is a field, $\tilde{H}_i$ and $\tilde{H}^i$ are dual over K , and similarly for H_i , H^i .

N.B. By our definition, $\tilde{H}^i(\Delta) = 0$ if $i < 0$ with one exception, which is important for us:

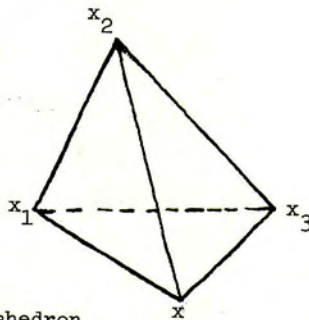
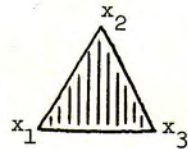
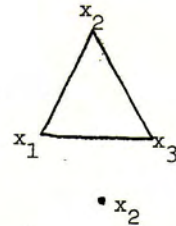
Examples. (a)



(b)

 $L(x) =$ 

(c)

 $L_{\Delta}(x) =$  $L_{\Delta_2}(x) =$  $L_{\Delta_1}(x) =$  $L_{\Delta_0}(x) = \emptyset$ Δ = solid tetrahedron Δ_i = i -skeleton, $i = 2, 1, 0$

Note that if $\sigma = \{x_{i_0}, \dots, x_{i_r}\}$, $i_0 < \dots < i_r$, and $\sigma' = \sigma \setminus \{x_{i_r}\}$, then

$$L_{\Delta}(\sigma) = L_{L_{\Delta}(\sigma')}^{(x_{i_r})},$$

and iterating gives formulas like

$$L_{\Delta}(\{x_{i_0}, x_{i_1}, x_{i_2}\}) = L_{L_{L_{\Delta}(x_{i_0})}^{(x_{i_1})}}^{(x_{i_2})}.$$

Note that if $\sigma = \emptyset$, $L(\sigma) = \Delta$.

For general facts about Cohen-Macaulayness, we refer the reader to (§3). For the moment we recall simply that R is C-M, or Cohen-Macaulay, if for every prime P , $\text{height}(P) = \text{depth}(P)$ (there is an R -sequence of length $\text{height}(P)$ in P).

Theorem (2.1). (G. Reisner). Let K be a field, and Δ a finite simplicial complex. Then $K[\Delta]$ is C-M if and only if for every link L of every σ in Δ , including $\Delta = L_{\Delta}(\emptyset)$,

$$\tilde{H}^i(L; K) = 0, \quad 0 \leq i < \dim(L).$$

If $|\Delta|$ is a (closed) manifold, the condition on the links (other than Δ) holds automatically. Hence,

Corollary (2.2). (G. Reisner). If $|\Delta|$ is a closed manifold, then $K[\Delta]$ is Cohen-Macaulay if and only if

$$\tilde{H}^i(\Delta; K) = 0, \quad 0 \leq i < \dim(\Delta).$$

To understand this result better, let us examine the connection between $K[\Delta]$ and $|\Delta|$ in greater detail.

We recall [4], [28] that the set of ideals I generated by monomials in $x_0, \dots, x_n$ is a distributive lattice under $+$, $\cap$, and each such I is an irredundant intersection of ideals of the form

$\{x_{i_0}^{m_0}, \dots, x_{i_r}^{m_r}\}$, $i_0 < \dots < i_r$. If K is a domain, this decomposition is an irredundant primary decomposition. The ideals generated by the square-free monomials form a distributive sublattice, and if K is reduced (no nilpotents but 0), these are the radical ideals generated by monomials.

If K is a field, $S = K[x_0, \dots, x_n]$, $I \subset S$, let $V(I)$ be the subset of K^{n+1} where the elements of I vanish. If $I = \{x_{i_0}, \dots, x_{i_r}\}$, we refer to $V(I)$ as a coordinate hyperplane. Let Σ be the set of all subsets of $\{x_0, \dots, x_n\}$. Then we get one-one correspondences between:

- (1) $S_1 = \{\text{subcomplexes of } \Sigma\}$.
- (2) $S_2 = \{\text{ideals generated by square-free monomials in } x_0, \dots, x_n\}$.
- (3) $S_3 = \{\text{unions of coordinate hyperplanes}\}$.

To go from S_1 to S_2 define I_Δ from Δ precisely as in the first paragraph of the Introduction. Henceforth, we relax the restriction

that each of $x_0, \dots, x_n$ be a vertex of Δ : we allow some extra indeterminates corresponding to "extra" vertices (not occurring in Δ).

We can go back from S_2 to S_1 by

$$I \mapsto \{ \{x_{i_0}, \dots, x_{i_r}\} \subset \Sigma : x_{i_0} \dots x_{i_r} \notin I \} .$$

The correspondence of S_2 to S_3 is $I \mapsto V(I)$. We write H_Δ for $V(I_\Delta)$. Note that we can recover Δ from H in S_3 , i.e., map S_3 to S_1) thus: $\{x_{i_0}, \dots, x_{i_r}\}$ is in Δ if and only if the element of K^{n+1} whose $x_{i_0}, \dots, x_{i_r}$ coordinates are 1 and whose other coordinates are 0 is in H .

If $K = \mathbb{J}R$, we can view the correspondence S_3 to S_1 as

$$H \mapsto H \cap |\Sigma| = |\Delta| .$$

Now assume that $x_0, \dots, x_n$ occur as vertices of Δ . Then:

maximal faces of Δ correspond to minimal primes of Δ

and

$$\begin{aligned} \dim(K[\Delta]) &= \dim(H_\Delta) \\ &= \dim(\Delta + 1) . \end{aligned}$$

Notice that if $K[\Delta]$ is C-M then $K[\Delta]$ has pure dimension (all minimal primes have the same coheight), whence Δ is of pure dimension (all maximal faces have the same dimension).

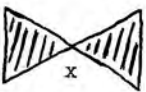
Let us say that Δ is C-M (over K) if $K[\Delta]$ is C-M.

The following are examples of the applications of Reisner's theorem.

0. $\dim(\Delta) = 0$. $K[\Delta]$ is always C-M.
1. $\dim(\Delta) = 1$. $K[\Delta]$ is C-M if and only if Δ is connected.
2. $\dim(\Delta) = 2$. $K[\Delta]$ is C-M if and only if
 - (a) Δ has pure dimension.
 - (b) $\tilde{H}^i(\Delta) = 0$, $i = 0, 1$.
 - (c) Every point of $|\Delta|$ has arbitrarily small neighborhoods which are connected even with an arbitrary finite subset removed.

Note: (c) follows from the condition that links be connected.

More specific examples:

- A.  is not C-M ((c) fails at x).

Note: $L(x) = \{ \} \{ \}$ is not C-M, for $\tilde{H}^0(L(x)) \neq 0$.

- B. 2-spheres are C-M.

- C. A (hollow) torus  is not C-M, since $\tilde{H}^1 \neq 0$.

- D. A cylinder  is not C-M! ($\tilde{H}^1 \neq 0$.) (Of course, a circle is C-M.)

- E. A real projective plane is C-M if $\text{char}(K) \neq 2$ (for then $\tilde{H}^1 = 0$). But if $\text{char}(K) = 2$, $\tilde{H}^1 \cong K$, and the real projective plane is not C-M over K .

If one looks at a few simple examples, one is tempted to believe that the projective resolution of an ideal generated by monomials in a permutable R-sequence depends only on the "form" of the set of monomials:

not on the R-sequence. (See [4], [17], and [28] for some positive results along these lines.)

However, as observed in [23], not only the form of the resolution but even the projective dimension may vary, even for square-free monomials in indeterminates over a field. There is a dependence on the characteristic. One gets examples from Example 2.E (the real projective plane) above: take the ideal corresponding to any triangulation of the real projective plane.

We make a few remarks about what happens when K is not a field. We first note a trivial fact: the Cohen-Macaulayness of $K[\Delta]$, K a field, depends only on $\text{char}(K)$. Let us write $Z_p = \mathbb{Z}/p\mathbb{Z}$ when p is a positive prime and $Z_0 = \mathbb{Q}$, the rationals, i.e., Z_p is the smallest field of characteristic p .

Proposition (2.3). Let K be a commutative ring and Δ a simplicial complex. The following conditions are equivalent:

- (a) $K[\Delta]$ is C-M.
- (b) K is C-M and for every field F which is a K -algebra, $F[\Delta]$ is C-M.
- (c) K is C-M and $Z_p[\Delta]$ is C-M for each positive prime integer p such that $1/p \notin K$, and, if $Z \subset K$, for $p = 0$ as well.

This follows easily from the following proposition.

Proposition (2.4). (a) The following conditions on a simplicial complex

Δ are equivalent:

- (1) $Z[\Delta]$ is C-M.
- (2) $K[\Delta]$ is C-M for every C-M ring K .
- (3) $Z_p[\Delta]$ is C-M for all primes $p > 0$.

(b) Moreover, the following conditions are equivalent:

- (1) $Q[\Delta]$ is C-M.
- (2) $K[\Delta]$ is C-M for some field K .
- (3) For all but finitely many positive prime integers $p_1, \dots, p_r$, $K[\Delta]$ is C-M if K is a field with $\text{char}(K)$ not in $\{p_1, \dots, p_r\}$.

Part (b) can be deduced from "universal coefficient" theorems.

The case of prime fields is evidently crucial. Or, as in algebraic topology, one may use universal coefficient theorems to get the information from the case $K = Z$.

For our purposes here it is simpler to stick with fields, particularly positive characteristic fields - the Frobenius comes in handy.

