Math 412. The Symmetric Group $\mathcal{S}_n$.

DEFINITION: The **symmetric group** $\mathcal{S}_n$ is the group of bijections from any set of n objects, which we usually call simply $\{1, 2, \ldots, n\}$, to itself. An element of this group is called a **permutation** of $\{1, 2, \ldots, n\}$. The group operation in $\mathcal{S}_n$ is *composition* of mappings.

PERMUTATION STACK NOTATION: The notation $\begin{pmatrix} 1 & 2 & \cdots & n \\ k_1 & k_2 & \cdots & k_n \end{pmatrix}$ denotes the permutation that sends i to k_i for each i.

CYCLE NOTATION: The notation $(a_1\ a_2\ \cdots\ a_t)$ refers to the (special kind of!) permutation that sends a_i to a_{i+1} for $i < t$, a_t to a_1, and fixes any element other than the a_i's. A permutation of this form is called a t**-cycle.** A 2-cycle is also called a **transposition.**

THEOREM 7.24: Every permutation can be written as a product of *disjoint cycles* — cycles that all have no elements in common. Disjoint cycles commute.

THEOREM 7.26: Every permutation can be written as a product of *transpositions,* not necessarily disjoint.

A. WARM-UP WITH ELEMENTS OF $\mathcal{S}_n$

(1) Write the permutation $(1\ 3\ 5)(2\ 7) \in \mathcal{S}_7$ in permutation stack notation.
(2) Write the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 6 & 1 & 2 & 4 & 7 & 5 \end{pmatrix} \in \mathcal{S}_7$ in cycle notation.
(3) If $\sigma = (1\ 2\ 3)(4\ 6)$ and $\tau = (2\ 3\ 4\ 5\ 6)$ in $\mathcal{S}_7$, compute $\sigma\tau$; write your answer in stack notation. Now also write it as a product of disjoint cycles.
(4) With σ and τ as in (4), compute $\tau\sigma$. Is $\mathcal{S}_7$ abelian?
(5) List all elements of $\mathcal{S}_3$ in **cycle notation**. What is the order of each? Verify Lagrange's Theorem for each element of $\mathcal{S}_3$.
(6) What is the inverse of $(1\ 2\ 3)$? What is the inverse of $(1\ 2\ 3\ 4)$? How about $(1\ 2\ 3\ 4\ 5)^{-1}$? How about $[(1\ 2\ 3)(3\ 4\ 5)]^{-1}$?

B. THE SYMMETRIC GROUP $\mathcal{S}_4$

(1) What is the order of $\mathcal{S}_4$?
(2) List all 2-cycles in $\mathcal{S}_4$. How many are there?
(3) List all 3-cycles in $\mathcal{S}_4$. How many are there?
(4) List all 4-cycles in $\mathcal{S}_4$. How many are there?
(5) List all 5-cycles in $\mathcal{S}_4$.
(6) How many permutations in $\mathcal{S}_4$ are not cycles? Find them all.
(7) Find the order of each element in $\mathcal{S}_4$. Why are the orders the same for permutations with the same "cycle type"?
(8) Find cyclic subgroups of $\mathcal{S}_4$ of orders 2, 3, and 4.
(9) Find a subgroup of $\mathcal{S}_4$ isomorphic to the Klein 4-group. List out its elements.
(10) List out all elements in the subgroup $H = \langle (1\ 2\ 3), (2\ 3) \rangle$ of $\mathcal{S}_4$ generated by $(1\ 2\ 3)$ and $(2\ 3)$. What familiar group is this isomorphic to? Can you find four different subgroups of $\mathcal{S}_4$ isomorphic to $\mathcal{S}_3$?

C. EVEN AND ODD PERMUTATIONS.

A permutation is **odd** if it is a composition of an odd number of transposition, and **even** if it is a product of an even number of transpositions.[1]

[1] Skeptic mathematicians that you have been trained to be, you realize, perhaps, that this definition may be problematic. Why? What should be proved to ensure the definition is well-defined? This is handled in the book, in Section 7.5.

(1) Write the permutation (123) as a product of transpositions. Is (123) even or odd ?
(2) Write the permutation (1234) as a product of transpositions. Is (1234) even or odd ?
(3) Write the $\sigma = (12)(345)$ a product of transpositions in two different ways. Is σ even or odd ?
(4) Prove that every 3-cycle is an even permutation.

D. THE ALTERNATING GROUPS

(1) Prove that the subset of even permutations in S_n is a subgroup. This is the called the **alternating group** A_n.
(2) List out the elements of A_2. What group is this?
(3) List out the elements of A_3. To what group is this isomorphic?
(4) How many elements in A_4? Is A_4 abelian? What about A_n?

E. THE SYMMETRIC GROUP $\mathcal{S}_5$

(1) Find one example of each type of element in S_5 or explain why there is none:
  (a) A 2-cycle
  (b) A 3-cycle
  (c) A 4-cycle
  (d) A 5-cycle
  (e) A 6-cycle
  (f) A product of disjoint transpositions
  (g) A product of 3-cycle and a disjoint 2-cycle.
  (h) A product of 2 disjoint 3 cycles.
(2) For each example in (1), find the order of the element.
(3) What are all possible orders of elements in $\mathcal{S}_5$?
(4) What are all possible orders of cyclic subgroups of $\mathcal{S}_5$.
(5) For each example in (1), write the element as a product of transpositions. Which are even and which are odd?

F. Discuss with your workmates how one might prove Theorem 7.26. One way is by using Theorem 7.24 and the calculations you did in E (5). Another is by pure thinking.[2]

G. PERMUTATION MATRICES. We say that an $n \times n$ matrix is a **permutation matrix** if it can be obtained from the $n \times n$ identity matrix by swapping columns (or rows).

(1) List out all 3×3 permutation matrices.
(2) Prove that the set $\mathcal{P}_3$ of 3×3 permutation matrices is a subgroup of $GL_3(\mathbb{R})$.
(3) Find an isomorphism between $\mathcal{S}_3$ and $\mathcal{P}_3$. Prove that under your isomorphism, the alternating subgroup A_3 of $\mathcal{S}_3$ is identified with $\mathcal{P}_3 \cap SL_3(\mathbb{R})$.
(4) Find the order of each element of the subgroup of permutation matrices in $GL_3(\mathbb{R})$.
(5) For any n, the set $\mathcal{P}_n$ of all $n \times n$ permutation matrices is a group isomorphic to $\mathcal{S}_n$. Can you explain why?

BONUS. What are the possible orders of elements in $\mathcal{S}_6$? What are the possible orders of elements in $\mathcal{S}_7$? Prove that $\mathcal{S}_{12}$ has an element of order 35.

[2]Hint: Imagine lining everyone in the class up in a straight line. How can we put the class in alphabetical order by a sequence of swaps?