Math 217: Proof of Multiplicative Property of Determinant

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Two of the most important theorems about determinants are yet to be proved:

Theorem 1: *If A and B are both $n \times n$ matrices, then* $\det A \det B = \det(AB)$.

Theorem 2: *A square matrix is invertible if and only if its determinant is non-zero.*

A. The proof of Theorem 2.

1. Use the **multiplicative property of determinants** (Theorem 1) to give a one line proof that if A is invertible, then $\det A \neq 0$.
2. Recall the three types of elementary row operations on a matrix:
   (a) Swap two rows;
   (b) Multiply one row by a non-zero scalar
   (c) Replace 'Row i' by 'Row i plus a scalar multiple of Row j".

   Now, using the **multilinearity and alternating properties** of determinants, explain exactly what happens to the determinant of a square matrix A under each type of elementary operation.
3. Prove that if the determinant of A is non-zero, then A is invertible. [Hint: Recall that A is invertible if and only if a series of elementary row operations can bring it to the identity matrix.]

Solution note:

1. Suppose A is invertible. Thus there exists an inverse matrix B such that $AB = BA = I_n$. Take the determinant of both sides. Using Theorem 1, we have $\det(AB) = \det A \det B = \det I_n = 1$. So we can not have $\det A = 0$, as this would imply $0 = 1$. QED.
2. Swapping two rows: determinant changes sign. Multiplying ONE row by c multiplies the determinant by c. Replacing "Row i" by "Row i + c Row j" leaves the determinant unchanged, since
$$\det \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_i + cR_j \\ \vdots \\ R_j \\ \vdots \\ R_n \end{bmatrix} = \det \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_i \\ \vdots \\ R_j \\ \vdots \\ R_n \end{bmatrix} + c\det \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_j \\ \vdots \\ R_j \\ \vdots \\ R_n \end{bmatrix} = \det \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_i \\ \vdots \\ R_j \\ \vdots \\ R_n \end{bmatrix} + 0.$$
   Note that the last equality holds because two rows are the same in one of the matrices.
3. Suppose that A has determinant D. We do elementary row ops on A, and each step, the new determinant is a NON-ZERO multiple of the previous (either -1 if we swap rows, or c if we scaled a row by c, or 1 if we added a multiple of a row to another). So A has deteminant $D = 0$ if and only if rref A also has determinant zero. So A is invertible if and only if $D \neq 0$.

B. The proof of Theorem 1.

1. Suppose that A and B are $n \times n$ upper triangular matrices. Verify the theorem in this case.
2. Prove that if A is not invertible, then neither is AB (without using Theorem 1 or 2, but rather the definition of invertible).
3. Prove the theorem in the case A is not invertible.
4. Prove the Theorem in the case A is invertible. [Hint: Recall that each elementary row operation is multiplication by an elementary matrix, so that an invertible matrix is a product of elementary matrices. Now do induction on the number of elementary matrices in the product.]

Solution note:

1. The product of upper triangle matrices is also upper triangular, and the diagonal entries are the corresponding products. So the determinant of AB is the product of the diagonal entries of A and B.
2. Suppose, on the contrary, that AB is invertible. Suppose C is the inverse (also $n \times n$). That means $(AB)C = I_n$. So $A(BC) = I_n$. Since A is $n \times n$, this means that the $n \times n$ matrix BC is the inverse of A. This contradicts A non-invertible.
3. Suppose A is not invertible. This means the determinant of A is zero. Similarly, AB is not invertible, so its determinant is 0. Obviously, then $\det A \det B = \det AB$.
4. Write A as a product of (say,) t elementary matrices. We show by induction on t that $\det A \det B = \det AB$. Base case: Suppose $A = E_1$ where E_1 is an elementary matrix. Then $\det E_1 \det B = \det E_1 B$ was checked in Problem A. Inductive step: Assume that if A' is a product of $t-1$ elementary matrices, then $\det A' \det B = \det(A'B)$. We need to prove the result for a EA' where E is an elementary matrix. But we know how multlplying by E changes the determinant from problem A, and it is the same on both sides, since $(EA')B = E(AB)$.