§3. Depth, C-M Rings, Projective Resolutions, Local Cohomology, and Gorenstein Rings. This section is a lumping together of some basic facts. Let R be a Noetherian ring, $I \subset R$ an ideal, and M an R -module of finite type. A sequence $x_1, \dots, x_n$ in R is called regular on M , an M-sequence, or an R-sequence on M , if $(x_1, \dots, x_n)_M \neq M$

and x_{i+1} is not a zero divisor on $M/(x_1, \dots, x_i)M$, $0 \leq i < n$.
 If $IM \neq M$, then $\text{depth}_I(M)$, the depth of M on I , denotes the length of any maximal M -sequence in I . By a theorem, any M -sequence in I can be enlarged to one of length $\text{depth}_I(M)$.

Proposition (3.1). If M, N are modules of finite type over a Noetherian ring R with $\text{Ann}_R(M) = I$, $\text{Ann}_R(N) = J$, and $I + J \neq R$ (equivalently, $IN \neq N$ or $M \otimes_R N \neq 0$), then

$$\begin{aligned} \text{Ext}_R^i(M, N) &= 0 & \text{if } i < \text{depth}_I(N) \\ \text{Ext}_R^i(M, N) &\neq 0 & \text{if } i = \text{depth}_I(N). \end{aligned}$$

Theorem (3.2). (Auslander-Buchsbaum). Let $M \neq 0$ be a module of finite type over a local ring $(R, \mathfrak{m})$. Then if M has a finite projective resolution, $\text{pd}_R(M)$, the length of any shortest such resolution, is

$$\text{depth}_{\mathfrak{m}}(R) - \text{depth}_{\mathfrak{m}}(M).$$

Proposition (3.3). (Rees). For any R -module $M \neq 0$ of finite type, if $I = \text{Ann}_R(M)$ then

$$\text{depth}_I(R) \leq \text{pd}_R(M).$$

Proof. Apply (3.1) with $N = R$.

Let (R, m) be either a local ring with residue class field $K = R/m$ or else let $R = R_0 \oplus R_1 \oplus R_2 \oplus \dots$ be a finitely generated nonnegatively graded K -algebra, K a field, such that $R_0 = K$ and R_1 generates R over K , and let $m = R_1 \oplus R_2 \oplus \dots$. (We insert certain adjectives parenthetically in the sequel which are needed in the graded case but not in the local case.)

Let M be a finitely generated (graded) R -module. Then M has a minimal (graded) resolution by finitely generated (graded) free modules (and maps of degree 0), say

$$\dots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \dots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \longrightarrow 0,$$

where the minimality means, equivalently, that

- (i) for each $i \geq 0$, d_i maps the free generators of F_i onto a minimal (homogeneous) basis for $\text{Im}(d_i)$; or
- (ii) for each $i \geq 1$, $\text{Im}(d_i) \subset mF_{i-1}$.

Regard K as the R -module R/m . The numbers $b_i = \dim_K(\text{Tor}_i^R(K, M))$ are called the Betti numbers of M . Let $F. = \dots \rightarrow F_i \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow 0$. Then

$$\text{Tor}_i^R(K, M) \cong H_i(K \otimes F.) = H_i(\dots \xrightarrow{0} K \otimes F_i \xrightarrow{0} \dots \longrightarrow K \otimes F_0 \xrightarrow{0} 0)$$

$$\cong K \otimes F_i \quad (\text{all the maps are } 0 \text{ by condition (ii) above}),$$

whence $b_i = \text{rank } F_i$.

It is usually difficult, given a specific R and M to find an explicit minimal resolution of M or even to determine the Betti numbers.

Let $x_1, \dots, x_n$ denote n elements of a ring R . Let $R^n = Ru_1 \oplus \dots \oplus Ru_n$ be the free module on n generators. The exterior algebra $\Lambda(R^n)$ becomes a free complex, called the Koszul complex $K.(x_1, \dots, x_n; R)$,

$$0 \longrightarrow \Lambda^n R^n \xrightarrow{d_n} \dots \longrightarrow \Lambda^1(R^n) \xrightarrow{d_1} R \longrightarrow 0$$

where $d_r(u_{i_1} \wedge \dots \wedge u_{i_j} \wedge \dots \wedge u_{i_r}) = \sum_{j=0}^r (-1)^j x_{i_j} (u_{i_1} \wedge \dots \wedge \hat{u}_{i_j} \wedge \dots \wedge u_{i_r})$ ($\hat{}$ denotes omission). If $\underline{x} = x_1, \dots, x_n$ is an R -sequence, then $K.(\underline{x}; R)$ is acyclic: its augmentation is $M = R/(x_1, \dots, x_n)R$ and it gives a shortest projective resolution of M . If R is regular local and $x_1, \dots, x_n$ is a regular system of parameters, or if $R = K[x_1, \dots, x_n]$ and the x_i are indeterminates, $K.(\underline{x}; R)$ gives a minimal free resolution of K , and hence may be used to compute $\text{Tor}_R^i(K,)$ and $\text{Ext}_R^i(K,)$.

We briefly review what it means for a Noetherian ring R to be Cohen-Macaulay (C-M).

Definition-Proposition (3.4). The following conditions on a Noetherian ring R are equivalent:

- (1) For every ideal $I \subsetneq R$, $\text{depth}(I) = \text{ht}(I)$ ($\text{ht}(I) = \min\{\dim(R_P) : P \supset I, P \text{ prime}\}$).
- (2) For every maximal ideal m , $\text{depth}(m) = \text{ht}(m)$.
- (3) For every maximal ideal m , some system of parameters for R_m is an R_m -sequence.
- (4) For every prime ideal P , every system of parameters for R_P is an R_P -sequence.

A ring R satisfying these equivalent conditions is called Cohen-Macaulay.

Theorem (3.5). Let S be a polynomial ring over a field K (resp., a regular local ring) and let I be a homogeneous (resp., arbitrary) proper ideal. Then S/I is C-M if and only if $\text{pd}_S(S/I) = \text{depth}(I)$ ($= \text{height}(I)$).

Moreover, in the graded case, if $\mathfrak{m}$ is the homogeneous maximal ideal of $R = S/I$, then R is C-M if and only if $R_{\mathfrak{m}}$ is C-M.

Proof. The first part is essentially a consequence of (3.2) and the Cohen-Macaulayness of regular rings. For the second part, see [19] or [11].

(3.6) Examples of C-M and non-C-M rings.

0. 0-dimensional rings are C-M.

1. A. 1-dimensional reduced rings are C-M.

B. Some 1-dimensional nonreduced rings, e.g., $K[x,y]/(y^2)$ are C-M.

C. $K[x,y]/(x^2, xy)$ is not C-M.

2. A. 2-dimensional integrally closed rings are C-M, e.g.,

$K[x^2, xy, y^2] \subset K[x,y]$ is C-M.

B. $K[x^2, x^3, y]$ is C-M, even though it is not integrally closed.

C. $K[x^2, x^3, y, xy]$ is not C-M.

3. Let $R = K[x,y,z] = K[X,Y,Z]/(X^3 + Y^3 + Z^3)$. Then

$K[xu, yu, zu, xv, yv, zv] \subset R[u, v]$ is integrally closed, 3-dimensional, but not C-M. See [7].

4. A. A regular ring is C-M.

B. If R is C-M and $\text{ht}(x_1, \dots, x_n) = n$, then $x_1, \dots, x_n$ is an R-sequence and $R/(x_1, \dots, x_n)$ is C-M.

C. If R is C-M, $X = (x_{ij})$ is an r by s matrix of indeterminates over R , and I is the ideal generated by the $(t+1)$ -size minors in $S = R[x_{ij} : i, j]$, then S/I is C-M. Moreover, $\text{ht}(I) = (r-t)(s-t)$. See [10].

The notion of Cohen-Macaulayness is subtle and hard to understand. It's an "arithmetic" condition with geometric consequences.

One key point is that ideals generated by R-sequences are unmixed. Another is that when a Noetherian ring R can be represented as a finite module over a regular ring A , $A \subset R$, then R is C-M if and only if $R_{\mathfrak{m}}$ is $A_{\mathfrak{m}}$ -free for each maximal ideal of A . Many theorems for "curves" (1-dimensional reduced rings) can be generalized in a useful way to C-M rings of arbitrary dimension.

One important example is Serre-Grothendieck duality which asserts:

Theorem (3.7). If $(X, \mathcal{O}_X)$ is a projective K -variety of pure dimension r and is C-M (all its local rings are C-M), then there is a coherent sheaf ω_X on X , such that for every locally free coherent sheaf F on X

$$H^i(X, F)^* \cong H^{r-i}(X, \omega_X \otimes_{\mathcal{O}_X} F^V),$$

where $^* = \text{Hom}_K(, K)$ and $^V = \text{Hom}_{\mathcal{O}_X}(, \mathcal{O}_X)$.

If X is nonsingular $\omega_X = \Omega_{X/K}^r$ = the sheaf of highest order Kähler differentials (the determinant of the cotangent bundle). If $X = \mathbb{P}^n$, $\omega_X = \mathcal{O}_X(-n-1)$. If X is a closed subvariety of a nonsingular n -dimensional Y (e.g., $Y = \mathbb{P}^n$), $\omega_X = \text{Ext}_{\mathcal{O}_Y}^{n-r}(\mathcal{O}_X, \omega_Y)$.

See [1] for further details.

This duality is a generalization of part of the Riemann-Roch theorem for curves.

For many other uses of C-M rings, e.g., the construction of "depth-sensitive" complexes analogous to the Koszul complex, vanishing of certain cohomology of sheaves, and extending sections of sheaves, we refer the reader to [10], [8], [9], [12], [13], and [25].

