Visualizing Structures on the Torus and Pair of Pants.

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1. Introduction

- The 2-torus T^2, familiarly known as a donut, is the 2-dimensional manifold formed by gluing opposite sides of a parallelogram together. Mathematically, we define it by $T^2 := \mathbb{R}^2 / \mathbb{Z}^2$.
- One can measure the distance between two points $\mathbf{p}, \mathbf{q}$ on T^2 by first "unrolling" the torus, then measuring the Euclidean distance between $\mathbf{p}$ and $\mathbf{q}$ in the resulting parallelogram. This method of measuring distance gives T^2 a *flat* metric, essentially meaning that locally the geometry of the torus looks like the geometry of the familiar euclidean plane.
- Say two tori are "equivalent" if one can be smoothly transformed into the other while preserving the distances between points. How many different unmarked lattices, hence nonisometric tori, are there?
- In this project, we studied the connections between two methods of parametrizing all flat tori: the upper half-plane $\mathbb{H}^2$ under an action of SL$_2(\mathbb{Z})$, and lattices in $\mathbb{R}^2$ that are identical up to rotation and uniform scaling.

2. The Upper Half Plane and Euclidean Lattices

The *upper half plane* $\mathbb{H}^2$ is defined to be $\{z \in \mathbb{C} : \Im(z) > 0\}$. The group SL$_2(\mathbb{R}) := \{M \in$ GL$_2(\mathbb{R}) : \det(M) = 1\}$ acts on $\mathbb{H}^2$ via fractional linear transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

This action is transitive: for any point $\tau = x + iy \in \mathbb{H}^2$, the transformation

$$M_\tau := \begin{pmatrix} 1/\sqrt{y} & -x/\sqrt{y} \\ 0 & \sqrt{y} \end{pmatrix}$$

maps the point τ to the point i. The columns of M_τ determine a lattice in $\mathbb{R}^2$. As $\det M_\tau = 1$, the fundamental parallelogram $\mathcal{P}$ of the resulting lattice will have area 1. Thus, we identify τ with the torus that results from gluing the opposite sides of $\mathcal{P}$.

Example 1 (i versus $-1+i$). *The lattices determined by i and $-1+i$ are spanned by $\{(1,0)^t, (0,1)^t\}$ and $\{(1,0)^t, (1,1)^t\}$, respectively. While the* lattice bases *are different, the lattice points they determine are identical. We conclude that i and $i+1$ determine the same torus up to isometry.*

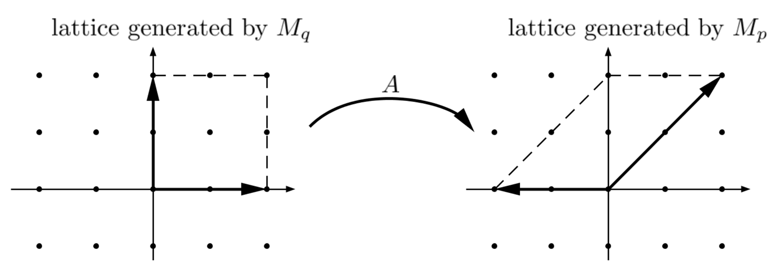

Figure 1: *Two lattice bases determining the same lattice*

To handle this double-counting, we identity points $\tau, \rho \in \mathbb{H}^2$ if they determine the same lattice up to choice of basis. The action of SL$_2(\mathbb{Z})$ sends lattice bases to lattice bases of the same lattice. So, we may quotient $\mathbb{H}^2$ by the action of SL$_2(\mathbb{Z})$ to find a sole representative for each *unmarked* lattice, hence nonisometric torus.

