Math 217: MULTILINEARITY AND ALTERNATING PROPERTIES OF DETERMINANTS

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A. **Theorem:** *An $n \times n$ matrix A is invertible if and only if* $\det A \neq 0$.

1. One direction of the Theorem is easy to prove: prove it.
2. Prove that a square matrix A is invertible if and only if A^T is invertible.
3. Prove that A has rank n if and only if $\det A \neq 0$.
4. Give a *geometric* explanation why a 3×3 matrix whose columns are linearly dependent has determinant 0.
5. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & k & 0 \\ 0 & k & k \end{bmatrix}$. Find all values of k for which the system of equations $A\vec{x} = \vec{0}$ has exactly one solution.

Solution note: (1) If A is invertible, then $A\,A^{-1} = I_n$, so by the multiplicative property of determinants, $\det A \ \det(A^{-1}) = \det I_n = 1$. So $\det A$ can not be zero.
(2). Since $\det A = \det A^T$, either both are zero or both or not.
(3). The rank is n if and only if the matrix is invertible. So this follows from the theorem.
(4). If the columns are dependent, then the image has dimension 2 or less. So the image of the unit cube will be a parallelogram in the image plane (or line) of T_A. This has volume zero.
(5) $\det A = k^2 + k$). This is zero if and only if $k = 0, -1$. So the system has exactly one solution when $k \neq 0, -1$.

B. **Theorem:** *The determinant is* **multilinear** *in the columns. The determinant is* **multilinear** *in the rows.* This means that if we fix all but one column of an $n \times n$ matrix, the determinant function is linear in the remaining column. Ditto for rows.

1. Consider the map $\mathbb{R}^3 \to \mathbb{R}$ sending $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \det \begin{bmatrix} x & 1 & 2 \\ y & -1 & 0 \\ z & 0 & 1 \end{bmatrix}$. Is it linear? Why? If so, find its matrix.
2. What does the theorem say about the determinant of $\begin{bmatrix} a & b & c \\ x_1 + y_1 & x_2 + y_2 & z_1 + z_2 \\ d & e & f \end{bmatrix}$? Similarly, what other statement does the linearity in the second row give us for 3×3 matrices?
3. Prove the Theorem in the 2×2 case for the first column. That is: Show that the determinant of a 2×2 matrix is linear in the first column.
4. Compute $\det \begin{bmatrix} 2a + 5p & p \\ 5q + 2d & q \end{bmatrix}$ given that $\det \begin{bmatrix} a & -d \\ p & -q \end{bmatrix} = 17$.

5. Suppose $\vec{v}_1, \ldots, \vec{v}_{n-1}$ are vectors in $\mathbb{R}^n$ and that

$$\det[\vec{v}_1 \;\; \vec{v}_2 \;\; \ldots \vec{v}_{n-1} \;\; \vec{a}] = 5, \quad \det[\vec{v}_1 \;\; \vec{v}_2 \;\; \ldots \vec{v}_{n-1} \;\; \vec{b}] = 7, \quad \det[\vec{v}_1 \;\; \vec{v}_2 \;\; \ldots \vec{v}_{n-1} \;\; \vec{c}] = -3.$$

   Find $\det[\vec{v}_1 \;\; \vec{v}_2 \;\; \ldots \vec{v}_{n-1} \;\; (2\vec{a} + 4\vec{b} - 6\vec{c} + 17\vec{v}_1)]$.

6. TRUE OR FALSE, Justify: The determinant map $\mathbb{R}^{n \times n} \longrightarrow \mathbb{R}$ is linear. [HERE AND ESPECIALLY ON EXAM 1: give a simple counterexample to justify "false".]

7. Prove that the determinant function of an $n \times n$ matrix is linear in the **last column.** [Hint: Use Laplace expansion along the last column.] What about other rows or columns?

Solution note:

1. Yes, the map is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}$. This is the content of the theorem! You can (and perhaps should!) also verify this directly by checking it preserves addition and scalate multiplication. Its matrix is 1×3 and we find it by seeing where each of the $\vec{e}_i$ go. To find this, we compute the three given determinants, replacing the first column by each of $\vec{e}_1, \vec{e}_2, \vec{e}_3$. We get $A = \begin{bmatrix} -1 & -1 & 2 \end{bmatrix}$.