We next recall some basic facts about local cohomology. Let R be Noetherian, I an ideal, and M an R -module. We define $H_I^*(M) = \varinjlim_t \text{Ext}_R^*(R/I^t, M)$. If $x_1, \dots, x_r$ generates an ideal with the same radical as I , and $K^*(\underline{x}^\infty; M)$ denotes the right complex

$$\begin{aligned} 0 \rightarrow M \rightarrow \bigoplus M_{x_i} \rightarrow \bigoplus M_{x_i x_j} \rightarrow \cdots \rightarrow M_{x_1 \cdots x_r} \rightarrow 0 \\ = (\bigotimes_i (0 \rightarrow R \rightarrow R_{x_i} \rightarrow 0)) \otimes M, \end{aligned}$$

then $H_I^*(M) \cong H^*(K^*(\underline{x}^\infty; M))$. See [5], [12], and [13] for more details.

$H_I^*(M)$ depends only on $\text{Rad}(I)$. The following result (see [5]) summarizes some basic facts.

Theorem (3.8). Let $\mathfrak{m}$ be a maximal ideal of R , and let M be an R -module of finite type. Let $I = \text{Ann}(M)$, and assume $I \subset \mathfrak{m}$ (otherwise $H_m^*(M) = 0$). Then:

- (1) The modules $H_m^*(M)$ are Artinian (have D.C.C.). [Note: They need not have finite length!].
- (2) Let $d = \text{depth}_m(M)$ and $h = \dim(R_m/I_m)$. Then $H_m^d(M) \neq 0$, $H_m^h(M) \neq 0$, but $H_m^i(M) = 0$ if $i < d$ or if $i > h$.
- (3) $H_m^*(M) = H_{mR_m}^*(M_m) = H_{m\hat{R}_m}^*(\hat{M}_m)$.

The fact that $H_m^*(M)$ measures $\text{depth}_m(M)$ is particularly important. In fact, quite generally:

(3.8⁰). If M is Noetherian and $IM \neq M$, the least integer d such that

$$H_I^d(M) \neq 0$$

is $\text{depth}_I(M)$.

We conclude this section with a brief discussion of Gorenstein rings. A Noetherian ring R is called Gorenstein if $\text{id}_R(R)$ is finite (id_R denotes the length of a shortest injective resolution). The following characterization is often more useful.

Theorem (3.9). A Noetherian ring R is Gorenstein if and only if R is C-M and for every maximal (and even prime) ideal m of R and some (resp., every) system of parameters $x_1, \dots, x_d$ for R_m , $R_m/(x_1, \dots, x_d)R_m$ is injective as a module over itself.

We refer the reader to [2], [16] and [5] for details.

We note that a 0-dimensional local ring (R, m) with $K = R/m$ is injective if and only if $\text{Hom}_R(K, R) \cong K$. ($\text{Hom}_R(K, R) \cong \text{Ann}_R(m)$.) Moreover, if R is C-M and $x_1, \dots, x_d$ is a system of parameters,

$$\text{Hom}_R(K, R/(x_1, \dots, x_d)R) \cong \text{Ext}_R^d(K, R).$$

Thus we obtain the following.

Proposition (3.10). A local ring R with residue class field K is Gorenstein if and only if

$$\text{Ext}_R^i(K, R) \cong \begin{cases} 0, & 0 \leq i < \dim(R) \\ K, & i = \dim(R) \end{cases}.$$

In the graded case we may write $R = S/I$, where S is a polynomial ring and I is a homogeneous ideal. Likewise, if R is local, in most cases which come up we may write $R = S/I$, where S is regular local. In either case we obtain the following proposition.

Proposition (3.11). In either of the above cases, $R = S/I$ is Gorenstein

if and only if the highest nonzero Betti number b_i for R occurs when $i = \text{ht}(I)$ and is 1.

Proof. The condition $b_j = 0$ for $j > \text{ht}(I) = \text{depth}_I(S) = d$ means that $\text{pd}_S(R) = d$, i.e., $\text{depth}_m(R) = \dim(S - d) = \dim(R)$. Thus, this part of the condition asserts that R is C-M. For the rest, we refer the reader to [2].

It is worth noting that if R is C-M of pure dimension d , $\text{Spec}(R)$ is connected, and R is a homomorphic image of a Gorenstein ring S , one may associate with R the module $\Omega_R = \text{Ext}_S^{n-d}(R, S)$, which is unique up to tensoring with rank one projectives. If we restrict R to being an algebra of finite type over a field K and S to being a polynomial ring over K , or if we assume R is local, Ω_R is unique, up to isomorphism.

We note that the restriction of ω_X to an open affine $\text{Spec}(R)$ of X will correspond to a canonical module Ω_R for the coordinate ring of that open affine.

One important fact we need for (§6) is that a C-M ring R is Gorenstein if and only if Ω_R is a rank one projective.

In the situation of (3.7), a C-M X is (locally) Gorenstein if and only if ω_X is locally free of rank one.

§4. R. Stanley's Results on the Upper Bound Conjecture. The original Upper Bound Conjecture (UBC), due to Motzkin [22], gives an upper bound for the number of i -faces of a $(d + 1)$ -dimensional convex polytope with $n + 1$ vertices: To wit, that the maximum possible is the number of i -faces of a cyclic polytope, which is defined as the convex hull of any $n + 1$ distinct points on the curve

$$\{(t, t^2, \dots, t^{d+1}) : t \text{ is in } \mathbb{R}\}.$$

This conjecture may be reduced to the simplicial case (the faces are simplices) by a trick of "pulling vertices." It then can be extended to a conjecture about arbitrary triangulations of d -spheres or even d -manifolds — and this is done, by Klee [17].

In 1970 McMullen (see [20], [21]) proved UBC for polytopes using a result of Brugesser and Mani [3] that the boundary complex of a convex polytope is "shellable" in a certain strong sense. However, not all triangulations of spheres [30] are shellable (or even of cells [24]) and the general question remained open, even for spheres.

R. Stanley has solved UBC for spheres now and, more generally, for all simplicial complexes Δ such that $K[\Delta]$ is C-M for some field K (equivalently, such that $Q[\Delta]$ is C-M). See [27].

We sketch the argument. Let $x_0, \dots, x_n$ be indeterminates over a field K , let $S = K[x_0, \dots, x_n]$, and let I be a homogeneous ideal of S . Then $R = S/I$ is a nonnegatively graded finitely generated K -algebra generated by R_1 , and with $R_0 = K$. Such an R has a Hilbert function H_R defined by

$$H_R(m) = \dim_K(R_m)$$

(H_R agrees with a certain polynomial in m of degree $\text{Knull}(\dim(R) - 1)$ for all sufficiently large m).

When R is C-M one can get inequalities on the values of H_R by computing in two different ways (here, Stanley is following Macaulay, [18]). On the one hand, one can represent R as a graded free module over a polynomial subring generated by forms of equal degree, while on the other hand one can use the fact that R has a K -vector space basis consisting of the images of certain monomials $x_1^{i_1} \dots x_n^{i_n}$ such that if the monomial v occurs and $\mu|v$ (formally, in S), then μ occurs.

Now, when $R = K[\Delta] = S/I_\Delta$ it is very easy to compute $H_\Delta = H_{R_\Delta}$ ([27], Proposition 3.2): in fact,

$$H_\Delta(m) = \begin{cases} 1 & \text{if } m = 0 \\ \sum_{i=0}^{\dim(\Delta)} f_i \binom{m-1}{i} & \text{if } m \geq 1, \end{cases}$$

where f_i is the number of i -faces of Δ .

The inequalities on the values of H_Δ for the C-M case mentioned above yield UBC, in the form considered by McMullen, [20], [21], at once.

§5. The Betti Numbers of $K[\Delta]$. Throughout this section Δ is a simplicial complex with vertices $x_0, \dots, x_n$, unless otherwise specified K is a field, $S = K[x_0, \dots, x_n]$, and $R = K[\Delta] = S/I$. Let b_i be the i -th Betti number of R over S , i.e., the rank of the i -th free

module in a minimal graded free resolution of R over S . Thus,

$$b_i = \dim_K(\text{Tor}_i^S(K, K[\Delta])) , \text{ where } K = S/(x_0, \dots, x_n) .$$

If $T \subset \{x_0, \dots, x_n\} = V$, let Δ/T denote the largest subcomplex of Δ whose vertices are $V - T$ (i.e., $\Delta/T = \Delta \cap 2^{V-T}$).

Theorem (5.1). The i -th Betti number b_i of $K[\Delta]$ over S is given by

$$b_i = \sum_{\substack{-1 \leq q \leq n-i \\ T \subset V, \text{card}(T+q) = n-i}} \dim_K \tilde{H}^q(\Delta/T; K) .$$

To see why this is true, we compute $\text{Tor}_i^S(K, K[\Delta])$ by using the Koszul complex to resolve K . Let Σ be the simplicial complex consisting of all subsets of $V = \{x_0, \dots, x_n\}$, i.e., $\Sigma = 2^V$. Then we may view $K_i = K_i(x_0, \dots, x_n; S)$ as the free S -module on the $(n-i)$ -faces of Σ (where, if $V - \sigma = \{j_1, \dots, j_i\}$, $j_1 < \dots < j_i$, then σ corresponds to $u_{j_1} \wedge \dots \wedge u_{j_i}$). In terms of this basis for K , we have that

$$d\sigma = \sum (-1)^{\alpha(j, \sigma)} x_j (\sigma \cup \{x_j\})$$

where the sum is taken over those x_j in $V - \sigma$, and where $\alpha(j, \sigma) = \text{card}\{t : t < j \text{ and } x_t \text{ in } V - \sigma\}$.