3. Results

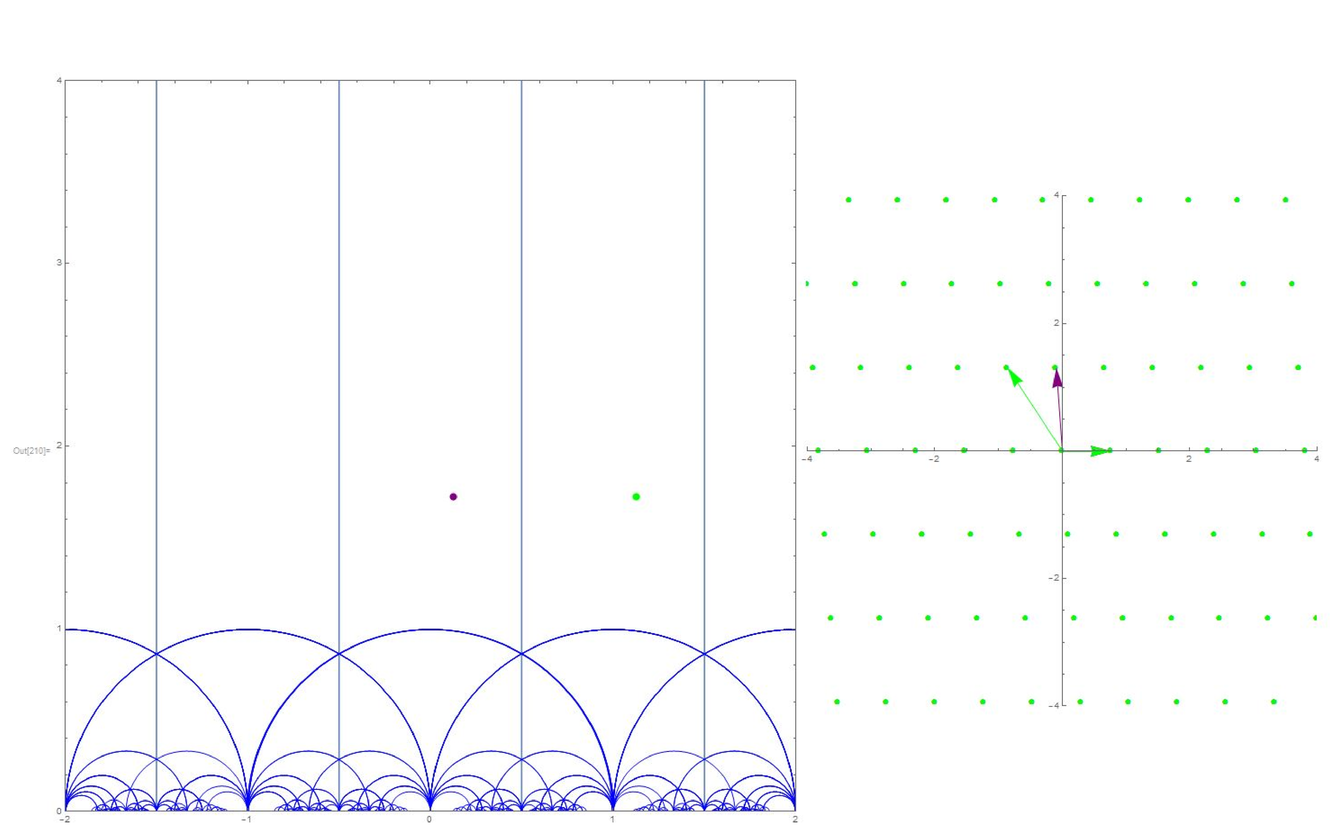

Figure 2: *Two points in the same SL$_2(\mathbb{Z})$-orbit determine isometric tori*

How our program works:

- The user clicks a point $\mathbf{p}$ in the upper half-plane. Using the SL$_2(\mathbb{Z})$ action, a shadow point $\mathbf{q}$ is generated in the *fundamental domain* $\mathcal{F} := \{z \in \mathbb{C} : -1/2 \leq \Re(z) \leq 1/2, |z| \geq 1\}$. To compute the lattice associated to $\mathbf{p}$, it suffices to generate the lattice associated to $\mathbf{q}$ and potentially apply a rotation (see Figure 3).

