Visualizing the Birman-Series Set on the Punctured Torus

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Introduction

Goals

• Understand the distribution of simple, closed geodesics on hyperbolic surfaces

• Create pictures to demonstrate phenomena in hyperbolic geometry

Example 1 (The Punctured Torus)
The punctured torus \( T^*_0 \) can be obtained by gluing opposite sides of an isometric quadrilateral in the hyperbolic plane. This allows us to equip \( \Gamma \) with a hyperbolic metric and identify the universal cover of \( T^*_0 \) with the hyperbolic plane \( \mathbb{H}^2 \).

Definition. A path between \( a, b \) is a geodesic if it locally minimizes distance.

2. A path is simple provided it is non-self-intersecting.

3. The Birman-Series set is the union of all simple, closed geodesics on a finite area hyperbolic surface.

Theorem 1

1. (Birman-Serre 1985) The Birman-Series set has Hausdorff dimension 1; in particular, it is nowhere dense.

2. (Buser-Parlier 2008) If \( \Gamma \) is a finite-area orientable hyperbolic surface, then there exists a constant \( c > 0 \) (depending only on the topology of the surface) and a hyperbolic ball of radius \( r \) embedded in \( \Lambda \) disjoint from the Birman-Series set.

Fundamental Domains

Example 2 (Definition of \( g \) and \( \Gamma \).) Let \( \Gamma' = \left\{ (z \rightarrow \lambda z, \lambda \in \mathbb{Z} \} \right\} \) and let \( \Gamma'' = \left\{ (z \rightarrow \lambda z, \lambda \in \mathbb{Z} \} \right\} \) be the index 2 subgroup of \( \Gamma \) consisting of orientation preserving transformations. Each generator of \( \Gamma \) is a reflection about a geodesic, and the union of these geodesics bounds a fundamental triangle \( \hat{T} \subseteq \mathbb{H}^2 \) satisfying:

- \( \partial T = \partial \hat{T} \)

- \( \text{Int}(T) \cap \text{Int}(\hat{T}) = \emptyset \)

We say that \( \hat{T} \) is tessellated by the images of \( \Gamma \) under \( \Gamma \) and that \( \Gamma \) is a fundamental domain for the action of \( \Gamma \) on \( \mathbb{H}^2 \). For any point \( \gamma \in \Gamma \), \( \Gamma \) is an example of what is called a Fuchsian Group.

Definition. An action of \( \Gamma \subseteq \text{PSL}(2, \mathbb{R}) \) is discrete if given a sequence \( \{ \gamma_n \} \subseteq \Gamma \) satisfying \( \gamma_n \rightarrow \gamma \in \Gamma \), there exists \( N \) such that \( \gamma_n = \gamma \) for all \( n > N \). A Fuchsian group is a discrete subgroup of \( \text{PSL}(2, \mathbb{R}) \).

The tesselation of an ideal triangle by \( \Gamma \) in \( \mathbb{H} \) (top left), in \( \mathbb{H} \) (top right). Any one triangle defines a fundamental domain.

Theorem. Fix generators \( a, b \) of \( \Gamma \), and let \( \Gamma = \mathbb{Z}_2 \), as in Figure 1. Given \( \lambda \in (1, \infty) \), let \( \Gamma_\lambda = \left\{ (a \rightarrow \lambda a, \lambda \in \mathbb{Z} \} \right\} \) be the ideal quadrilateral with vertices \( 0, 1, \lambda \), and \( \infty \). Define \( \rho_\lambda : \Gamma_\lambda \rightarrow \text{PSL}(2, \mathbb{R}) \) by setting \( \rho_\lambda(a) = \text{hyperbolic translation along the geodesic orthogonal to the geodesics at } [0,\infty] \) and \( \lambda \), such that \( \rho_\lambda(a) = 1 \) and \( \rho_\lambda(\lambda a) = \lambda \). Similarly define \( \rho_\lambda(b) \) with \( \rho_\lambda(b)(\infty) = \infty \) and \( \rho_\lambda(b)(\lambda) = 1 \). If \( \lambda \in (1, \infty) \), then:

- \( \rho_\lambda \) is injective,

- \( \Gamma_\lambda = \rho_\lambda(\Gamma_\lambda) \) is Fuchsian, and

- \( \Gamma_\lambda \) is conjugate to \( \Gamma_\lambda \) in \( \text{PSL}(2, \mathbb{R}) \).