If $\mu = x_{i_1}^{t_1} \dots x_{i_h}^{t_h}$ in S is a monomial, we write $\text{Supp}(\mu)$ for $\{x_i : x_i | \mu\} = \{x_{i_v} : t_v \geq 1\}$. Then $\{\mu : \text{Supp}(\mu) \text{ is in } \Delta\}$ is a free K -basis for $K[\Delta]$ and $\{(\mu, \sigma) : \text{Supp}(\mu) \text{ is in } \Delta \text{ and } \sigma \text{ is an } (n-i)\text{-}$

face of Σ is a free K -basis for $K[\Delta] \otimes_S K_i$. Now,

$\text{Tor}_1^S(K, K[\Delta]) \cong H_1(K[\Delta] \otimes_S K.)$ and $d|_{K[\Delta] \otimes_S K_i}$ is given by

$$(*) \quad d(\mu, \sigma) = \sum (-1)^{\alpha(j, \sigma)} (x_j \mu, \sigma \cup \{x_j\}) .$$

where the sum is taken over those x_j in $V - \sigma$ with $\{x_j\} \cup \text{Supp}(\mu)$ in Δ .

We call (μ, σ) an environment if $\text{Supp}(\mu) \cap \sigma = \emptyset$. Given (μ, σ) , we define an environment $\text{env}(\mu, \sigma)$ as $(\mu / \Pi x_i, \sigma - \text{Supp}(\mu))$, where the product is taken over those x_i in $\text{Supp}(\mu) \cap \sigma$. Note that if $\{x_j\} \cup \text{Supp}(\mu)$ is in Δ , then

$$(\#) \quad \text{env}(x_j \mu, \sigma \cup \{x_j\}) = \text{env}(\mu, \sigma) .$$

Following G. Reisner [23], we split the complex $K[\Delta] \otimes_S K.$ into a direct sum of subcomplexes L^ϵ , one for each environment $\epsilon = (\nu, \tau)$. In fact, if E is the set of environments and ϵ is in E , L_0^ϵ is spanned, as a free K -module, by the set

$$W_\epsilon = \{(\mu, \sigma) : \text{env}(\mu, \sigma) = \epsilon\} .$$

Clearly, $K[\Delta] \otimes_S K. = \bigoplus_{\epsilon \in E} L^\epsilon$ as K -modules. But d maps L^ϵ into itself, by $(*)$ and $(\#)$, and we have a splitting as complexes. Let $T_i^\epsilon(\Delta) = H_i(L^\epsilon)$. Thus, $\text{Tor}_1^S(K, K[\Delta]) = \bigoplus_{\epsilon \in E} T_1^\epsilon(\Delta)$, and it remains to compute $T_i^\epsilon(\Delta)$.

We next note that each element (μ, σ) in W_ϵ is uniquely determined, given $E = (\nu, \tau)$, by $\rho = \sigma - \tau$. In fact, if we define

$$\rho_\varepsilon = (\vee \Pi_{x_j \text{ in } \rho} x_j, \tau \cup \rho) ,$$

then $(\mu, \sigma) = \rho_\varepsilon$, where $\varepsilon = \text{env}(\mu, \sigma)$ and $\rho = \sigma - \tau$. Let $\Delta_\varepsilon = \{\rho \in \Delta : \text{Supp}(\nu) \cup \rho \in \Delta \text{ and } \tau \cap \rho = \emptyset\}$. Δ_ε is a subcomplex of Δ . For each ρ , let $\gamma(\rho) = \sum_{x_j \in \rho} \alpha(j, \tau)$. Thus, if $x_j \notin \rho$, $\rho' = \rho \cup \{x_j\}$, then

$$\gamma(\rho') = \gamma(\rho) + \alpha(j, \tau) .$$

Now, $\rho \mapsto \rho_\varepsilon$ is a bijection of Δ_ε with a free basis for L^ε (i-faces of Δ_ε map to a free basis for $L_{n-i-\text{card}(\tau)}^\varepsilon$). Let $\{\rho^*\}$ be the dual basis for $C^i(\Delta_\varepsilon)$ to the basis of i-faces for $C_i(\Delta_\varepsilon)$ (i.e., $\rho^*(\sigma) = \delta_{\rho\sigma}$). Thus, $\phi : \rho_\varepsilon \mapsto (-1)^{\alpha(\rho)} \rho^*$ is an isomorphism of L^ε with $C^*(\Delta_\varepsilon)$ as K -modules which takes $L_{n-i-\text{card}(\tau)}^\varepsilon$ isomorphically onto $C^i(\Delta_\varepsilon)$. Moreover, for each i the diagram

$$\begin{array}{ccc} C^i(\Delta_\varepsilon) & \xrightarrow{\delta} & C^{i+1}(\Delta_\varepsilon) \\ \phi \uparrow & & \uparrow \phi \\ L_{n-i-\text{card}(\tau)}^\varepsilon & \xrightarrow{d} & L_{n-i-1-\text{card}(\tau)}^\varepsilon \end{array}$$

$$\begin{aligned} \text{commutes: } \delta(\phi(\rho_\varepsilon)) &= \delta((-1)^{\gamma(\rho)} \rho^*) \\ &= (-1)^{\gamma(\rho)} \sum (-1)^{\beta(j, \rho)} (\rho \cup \{x_j\})^* , \end{aligned}$$

where $\beta(\rho, j) = \text{card}\{t < j : x_t \text{ in } \rho\} = \alpha(j, \tau) - \alpha(j, \tau \cup \rho)$, i.e., (with all sums taken over x_j not in ρ , $\rho \cup \{x_j\}$ in Δ_ε)

$$\begin{aligned} \delta(\phi(\rho_\varepsilon)) &= (-1)^{\gamma(\rho)} \sum_{x_j} (-1)^{\alpha(j, \tau) - \alpha(j, \tau \cup \rho)} (-1)^{\gamma(\rho \cup \{x_j\})} \phi((\rho \cup \{x_j\})_\varepsilon) \\ &= \phi((-1)^{\gamma(\rho)} \sum_{x_j} (-1)^{\alpha(j, \tau) - \alpha(j, \tau \cup \rho) + \gamma(\rho) + \alpha(j, \tau)} (\rho \cup \{x_j\})_\varepsilon) \end{aligned}$$

$$= \phi(\Sigma_{x_j} (-1)^{\alpha(j, \tau \cup \rho)} (\rho \cup \{x_j\})_\epsilon) = \phi(d\rho_\epsilon), \text{ since}$$

$$d\rho_\epsilon = d(v\Pi_{x_h \text{ in } \rho} x_h, \tau \cup \rho) = (\text{by } (*))$$

$$\Sigma_{\substack{\{x_j\} \cup \text{Supp}(v\Pi_{x_h \text{ in } \rho} x_h) \text{ in } \Delta \\ x_j \text{ not in } \tau \cup \rho}} (-1)^{\alpha(j, \tau \cup \rho)} (x_j v\Pi_{x_h \text{ in } \rho} x_h, \tau \cup \rho \cup \{x_j\})$$

$$= \Sigma_{\substack{\{x_j\} \cup \text{Supp}(v) \cup \rho \text{ in } \Delta \\ x_j \text{ not in } \tau \cup \rho}} (-1)^{\alpha(j, \tau \cup \rho)} (\rho \cup \{x_j\})_\epsilon$$

$$= \Sigma_{x_j \text{ not in } \rho, \rho \cup \{x_j\} \text{ in } \Delta_\epsilon} (-1)^{\alpha(j, \tau \cup \rho)} (\rho \cup \{x_j\})_\epsilon, \text{ as}$$

required. Thus, ϕ induces isomorphisms

$$(**) \quad H_{n-i-\text{card}(\tau)}(L_\epsilon^\epsilon) \cong \tilde{H}^i(\Delta_\epsilon).$$

Now, if $v \neq 1$, say x_h is in $\text{Supp}(v)$, then Δ_ϵ is a cone with x_h as vertex: if ρ is in Δ_ϵ , so does $\rho \cup \{x_h\}$. In this case, $\tilde{H}^*(\Delta_\epsilon) = 0$, and so $T^\epsilon(\Delta) = H.(L_\epsilon^\epsilon) = 0$. If $v = 1$, $\epsilon = (1, \tau)$, then $\Delta_\epsilon = \Delta/\tau$. Now, this entire discussion is valid whether K is a field or not, so that we now have shown the following.