2. $\det \begin{bmatrix} a & b & c \\ x_1 + y_1 & x_2 + y_2 & z_1 + z_2 \\ d & e & f \end{bmatrix} = \det \begin{bmatrix} a & b & c \\ x_1 & x_2 & z_1 \\ d & e & f \end{bmatrix} + \det \begin{bmatrix} a & b & c \\ y_1 & y_2 & z_2 \\ d & e & f \end{bmatrix}$ and
$\det \begin{bmatrix} a & b & c \\ kx_1 & kx_2 & kz_1 \\ d & e & f \end{bmatrix} = k \det \begin{bmatrix} a & b & c \\ x_1 & x_2 & z_1 \\ d & e & f \end{bmatrix}$.

3. We need to show
$$\det \begin{bmatrix} a + a' & b \\ c + c' & d \end{bmatrix} = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \det \begin{bmatrix} a' & b \\ c' & d \end{bmatrix}, \quad \det \begin{bmatrix} ka & b \\ kc & d \end{bmatrix} = k \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$
Compute: $(a+a')d - (c+c')b = ad + a'd - cb -' cb = (ad - bc) + (a'd -' cb)$ and $(ka)d - (kc)b = k(ad - bc)$. QED.

4. $\det \begin{bmatrix} 2a + 5p & 5q + 2d \\ p & q \end{bmatrix} = \det \begin{bmatrix} 2a & 2d \\ p & q \end{bmatrix} + \det \begin{bmatrix} 5p & 5q \\ p & q \end{bmatrix}$ by linearity in the first row and the fact that the determinant is zero if there is a relation on the rows. This further becomes $2 \det \begin{bmatrix} a & d \\ p & q \end{bmatrix}$, again by linearity in the first row. On the other hand, using linearity in the second column, $\det \begin{bmatrix} a & -d \\ p & -q \end{bmatrix} = -\det \begin{bmatrix} a & d \\ p & q \end{bmatrix} = -17$. So the answer is -34.

5. Using multilinearity, we have
$$\begin{aligned} &\det[\vec{v}_1 \ \vec{v}_2 \ \ldots \vec{v}_{n-1} \ \ 2\vec{a}] + \det[\vec{v}_1 \ \vec{v}_2 \ \ldots \vec{v}_{n-1} \ \ 4\vec{b}] + \det[\vec{v}_1 \ \vec{v}_2 \ \ldots \vec{v}_{n-1} \ \ -6\vec{c}] + \det[\vec{v}_1 \ \vec{v}_2 \ \ldots \vec{v}_{n-1} \ \ 17\vec{v}_1)] \\ =&2\det[\vec{v}_1 \ \vec{v}_2 \ \ldots \vec{v}_{n-1} \ \ \vec{a}] + 4\det[\vec{v}_1 \ \vec{v}_2 \ \ldots \vec{v}_{n-1} \ \ \vec{b}] - 6\det[\vec{v}_1 \ \vec{v}_2 \ \ldots \vec{v}_{n-1} \ \ \vec{c}] + 17\det[\vec{v}_1 \ \vec{v}_2 \ \ldots \vec{v}_{n-1} \ \ \vec{v}_1)] \\ =&10 + 28 + 18 + 0 = 56. \end{aligned}$$
The final zero is because a matrix with dependent columns always has determinant zero.