Analog to Euclidean Space

In both pictures above, we have a red square. Any two-colored lines represent the action of \( \rho_\lambda(a) \) and \( \rho_\lambda(b) \). If the blue line corresponds to \( \rho_\lambda(a) \), then, in the flat case (left), the square's left side is transformed by means of translation to its right side, and each other point in the interior is transformed accordingly to the right. In the hyperbolic case (right), the square's upper-left side is transformed by means of \( \rho_\lambda(a) \) to the bottom-right, and each other point in the interior is transformed accordingly to the transformation group.

We may now visualize the union of all closed geodesics in the surface \( \Gamma_\lambda \). This allows us to equip \( \Gamma \) with a hyperbolic metric and identify the universal cover of \( T^*_0 \) with the hyperbolic plane \( \mathbb{H}^2 \).

Figure 1: The punctured torus has a fundamental domain generated by two curves (shown above)

Representing Simple Closed Geodesics

Given a group \( \mathcal{C} \) and a set of generators \( \mathcal{S} \), we say the word length of an element \( g \in \mathcal{G} \) is the minimal word length in \( \mathcal{S} \) such that it is equal to \( g \). As there are no relations in a free group, the length of any reduced word in \( \mathcal{F} \) is computed by counting the number of \( a \) and \( b \).

Generating Closed Geodesics

To generate all closed geodesics in \( \mathcal{F} \) we first build a list of all words up to a fixed word length \( n \). We then need to throw out any words that correspond to loops about a puncture. This is done by fixing a \( \lambda \), taking a word \( w \) in \( \mathcal{F} \) and calculating the determinant of \( \rho_\lambda(w) \). As long as \( |\text{det}(\rho_\lambda(w))| \geq 2 \), then \( \rho_\lambda(w) \) will correspond to a closed geodesic in \( \Gamma_\lambda \). This is described for every \( \lambda \in (1, \infty) \).

Generating Simple Closed Geodesics

In order to pick out the simple closed geodesics from the list of closed geodesics, we use a characterization of simplicity given by Buser.

Definition. (Small Variation) A finite sequence of nonzero integers \( N_1, \ldots, N_p \) is said to have small variation provided that sums of \( m \) consecutive elements (indices \( k \) and \( j+m \)) never differ by more than \( \pm 1 \).

Theorem 2: (Buser 1988)

Every nontrivial simple, closed curve can, after suitably renaming generators, be represented by one of the following words:

- \( a^m b^m \) \( \langle m \rangle \), where the sequence \( N_1, \ldots, N_p \) has small variation.

Conversely, each of these words is homotopic to a multiple of a simple closed curve.

Since we had already generated a list of the closed geodesic curves of length up to \( n \), to draw the simple, closed geodesics, we need to filter out the non-simple curves. To do this we took a word, cyclically reduced it, and checked if it met Buser’s criteria.

The following pictures represent geodesics of word length up to \( n = 6 \).

Areas Avoid of Simple Closed Geodesics

Cuts the punctured torus along a simple closed geodesic; \( \gamma \) yields a thrice-punctured sphere with boundary. Every geodesic intersecting \( \gamma \) must intersect this copy of \( S^3 \) in simple arcs; hence, when \( \gamma \) is short, the simple closed geodesics of the torus must disjoint from the 12 regions (of definite diameter) described previously. This observation is essential to the proof of the Buser-Parlier result.

References