Theorem (5.2). For any ring K and any Δ ,

$$\text{Tor}_1^S(K, K[\Delta]) \cong \bigoplus_{\substack{-1 \leq q \leq n-i \\ T \subset V, \text{card}(T)+q=n-i}} \tilde{H}^q(\Delta/T; K)$$

and

$$\text{Ext}_S^{j+1}(K, K[\Delta]) \cong \bigoplus_{\substack{-1 \leq q \leq j \\ T \subset V, \text{card}(T)+q=j}} \tilde{H}^q(\Delta/T; K) .$$

The first statement is clear since $\text{Tor}_1^S(K, K[\Delta]) = \bigoplus_{\epsilon} \text{in } E H_1(L^\epsilon) = \bigoplus_{(1, \tau) \text{ in } E} \tilde{H}^{n-i-\text{card}(\tau)}(\Delta/\tau; K)$: let $q = n - i - \text{card}(\tau)$ and $T = \tau$. The second statement follows from the identification $\text{Ext}_S^{j+1}(K, N) \cong \text{Tor}_{n-j}^S(K, N)$ (N arbitrary). (For $\text{Ext}_S^{j+1}(K, N) = H^{j+1}(\text{Hom}_S(K, N)) = H^{j+1}(\text{Hom}_S(K, S) \otimes_S N)$ and $\text{Hom}_S(K, S)$, with the numbering reversed, is identical with K , so that

$$H^{j+1}(\text{Hom}_S(K, S) \otimes_S N) \cong H_{n+1-(j+1)}(K \otimes_S N) \cong \text{Tor}_{n-j}^S(K, N) .$$

Corollary (5.3). Let K be a field. Then

$$\text{depth}(K[\Delta]) \geq d$$

(depth means depth on the homogeneous maximal ideal) if and only if for all q and $T \subset V$ with $-1 \leq q < d - 1$ and $q + \text{card}(T) < d - 1$

$$\tilde{H}^q(\Delta/T; K) = 0 .$$

Hence, $K[\Delta]$ is Cohen-Macaulay if and only if for all q and $T \subset V$ with $-1 \leq q < \dim(\Delta)$ and $q + \text{card}(T) < \dim(\Delta)$,

$$\tilde{H}^q(\Delta/T; K) = 0 .$$

Proof. The first assertion is immediate from Theorem (5.2) and Proposition (3.1). The second follows from the fact that $\dim(K[\Delta]) = \dim(\Delta) + 1$.

The characterization of Cohen-Macaulayness given here is not nearly as useful in practice as Reisner's link characterization.

We also have:

Proposition (5.4). Let K be a field. If $K[\Delta]$ is Cohen-Macaulay and $\dim(\Delta) = d + 1$, then $K[\Delta]$ is Gorenstein if and only if

$$\bigoplus_{\substack{-1 \leq q \leq d \\ T \subset V, \text{card}(T)=d-q}} \tilde{H}^q(\Delta/T) \cong K.$$

Proof: By Proposition (3.11), $K[\Delta]$ will be Gorenstein if and only if $b_{(n+1)-(d+1)} = b_{n-d} = 1$, i.e., if and only if $\text{Tor}_{n-d}^S(K, K[\Delta]) \cong K$. The result is the immediate from Theorem (5.2).

Proposition (5.5). (a) Let Δ be 0-dimensional and consist of $n + 1$ points, $n \geq 0$. Then $K[\Delta]$ is Gorenstein if and only if $n \leq 1$.

(b) Let Δ be 1-dimensional, with $(n + 1)$ vertices. Then Δ is Gorenstein if and only if either $|\Delta|$ is a circle or $1 \leq n \leq 2$ and $|\Delta|$ is a line.

Before proving this, we note the following.

Proposition (5.6). (a) Let x be a vertex of Δ . Then

$$K[\Delta]_x \cong K[L_\Delta(x)][X, 1/X]$$

where X is an indeterminate.

(b) If $K[\Delta]$ is C-M (respectively, Gorenstein), so is $L_\Delta(\sigma)$ for every σ in Δ .

Proof. (a) $K[\Delta]_x$ is graded, and every nonzero form can be written uniquely as $F(x/1)^t$, where $\deg(F) = 0$ and $t = \deg(F)$ in Z . Let R be the subring of degree 0 elements of $K[\Delta]_x$. Then $K[\Delta]_x = R[X, 1/X]$ where X is the image of x . R is spanned over K by the monomials $(\bar{x}_{i_1}/\bar{x})^{h_1} \dots (\bar{x}_{i_v}/\bar{x})^{h_v}$, $x_{i_r} \neq x$, where $\bar{}$ denotes reduction modulo I_Δ , and the only relations on the $\bar{x}_i/\bar{x}$ are that $(\bar{x}_{i_1}/\bar{x}) \dots (\bar{x}_{i_v}/\bar{x}) = 0$ if $\bar{x}_1 \dots \bar{x}_{i_v} \bar{x} = 0$ in $K[\Delta]$, i.e., if $\{x_{i_1}, \dots, x_{i_v}, x\}$ is not in Δ , or, equivalently, $\{x_{i_1}, \dots, x_{i_v}\}$ is not in $L_\Delta(x)$. Thus, $R = K[\bar{x}_i/\bar{x} : x_i \neq x] \cong K[L_\Delta(x)]$.

(b) $K[\Delta]$ C-M or Gorenstein implies $K[\Delta]_x \cong K[L_\Delta(x)][X, 1/X]$ is as well, and hence, so is $K[L_\Delta(x)][X, 1/X]/(X - 1) = K[L_\Delta(x)]$. This proves the result for links of vertices. The general result now follows, since each $L_\Delta(\sigma)$ is an iterated link of vertices.

Proof of (5.5). Let the vertices of Δ be $\{x_0, \dots, x_n\}$.

(a) $d = 0$. If $n = 0$, $\tilde{H}^{-1}(\emptyset) + \tilde{H}^0(\Delta) \cong K \oplus 0 \cong K$. If $n \geq 1$, we have $\bigoplus_{x_i} \tilde{H}^{-1}(\Delta - \{x_i\}) \oplus \tilde{H}^0(\Delta) \cong (0^{n+1}) \oplus (K^n) \cong K^n$. Thus, $K[\Delta]$

is Gorenstein if $n = 1$ but not if $n \geq 2$.

(b) $d = 1$. We have

$$[\oplus_{x_i \neq x_j} \tilde{H}^{-1}(\Delta/\{x_i, x_j\})] \oplus [\oplus_{x_i} \tilde{H}^0(\Delta/x_i)] \oplus [\tilde{H}^1(\Delta)].$$

$K[\Delta]$ C-M if and only if Δ is connected. In order for $K[\Delta]$ to be Gorenstein, we must have $L_\Delta(x_i)$ is Gorenstein for each i , i.e., at most two edges meet at each vertex. Thus, Δ is not Gorenstein unless it is connected and at most two edges meet at each vertex. But this says that $|\Delta|$ is either a circle (and then $n \geq 2$) or a line.

If $|\Delta|$ is a circle then each $\Delta/\{x_i, x_j\} \neq \emptyset$, so that the first bracketed term vanishes, each $\Delta/\{x_i\}$ is a line, so that the second bracketed term vanishes, while $\tilde{H}^1(\Delta) \cong K$. Thus, circles are Gorenstein.

Now suppose $|\Delta|$ is a line with endpoints, say, x_0, x_n . If $n = 1$ the sum becomes

$$[K] \oplus [0] \oplus [0] \cong K.$$

If $n = 2$ the sum becomes

$$[0 \oplus 0 \oplus 0] \oplus [0 \oplus K \oplus 0] \oplus [0] \cong K.$$

(Note: $\tilde{H}^0(\Delta/\{x_i\}) \cong K$.)

If $n \geq 3$ then $n - 1 \neq 1$ and

$$\tilde{H}^0(\Delta/\{x_1\}) = \tilde{H}^0(\Delta/\{x_{n-1}\}) \cong K.$$

Thus, when $|\Delta|$ is a line, $K[\Delta]$ is Gorenstein precisely if there are at most 3 vertices.

Remarks. Thus, while I conjecture that the Cohen-Macaulayness of $K[\Delta]$ is actually a topological property of $|\Delta|$, this is evidently false for the Gorenstein property. This situation will be clarified in (§6), where we study the Gorenstein property further.

§6. Pic and Sheaf Cohomology on $\text{Proj}(K[\Delta])$, the Gorenstein Property for $K[\Delta]$, and Poincaré Duality on $|\Delta|$. Let K be a commutative ring, Δ a simplicial complex, and let $(X_{\Delta,K}, \mathcal{O}_{\Delta,K})$ or, briefly, $(X_{\Delta}, \mathcal{O}_{\Delta})$, denote $\text{Proj}_K(K[\Delta])$.

We shall compute $\text{Pic}(X_{\Delta})$ and then use the result to determine when $K[\Delta]$ is Gorenstein.

We say that K is seminormal if for all n and indeterminates $u_1, \dots, u_n$ over K ,

$$\text{Pic}(K) \rightarrow \text{Pic}(K[u_1, \dots, u_n])$$

is an isomorphism. (If K is a ring, $\text{Pic}(K)$ denotes the group of isomorphism classes of rank one projectives under $\otimes$. More generally, if X is scheme, $\text{Pic}(X)$ is the group of isomorphism classes of locally free sheaves of rank one under $\otimes$.)

Seminormality (defined differently) is studied in [29], where a number of equivalent properties are considered. If K is reduced

pseudogeometric Noetherian, then [6] K is seminormal if and only if for each λ in the total quotient ring of K , if λ^2, λ^3 are in K then λ is in K . In particular, if K is reduced pseudogeometric Noetherian, then K normal implies K seminormal.

Theorem (6.1). Let K be a seminormal reduced ring with $\text{Pic}(K) = 0$. Then for every Δ , $\text{Pic}(K[\Delta]) = 0$.

Before proving this we need the following.

Lemma (6.2). Let I_1, I_2 be ideals of R , and suppose $U(R/I_1 \cap I_2) \rightarrow U(R/I_1 + I_2)$ is surjective, where $U(S)$ denotes the group of units of the ring S . If $\text{Pic}(R/I_m) = 0$, $m = 1, 2$, then $\text{Pic}(R/I_1 \cap I_2) = 0$.