6. FALSE! For example $\det(2I_2) = 4$ not $2 \det I_2$.

7. Fix $\vec{v}_1, \ldots, \vec{v}_{n-1}$ in $\mathbb{R}^n$. We need to show
$$\det[\vec{v}_1 \ \vec{v}_2 \ \ldots \vec{v}_{n-1} \ \ \vec{x} + \vec{y}] = \det[\vec{v}_1 \ \vec{v}_2 \ \ldots \vec{v}_{n-1} \ \ \vec{x}] + \det[\vec{v}_1 \ \vec{v}_2 \ \ldots \vec{v}_{n-1} \ \ \vec{y}],$$
and similarly for scalar multiplication. Expanding along the last column:
$$\begin{aligned} \det[\vec{v}_1 \ \vec{v}_2 \ \ldots \vec{v}_{n-1} \ \ \vec{x} + \vec{y}] &= (-1)^{n+1}[(x_1 + y_1)A_{1n} - (x_2 + y_2)A_{2n} + \cdots \pm (x_n + y_n)A_{nn}] \\ &= (-1)^{n+1}[x_1 A_{1n} - x_2 A_{2n} + \cdots \pm x_n A_{nn}] + (-1)^{n+1}[y_1 A_{1n} - y_2 A_{2n} + \cdots \pm y_n A_{nn}] \\ &= \det[\vec{v}_1 \ \vec{v}_2 \ \ldots \vec{v}_{n-1} \ \ \vec{x}] + \det[\vec{v}_1 \ \vec{v}_2 \ \ldots \vec{v}_{n-1} \ \ \vec{y}]. \end{aligned}$$
A similar calculation shows the scalar multiplication is preserved too.

C. **Corollary:** *Let A be an $n \times n$ matrix. Then* $\det(kA) = k^n \det A$ *for any scalar k.* Prove this.

> *Solution note:* Let $\vec{v}_1, \ldots, \vec{v}_n$ be the rows of A. Then $\det kA = \det[k\vec{v}_1\ k\vec{v}_2\ \ \ldots k\vec{v}_n] = k\det[\vec{v}_1\ k\vec{v}_2\ \ \ldots k\vec{v}_n] = k^2\det[\vec{v}_1\ \vec{v}_2\ \ \ldots k\vec{v}_n] = k^n\det[\vec{v}_1\ \vec{v}_2\ \ \ldots \vec{v}_n]$. Each time we are pulling out one k, using linearity in the first, then second, then third, etc, column.

D **Theorem:** *The determinant is* **alternating** *in the columns. The determinant is* **alternating** *in the rows.* This means if we interchange two columns, the determinant changes sign. Ditto for rows.

1. Verify this for swapping the second two columns of $\begin{bmatrix} 0 & 1 & 2 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Also verify for swapping the first and third row.
2. Prove the theorem in the case of 2×2 matrices.
3. Prove that if two columns of a square matrix A are equal, then $\det A = 0$.
4. Fix three column vectors $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$. What is the dimension of the subspace of $\mathbb{R}^5$ spanned by the vectors
$$\begin{bmatrix} \det[\vec{a}\ \vec{b}\ \vec{c}] \\ \det[\vec{b}\ \vec{a}\ \vec{c}] \\ \det[\vec{c}\ \vec{c}\ \vec{a}] \\ \det[\vec{a}\ \vec{b}\ \vec{0}] \\ \det[\vec{a}+b\ \vec{b}\ \vec{c}] \end{bmatrix}, \begin{bmatrix} \det[\vec{b}\ \vec{a}\ \vec{c}] \\ \det[\vec{b}\ \vec{c}\ \vec{a}] \\ \det[\vec{c}\ \vec{b}\ \vec{b}] \\ \det[\vec{a}\ \vec{b}\ \vec{0}] \\ \det[\vec{a}\ \vec{c}\ \vec{b}+c] \end{bmatrix}, \begin{bmatrix} \det[\vec{a}\ \vec{a}\ \vec{c}] \\ \det[\vec{b}\ \vec{a}\ \vec{a}] \\ \det[\vec{c}\ \vec{b}\ \vec{b}] \\ \det[\vec{a}\ \vec{b}\ \vec{0}] \\ \det[\vec{a}\ \vec{0}\ \vec{c}] \end{bmatrix}$$
5. Let A be an $n \times n$ matrix. Let B be obtained from A by switching columns i and j. Prove that $\det A = -\det B$. [Hint: Write $A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \end{bmatrix}$. Compute the determinant of $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_i + \vec{v}_j & \ldots & \vec{v}_i + \vec{v}_j & \ldots & \vec{v}_n \end{bmatrix}$, where the $\vec{v}_i + \vec{v}_j$ is in BOTH the i-th spot, and the j-th spot.]