Proof. We may replace R by $R/I_1 \cap I_2$, I_m by $I_m/I_1 \cap I_2$, and hence assume $I_1 \cap I_2 = (0)$. Let P be a rank one projective R -module. $P \otimes R/I_m \cong P/I_m P$ is (R/I_m) -free and we may choose p_m in P so that its image in $P/I_m P$ is a free generator. Hence, the images of p_1, p_2 in $P \otimes R/I_1 \otimes R/I_2 \cong P/(I_1 + I_2)P$ are both free generators, and there is a unit α of $R/(I_1 + I_2)$ such that $p_1 \equiv \alpha p_2 \pmod{(I_1 + I_2)P}$. Since the map of units is surjective, we have that $\alpha = \bar{a}$ for a unit a of R , and then $p_1 - ap_2 = i_1 - i_2$, where i_m is in $I_m P$. Then $v = p_1 - i_1 = ap_2 - i_2$ has the property that its image in either $P/I_1 P$ or $P/I_2 P$ is a free generator (we only need that a be invertible modulo I_2 , in fact), so that $P = Rv + I_1 P = Rv + I_1(Rv + I_2 P) \subset Rv + I_1 I_2 P = Rv$, which implies that P is free.

Proof of (6.1). Let Δ have vertices $x_0, \dots, x_n$, and let $S = K[x_0, \dots, x_n]$. We prove the result that $\text{Pic}(S/I) = 0$ for any ideal I generated by square-free monomials in $x_0, \dots, x_n$, by induction on the size of the ideal: we assume the result for larger I . Note that the result is trivial if $I = (x_0, \dots, x_n)$, and if I is prime, then $I = (x_{i_0}, \dots, x_{i_r})$ for some subset $\{x_{i_0}, \dots, x_{i_r}\}$ of $\{x_0, \dots, x_n\}$, so that S/I is a polynomial ring, and the result follows from the facts that $\text{Pic}(K) = 0$ and K is seminormal.

Now suppose I is not prime. Then $I = I_1 \cap I_2$ where I_1, I_2 are strictly larger ideals of the same form (e.g., I_1 can be taken as one prime from the primary decomposition and I_2 as the intersection of the others). Since I_1, I_2 are strictly larger, the theorem follows from Lemma (6.2) ($\text{Pic}(S/I_m) = 0$, $m = 1, 2$, by the induction hypothesis) once we know that $U(S/I) \rightarrow U(S/I_1 + I_2)$ is surjective. Now $I_1 + I_2$ is also a radical ideal generated by monomials. But for any such ideal J , $U(K) \rightarrow U(S/J)$ is an isomorphism. In fact we leave it to the reader to check that if R is a nonnegatively graded reduced K -algebra with $R_0 = K$, then $U(K) \rightarrow U(R)$ is an isomorphism.

Theorem (6.3). Let K be seminormal reduced with $\text{Pic}(K) = 0$ and suppose $|\Delta|$ cannot be disconnected by removing finitely many points and $\dim(\Delta) \geq 1$. Let $U(K)$ be the group of units of K . Then

$$\text{Pic}(\text{Proj}(K[\Delta])) \cong H^1(\Delta; U(K)) \oplus \mathbb{Z}.$$

where the copy of $\mathbb{Z}$ is generated by $\mathcal{O}_\Delta(1)$, the very ample sheaf induced

by the homogeneous coordinate ring $K[\Delta]$.

Proof. Let f_i be the image of x_i in $R = K[\Delta]$, let $X = \text{Proj}(K[\Delta])$, and let $X_i = X_{x_i} = \text{Spec}([K[\Delta]_{x_i}]_0)$, where $[\]_0 = 0^{\text{th}}$ graded piece and, by Proposition (5.6)(a) and its proof, $[R_{x_i}]_0 = [R_{f_i}]_0 \cong K[L_\Delta(x_i)]$, whence, $\text{Pic}([R_{f_i}]_0) = 0$ (Theorem (6.1)). Thus, if F is a locally free sheaf of rank one on X , $F|_{X_i}$ corresponds to a free $[R_{f_i}]_0$ -module, and it follows that if $U = \{X_0, \dots, X_n\}$ (an open cover of X) and $\mathcal{W}$ is the sheaf of germs of units (i.e., if Y is an open affine then $\mathcal{W}(Y) = U(\Gamma(Y, \mathcal{O}_\Delta))$) then

$$\text{Pic}(X) \cong H^1(U; \mathcal{W}).$$

Note that $X_{i_1} \cap \dots \cap X_{i_r} = \emptyset$ if and only if $f_{i_1} \dots f_{i_r} = 0$ if and only if $x_{i_1} \dots x_{i_r}$ is in I_Δ if and only if $\{x_{i_1}, \dots, x_{i_r}\}$ is not in Δ . If T_v denotes the open star of the vertex x_v in $|\Delta|$, we also have $X_{i_1} \cap \dots \cap X_{i_r} = \emptyset$ if and only if $T_{i_1} \cap \dots \cap T_{i_r} = \emptyset$.

We next want to compute $U([R_{f_i f_j}]_0)$ for $i \neq j$. There are two cases. If $f_i f_j = 0$ (i.e., $X_i \cap X_j = \emptyset$), then $U([R_{f_i f_j}]_0) = U([0]_0) = U(0)$ is trivial (0 denotes the zero ring). If $f_i f_j \neq 0$ (i.e., $X_i \cap X_j \neq \emptyset$) then it is not hard to see that $U([R_{f_i f_j}]_0) \cong U(K) \oplus \mathbb{Z}$, where the copy of $\mathbb{Z}$ is generated by the image of x_i/x_j or f_i/f_j , i.e.,

$$U([R_{f_i f_j}]_0) = U(K) \{ (f_i/f_j)^m : m \text{ in } \mathbb{Z} \}$$

if $f_i f_j \neq 0$. Thus, a Čech 1-cocycle is determined by a pair of functions λ, μ on the set of pairs (i, j) such that $i \neq j$ and $X_i \cap X_j \neq \emptyset$, where

$$\lambda(i, j) \text{ is in } U(K)$$

$$\mu(i, j) \text{ is in } Z$$

and satisfying

- (i) λ determines a Čech 1-cocycle on $\{X_0, \dots, X_n\}$ with coefficients in the constant sheaf $U(K)$.
- (ii) $\mu(i, j) = \mu(j, i)$ (when defined) and if i, j, k are distinct and $X_i \cap X_j \cap X_k = \emptyset$, then $\mu(i, j) = \mu(j, k) = \mu(k, i)$.

(The cocycle is then obtained by sending $(i, j) \rightarrow \lambda(i, j)(f_i/f_j)^{\mu(i, j)}$ if $i \neq j$ and $X_i \cap X_j \neq \emptyset$. (i), (ii) are precisely what we need to guarantee that this is a cocycle.)

The pairs $\{i, j\}$ such that $i \neq j$ and $X_i \cap X_j \neq \emptyset$ correspond precisely to the edges (1-simplices) $\{x_i, x_j\}$ in Δ . Thus, we may view μ as a function on the edges, with the property that it is constant on the boundary of each 2-simplex of Δ . This easily implies that μ is constant on all the edges of any simplex σ in Δ . Consider the equivalence relation on the maximal simplices generated by requiring $\sigma \sim \tau$ if σ and τ have an edge in common. Clearly, if two edges lie in equivalent simplices, μ has the same value on both. For each equivalence class C of maximal simplices, let C^* be the set of points of $|\Delta|$ lying in one of these simplices. Then clearly, for distinct

equivalence classes C_1, C_2 , $C_1^* \cap C_2^*$ is finite. Hence, if there is more than one C_i^* of positive dimension, we can disconnect $|\Delta|$ by removing finitely many points. But no C^* can be zero-dimensional, for, since $\dim(\Delta) \geq 1$ and $|\Delta| - \emptyset = |\Delta|$ is connected, each maximal face of Δ is positive dimensional.

Thus, there is only one equivalence class, and μ must be constant, say $\mu(i, j) = m$ for all (i, j) such that $i \neq j$ and $X_i \cap X_j \neq \emptyset$.

It follows that there is a unique Čech 1-cocycle λ with values in the constant sheaf $U(K)$ and a unique integer m such that for all i, j

$$\gamma(i, j) = \lambda(i, j)(f_i/f_j)^m.$$

On the other hand, any coboundary actually comes from a 0-cochain with values in the constant sheaf $U(K)$, for $U([R_f]_0) = U(K[L_\Delta(x_i)]) = U(K)$.

Moreover, the cocycle

$$(i, j) \rightarrow (f_i/f_j)^m$$

corresponds to $\mathcal{O}_X(m) = \mathcal{O}_X(1)^{\otimes m}$. Thus,

$$\text{Pic}(X) \cong H^1(U; \mathbb{Z}) \cong H^1(U; U(K)) \oplus \mathbb{Z}$$

where the copy of $\mathbb{Z}$ corresponds to the powers of $\mathcal{O}_\Delta(1)$.

Let $\mathcal{T} = \{T_0, \dots, T_n\}$ be the open cover of $|\Delta|$ by open stars of vertices. Since $T_{i_1} \cap \dots \cap T_{i_r} = \emptyset$ if and only if $X_{i_1} \cap \dots \cap X_{i_r} = \emptyset$ we have

$$H^1(U; U(K)) \cong H^1(\mathcal{T}; U(K)) \cong H^1(|\Delta|; U(K)) \cong H^1(|\Delta|; U(K))$$

(simplicial or singular) $\cong H^1(\Delta; U(K))$ (simplicial).

We are now ready to apply this result to the study of when $K[\Delta]$ is Gorenstein. The case $\dim(\Delta) \leq 1$ was handled in §5.

Theorem (6.4). Let K be a seminormal reduced Gorenstein ring with $\text{Pic}(K) = 0$ and Δ a simplicial complex with vertices $x_0, \dots, x_n$. Let $S = K[x_0, \dots, x_n]$ and let

$$\underline{\omega}_\Delta = (\text{Ext}_S^{n-d}(K[\Delta], S))^{\sim},$$

where we assume $d = \dim(\Delta) \geq 1$ ($\underline{\omega}_\Delta$ is the canonical sheaf on $(X_\Delta, \mathcal{O}_\Delta) = \text{Proj}(K[\Delta])$, if X_Δ is C-M). Then $K[\Delta]$ is Gorenstein if and only if the following two conditions hold:

- (1) $K[\Delta]$ is C-M.
- (2) For some integer m , $\underline{\omega}_\Delta \cong \mathcal{O}_\Delta(m)$.

Before proving this, we state and prove the following important corollary.

Corollary (6.5). If K is seminormal reduced Gorenstein with $\text{Pic}(K) = 0$, Δ is a simplicial complex with $\dim(\Delta) \geq 2$, Δ is C-M over K with Gorenstein links of vertices, and $H^1(\Delta; U(K)) = 0$, then $K[\Delta]$ is Gorenstein.

Proof. The hypothesis on the links implies that X_Δ is (locally)

Gorenstein, i.e., that ω_Δ corresponds to an element of $\text{Pic}(X_\Delta) \cong H^1(\Delta; U(K)) \oplus \mathbb{Z} \cong 0 \oplus \mathbb{Z} \cong \mathbb{Z}$, by Theorem (6.3). (Note that since $K[\Delta]$ is C-M, $\dim(\Delta) \geq 2$, the link of each vertex is connected: this implies easily that $|\Delta|$ cannot be disconnected by removing finitely many points.) Since this copy of $\mathbb{Z}$ corresponds to the elements $\{O_X(m) : m \text{ in } \mathbb{Z}\}$, we have that $\omega_\Delta \cong O_X(m)$ for some m , and we may apply Theorem (6.4).

Proof of Theorem (6.4). The necessity of (1) is clear. Let

$$D_\Delta = \text{Ext}_S^{n-d}(K[\Delta], S),$$

so that $\omega_\Delta = D_\Delta^\sim$. If $K[\Delta]$ is Gorenstein, then since D_Δ is a canonical module, D_Δ is a (graded) rank one projective over $K[\Delta]$, and, hence, free, i.e., $D_\Delta \cong R(m)$ for some m , where $R = K[\Delta]$ and (m) indicates shift of grading:

$$[R(m)]_h = R_{m+h}.$$

But then $D_\Delta^r \cong R(m)^\sim = O_\Delta(m)$.

Now assume conditions (1) and (2). We must show that $K[\Delta]$ is Gorenstein. It suffices to show that the graded R -module D_Δ , which is a canonical module, is free (since $K[\Delta]$ is known already to be C-M), i.e., that $D_\Delta \cong R(m)$ for some positive integer m (as graded R -modules).

Now, with m as in condition (2), we know that $D_\Delta^\sim \cong O_\Delta(m) \cong R(m)^\sim$, and we know that D_Δ and $R(m)$ are both C-M modules of maximal dimension over R . Hence, if $p = \bigoplus_{i \geq 1} R_i$ we have (since $\dim(\Delta) \geq 2$)

that $\text{depth}_p(R(m)) = \text{depth}_p(D_\Delta) = \text{height}(p) = \dim(\Delta + 1) \geq 2 \Rightarrow H_p^0(E) = H_p^1(E) = 0$, where $E = D_\Delta$ or $R(m)$. Hence, by the discussion on the bottom of p. 134 of [13], we have

$$\begin{aligned} D_\Delta &\cong \bigoplus_{-\infty < t < \infty} \Gamma(X_\Delta, D_\Delta(t)^\sim) \\ &\cong \bigoplus_{-\infty < t < \infty} \Gamma(X_\Delta, R(m)(t)^\sim) \\ (\text{for } D_\Delta(t)^\sim &\cong O_\Delta(m+t) \cong R(m)(t)^\sim) \\ &\cong R(m) \end{aligned}$$

We next remark:

Proposition (6.6). Propositions (2.3) and (2.4) hold with "Gorenstein" replacing "C-M" throughout.

The next few propositions give some explicit, useful, sufficient (and, sometimes, necessary) conditions for $K[\Delta]$ to be Gorenstein.

Theorem (6.7). Let Δ be a simplicial complex, $\dim(\Delta) \geq 1$. The following conditions on Δ are equivalent:

- (1) $Z[\Delta]$ is Gorenstein.
- (2) $Z[\Delta]$ is C-M, and for each $(\dim(\Delta) - 2)$ -simplex σ in Δ , $L_\Delta(\sigma)$ is either a circle, or a line with at most 3 vertices.
- (3) For every prime field K of positive characteristic, $K[\Delta]$ is Gorenstein.

(4) For every Gorenstein ring K , $K[\Delta]$ is Gorenstein.

Proof. The equivalence of (1), (3) and (4) follows from (6.6). (2) is clearly necessary: $Z[L_\Delta(\sigma)]$ will be Gorenstein, and $L_\Delta(\sigma)$ is 1-dimensional. Thus, the interesting fact is that (2) implies (1). Thus, assume (2). We must show that $Z[\Delta]$ is Gorenstein. We use induction on $\dim(\Delta)$. If $\dim(\Delta) = 1$, $\Delta = L_\Delta(\emptyset)$. ($\emptyset$ is a (-1)-simplex) is a circle, or line with at most 3 vertices, and we are done.

Now assume $\dim(\Delta) \geq 2$. The hypothesis of (2) passes to links of vertices. Hence, for each vertex x , $K[L_\Delta(x)]$ is Gorenstein, by the induction hypothesis.

Thus, we may conclude from (6.5) that $Z[\Delta]$ is Gorenstein provided

$$H^1(\Delta; \mathbb{Z}_2) = 0$$

(since $U(Z) \cong \mathbb{Z}_2$). But $Z[\Delta]$ C-M implies $\mathbb{Z}_2[\Delta]$ C-M, which implies $\tilde{H}^1(\Delta; \mathbb{Z}_2) = H^1(\Delta; \mathbb{Z}_2) = 0$ if $\dim(\Delta) \geq 2$.

Corollary (6.8). If $|\Delta|$ is a sphere, $K[\Delta]$ is Gorenstein.

This answers a question of R. Stanley. The special role of $\mathbb{Z}_2$ here is analogous to the special role $\mathbb{Z}_2$ plays in the discussion of orientability of manifolds.

Proposition (6.9). Let K be a field of $\text{char}(2)$. Suppose $\dim(\Delta) \geq 1$.

Then $K[\Delta]$ is Gorenstein if and only if $K[\Delta]$ is C-M and for each $(\dim(\Delta - 2))$ -simplex σ in Δ , $L_\Delta(\sigma)$ is either a circle, or a line with at most 3 vertices.

In this case $R[\Delta]$ will be Gorenstein for each Gorenstein R such that the image of each odd integer in R is invertible.

Proof. For the first part we may assume $K = \mathbb{Z}_2$. The argument is then the same as in (2) implies (1) in the proof of Theorem (6.7), except that we conclude that

$$H^1(\Delta; U(K)) = 0$$

here because $U(K)$ is trivial!

Once we know that $\mathbb{Z}_2[\Delta]$ is Gorenstein, we also know that $Q[\Delta]$ is Gorenstein, and in fact, that $L[\Delta]$ is Gorenstein for all fields of characteristic 2 or 0. The hypothesis on R implies that every field R maps into has characteristic 2 or 0.

Proposition (6.10). Suppose $\dim(\Delta) \geq 1$, $Q[\Delta]$ is C-M, and for each $(\dim(\Delta - 2))$ -simplex σ in Δ , $L_\Delta(\sigma)$ is a circle or a line with at most 3 vertices.

Suppose also that if Δ_0 is either Δ or a link of a simplex in Δ and $\dim(\Delta_0) \geq 2$, then $H^1(\Delta_0; \mathbb{Z}_2) = 0$.

Then for every Gorenstein ring R which contains Q , $R[\Delta]$ is Gorenstein.

Proof. R maps only into fields of characteristic 0 and so it suffices to show that $Q[\Delta]$ is Gorenstein. It is enough to show that for some prime p , $Z_p[\Delta]$ is Gorenstein.

Let Δ_0 run through Δ and its links of dimension ≥ 2 . There are only finitely many primes q such that $H^1(\Delta_0; Z_2) \neq 0$ for some such Δ_0 , since, by the C-M property for $Q[\Delta]$, $H^1(\Delta_0; Q) = 0$ for any such Δ_0 . Let $q_1, \dots, q_t$ be all these primes. By hypothesis, all the q_i are odd, and so there is a prime p in the arithmetic progression

$$\{(2h+1)(q_1 \dots q_t) + 2 : h = 0, 1, 2, \dots\},$$

in fact, infinitely many. Hence we can choose p such that

$$q_i \nmid (p-1), \quad i = 1, \dots, t.$$

We shall show that $Z_p[\Delta]$ is Gorenstein. As before, using induction on $\dim(\Delta)$, the fact that the hypotheses pass to links, and so forth, we come down, by (6.5), to the problem of showing that

$$H^1(\Delta_0; U(Z_p)) = 0,$$

for links Δ_0 of dimension ≥ 2 , as above. Since $U(Z_p)$ has order $p-1$, it has a filtration by copies of Z_q , where q is not in $\{q_1, \dots, q_t\}$ is prime, and the result follows.

Although it is possible to make a substantially finer analysis of the Gorenstein property and related properties, we shall pursue the issue no further here.