Solution note: (1) Computing the determinant of the matrix in (1) we get 2. We get -2 after doing the intended swaps.
(2) We prove only alternating in rows, the case of columns being similar. We need to check that $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = -\det \begin{bmatrix} c & d \\ a & b \end{bmatrix}$. This is easy: $ad - cb = -(ac - bd)$.
(3) Say $A = [\vec{v}_1 \vec{v}_2 \ldots \vec{v}_n]$, and say its determinant is d. If $\vec{v}_i = \vec{v}_j$, then swapping them, we have the same matrix, so the determinant is also d. On the other hand, swapping them changes the determinant's sign, so it is also $-d$. The only way to get $d = -d$ is that $d = 0$. For another proof, note that the matrix can not be invertible, since the columns are not indepdent. So the determinant is 0.
(4). The first column is -1 times the second column and the third column is zero. So the dimension is 1.
(5) Note that $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_i + \vec{v}_j & \ldots & \vec{v}_i + \vec{v}_j & \ldots & \vec{v}_n \end{bmatrix}$ has determinant zero because two columns are the same. Now expand out using the multi-linearity: this is the same as

$$\det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_i & \ldots & \vec{v}_i + \vec{v}_j & \ldots & \vec{v}_n \end{bmatrix} + \det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_j & \ldots & \vec{v}_i + \vec{v}_j & \ldots & \vec{v}_n \end{bmatrix}.$$

Expanding in the other column and using the fact that the determinant is zero if two columns are the

$$0 = \det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_i & \ldots & \vec{v}_j & \ldots & \vec{v}_n \end{bmatrix} + \det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_j & \ldots & \vec{v}_i & \ldots & \vec{v}_n \end{bmatrix}.$$

F. Suppose that we have a linear transformation $T : \mathbb{R}^{2\times 3} \to \mathbb{R}^{2\times 3}$ whose matrix in the basis $(A_1, A_2, A_3, A_4, A_5, A_6)$ is $\begin{bmatrix} 0 & 1 & 2 & 3 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$.

1. What is the determinant of T?
2. What is the rank of T?
3. Find a basis for the kernel and image of T.
4. Suppose that $(B_1, B_2, B_3, B_4, B_5, B_6)$ is another basis, where

   $$B_1 = A_1 + A_2,\ B_2 = A_1 - A_2,\ B_3 = 3A_3,\ B_4 = 4A_4,\ B_5 = A_5 - A_4 - A_3 - A_2,\ B_6 = A_6.$$

   Find the change of basis matrix $S_{\mathcal{B}\to A}$. **Without computing:** explain why the change of basis matrix has non-zero determinant.
5. Find the $\mathcal{B}$-matrix of T (do not simplify!) **Without computing:** What is its determinant?

Solution note: The rank is 3, which is less than 6. So the determinant is 0.
(3) A basis for the image is represented by a maximal set of linearly independent columns. These can be taken to be the first, second and third columns. So a basis for the image is the set $(A_3, A_1 - A_2 - A_4 - A_5 - A_6, A_1)$. A basis for the kernel thus consists of three elements (rank nullity). To find it, we first use row-reduction to solve the system $A\vec{x} = \vec{0}$, and then re-interpret the column vectors as coordinates to get a basis in $\mathbb{R}^{2\times 3}$. Solving the system we get the solutions are spanned by $[3/2\ \ 0\ \ -3/2\ \ 1\ \ 0\ \ 0]^T, \vec{e}_5, \vec{e}_6$. Thus a basis for the kernel is $(3A_1 - 3A_3 + 2A_4, A_5, A_6)$.
(4) The change of basis matrix is

$$S_{\mathcal{B}\to A} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 3 & 0 & -1 & 0 \\ 0 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Its determinant is non-zero because change of basis matrices are always invertible! The determinant of the $\mathcal{B}$-matrix of T is zero, since it is the same as the determinant of the $\mathcal{A}$-matrix of T.