However, we do want to point out that there is a relationship between Serre-Grothendieck duality on $\text{Proj}(K[\Delta])$ and Poincaré duality on $|\Delta|$.

For the rest of this section, assume that K is a field, and that Δ is a simplicial complex of pure dimension $d \geq 2$. Assume also that each link of a vertex of Δ is C-M but not that $K[\Delta]$ is C-M.

Theorem (6.11). Under the hypotheses above

- (i) $H^i(X_\Delta; \mathcal{O}_\Delta) \cong H^i(\Delta; K)$, $1 \leq i < \dim(\Delta)$
- (ii) $H^i(X_\Delta; \mathcal{O}_\Delta(m)) = 0$, $m \neq 0$, $1 \leq i < \dim(\Delta)$
- (iii) $H^d(X_\Delta; \mathcal{O}_\Delta(m)) = 0$, $m > 0$, $d = \dim(\Delta)$.

Proof. Let $R = K[\Delta]$ and $P = \bigoplus_{i \geq 1} R_i$. One can reduce the characteristic 0 case to the characteristic p case and in characteristic p one may assume that K is perfect or even $\mathbb{Z}/p\mathbb{Z}$. We then deduce the result from Theorem 1.1 of [13], using the fact that

$$H_P^i(R) \cong [H^i(K^*(x_0, \dots, x_n; R))]_0,$$

where K^* denotes the Koszul complex with the numbering reversed and the fact that

$$[H_P^i(R)]_m \cong H^{i-1}(X, \mathcal{O}_X(m)), \quad i \geq 2.$$

The vanishing results are immediate from Theorem (2.1). Now, for $1 \leq i < \dim(\Delta)$ we have

$$\begin{aligned} H^i(X_\Delta, \mathcal{O}_\Delta) &\cong [H^{i+1}(K(x_0, \dots, x_n; R))]_0 \\ &\cong H^{n-i}([K(x_0, \dots, x_n; R)]_0) . \end{aligned}$$

The grading on $K(\underline{x}; R)$ is such that a K -basis for $[K^{i+1}(\underline{x}; R)]_0$ consists of products of monomials of degree $i + 1$ and elements of the free basis: hence, in $[K_{n-i}(x_0, \dots, x_n; R)]_0$ we allow only monomials of degree $i + 1$. If we trace this restriction through our identification of $\text{Tor}_{n-i}^S(K, K[\Delta])$ ($\cong H_{n-i}(K(\underline{x}; R))$) with

$$\bigoplus_{\substack{-1 \leq q \leq i \\ T \subset V, \text{card}(T+q)=i}} \tilde{H}^q(\Delta/T; K)$$

we see that the 0-th graded piece corresponds to the unique term for which $T = \emptyset$. (We are following the same argument as in [23].) Thus,

$$H^i(X_\Delta, \mathcal{O}_\Delta) \cong \tilde{H}^i(\Delta; K) \cong \tilde{H}^i(\Delta; K) , \quad 1 \leq i < \dim(\Delta) ,$$

as required.

Discussion. We can now relate Serre-Grothendieck duality on $\text{Proj}(K[\Delta])$ with Poincaré duality on $|\Delta|$.

Suppose that $|\Delta|$ is a manifold, or even merely that Δ is of pure dimension d and that for each vertex x , $K[L_\Delta(x)]$ is Gorenstein. (In a manifold, all proper links are homology spheres and hence all proper links are Gorenstein.) Suppose that $H^1(\Delta; U) = 0$, where U is the units of the prime field of K . (In fact, it is possible to allow $K = \mathbb{Z}$, in which case U is only $\mathbb{Z}_2$.)

The condition on the links of vertices implies that ω_Δ is locally free, while the condition on $H^1(\Delta; U)$ implies that $\omega_\Delta \cong \mathcal{O}_\Delta(m)$ for some m .

Now if F is any locally free sheaf on X_Δ we have

$$H^i(X, F)^* \cong H^{d-i}(X, \omega_\Delta \otimes F^V),$$

(see (3.7)) where $^* = \text{Hom}_K(_, K)$ and $^V = \text{Hom}_{\mathcal{O}_\Delta}(_, \mathcal{O}_\Delta)$. In particular,

$$H^i(X, \mathcal{O}_\Delta)^* \cong H^{d-i}(X, \mathcal{O}_\Delta(m)).$$

Thus, if $m \neq 0$, we obtain

$$H^i(X, \mathcal{O}_\Delta) = 0, \quad 1 \leq i < d$$

(and it is possible that $m \neq 0$).

If $m = 0$, we have

$$H^i(X, \mathcal{O}_\Delta)^* \cong H^{d-i}(X, \mathcal{O}_\Delta),$$

and, by (6.11), $H^i(X, \mathcal{O}_\Delta) \cong H^i(\Delta; K)$, $1 \leq i < d$. Thus, whether $m \neq 0$ or $m = 0$, we have

$$H_i(\Delta; K) \cong H^{d-i}(\Delta; K)^*, \quad 1 \leq i \leq d-1,$$

i.e., as claimed earlier, Serre-Grothendieck duality reduces to Poincaré duality.

§7. The Cohen-Macaulay Property for Ideals Generated by Monomials which may Involve Higher Powers. In this section we sketch quickly a technique for dealing with arbitrary ideals generated by monomials.

$$x_{i_1}^{h_1} \dots x_{i_t}^{h_t}$$

in the indeterminates $x_0, \dots, x_n$. For simplicity, we deal only with fields.

Let K be a field, let $x_0, \dots, x_n$ be indeterminates over K , and let $S = K[x_0, \dots, x_n]$. Let J be an ideal of S generated by monomials in the x 's. If f is another monomial in the x 's, f is not in J , we shall call $\text{Rad}(J:f)$ an associated radical ideal of J . $\text{Rad}(J:1) = \text{Rad}(J)$ is always one of these: let $S/\text{Rad}(J) = K[\Delta] = S/I_\Delta$, where Δ is the simplicial complex of supports of monomials not in $\text{Rad}(J)$. $\text{Rad}(J)$ is always one of the associated radical ideals of J , while the others correspond to certain subcomplexes of Δ . Hence, J has only finitely many associated radical ideals.

Theorem (7.1). With the above notation:

$$\text{depth}(S/J) = \min\{\text{depth}(S/I) : I \text{ is an associated radical ideal of } J\}.$$

Hence, S/J is Cohen-Macaulay if and only if all the S/I are Cohen-Macaulay of the same dimension.

Proof. One may reduce as in [13] to the case where K is a perfect

field of characteristic $p > 0$. Let ${}^e(S/J)$ denote S/J regarded as a module over itself via the e -th power of the Frobenius homomorphism. Then

$$\text{depth}_{S/J}(S/J) = \text{depth}_{S/J} {}^e(S/J).$$

It suffices to show that for large e , ${}^e(S/J)$ is a direct sum of cyclic modules with radical annihilators such that the set of annihilators which occur is precisely the set of associated radical ideals of J . To see this, first note that, for any e , the monomials

$$g = x_0^{h_0} \cdots x_n^{h_n}$$

such that each $h_i < p^e$ and g not in J give rise to a direct sum decomposition of ${}^e(S/J)$ into cyclic modules. The annihilator of the cyclic module arising from a given g is

$$\{r \text{ in } S/J : r^{p^e} g \text{ is in } J\}.$$

For fixed g and large e this is precisely

$$\text{Rad}(J:gR).$$

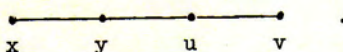
Thus, every associated radical ideal occurs for e sufficiently large.

Moreover, if p^e exceeds all integers which occur as exponents in a set of generating monomials for J , then

$$\begin{aligned} r^{mp^e} g \text{ in } J, \quad m \geq 1, \quad r \text{ a monomial,} \\ \text{implies } r^{p^e} g \text{ is in } J, \end{aligned}$$

regardless of which monomial g is. Thus, for large e all the annihilators are (associated) radical and all the associated radical ideals occur.

Example. Let $I = (xu, xv, yv)$ in $K[x, y, u, v]$, which corresponds to the simplicial complex



Thus, S/I is C-M. However, S/I^2 is not C-M, since

$$I^2 : xv = I + (yu) \text{ is radical}$$

$((xv)(yu) = (xy)(uv) \text{ in } I^2)$, and $I + (yu)$ corresponds to



which is disconnected and, hence, not C-M.

Added in Proof.

a) R. Stanley has written the author about the following new results: Let Δ be a simplicial complex of pure dimension d . Let K be a field (all cohomology if taken with coefficients in K).

i) J. Munkres has shown that $K[\Delta]$ is C-M if and only if for all x in $|\Delta|$, $H^i(|\Delta|, |\Delta| - \{x\}) = H^i(|\Delta|) = 0$, $i < d$, and has given a similar characterization of depth $K[\Delta]$.

ii) J. Munkres and R. Stanley have shown, independently, that $K[\Delta_1]$ is Gorenstein if and only if Δ_1 is the join of a simplex and a simplicial complex Δ as above such that $K[\Delta]$ is C-M and for all x in $|\Delta|$, $H^d(|\Delta|, |\Delta| - \{x\}) \cong H^d(|\Delta|) \cong K$.

b) The present author has given, independently, more complicated characterizations of the C-M property (and of depth) for $K[\Delta]$ in terms of purely topological properties of $|\Delta|$, along the lines indicated in [8] (but with cohomology over K replacing homotopy).

c) The author wishes to thank R. Stanley for correcting the reference to a non-shellable sphere, and L. Billera for correcting the inequalities in Cor. (5.3).

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