Generation of the shadow point

It is well-known that SL$_2(\mathbb{Z}) = \langle S, T \rangle$, where S and T are the fractional linear transformations

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

These transformations can be combined to build all elements of SL$_2(\mathbb{Z})$. We use an algorithm similar to the Euclidean algorithm for division to reverse-engineer the decomposition of the $A \in$ SL$_2(\mathbb{Z})$ such that $A\,\mathbf{p} = \mathbf{q}$.

Theorem. *The following procedure, given $\mathbf{p} \in \mathbb{H}^2$, will return $A \in$ SL$_2(\mathbb{Z})$ such that $A\,\mathbf{p} \in \mathcal{F}$:*

1. Apply T to $\mathbf{p}$ until $-1/2 \leq \Re(\mathbf{p}) \leq 1/2$. Update the value of $\mathbf{p}$.

2. Apply S to invert $\mathbf{p}$ about the unit circle. Update the value of $\mathbf{p}$. Note that $\Im(\mathbf{p})$ strictly increases under this inversion.

3. Repeat (1) and (2) until inverting with S moves $\mathbf{p}$ above the unit circle.

4. Apply T until $\mathbf{p}$ sits in $\mathcal{F}$.

As there are only finitely many regions in the tiling with real part between $-1/2$ and $1/2$ and imaginary part bounded below by some $k > 0$, this procedure will terminate.

Although $\mathbf{p}$ and the shadow point $\mathbf{q}$ are in the same SL$_2(\mathbb{Z})$-orbit, their associated lattices are only equivalent up to rotation. Why? The action of changing bases via SL$_2(\mathbb{Z})$ induces a rotation on the tangent space to the point $\mathbf{p}$. As we think of two lattices to be equivalent if they differ by rotation, we apply a rotation R which fixes i so that $M_\mathbf{p} = RM_\mathbf{q}A$. See Figure 2.

Explicitly, we have the equation

$$M_\mathbf{p}(\mathbf{p}) = (R \circ M_\mathbf{q} \circ A)(\mathbf{p}),$$

and we wish to solve for R so that we may correct the incidental rotation. An application of the chain rule yields $R'(i) = e^{-2i\theta}$, where θ is the unknown rotation. Thus $\theta = (1/2)\arg(A'(\mathbf{p}))$, so we can make the proper adjustment to the lattice associated with $M_\mathbf{q}A$.

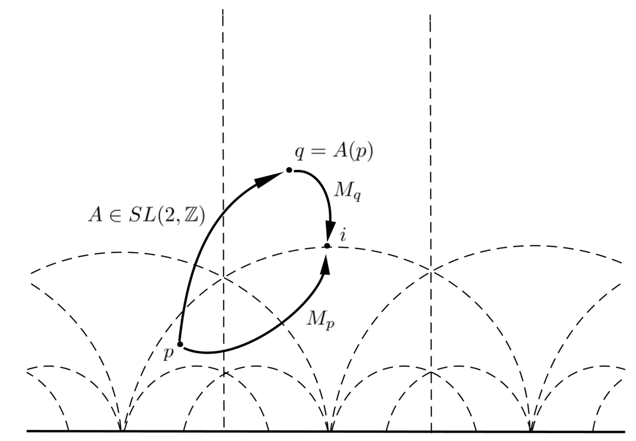

Figure 3: *Rotation is introduced by the A transformation, which we cancel by post-composing with R so that the diagram "commutes."*

4. Visualizing Other Spaces of Geometric Structures

The ideas used in this project can be used to parametrize non-Euclidean metrics on a variety of genus g surfaces. For example, how many hyperbolic metrics exist on a sphere with three disks removed, commonly referred to as a *pair of pants*? We are not restricted to hyperbolic geometry: we can consider convex projective structures as well. In this setting, cut a pair of pants into two triangles and unroll it onto a convex subset of the projective plane in a way analogous to unrolling a torus. We can then define a metric on this convex subset of the projective plane to define a metric on the pair of pants.

William Goldman has come up with a system of 8 coordinates which completely parametrizes the space of all convex projective structures on a pair of pants. In order to better understand these coordinates, we created a visualization tool which generates tilings of a convex subset of the projective plane given a choice of these coordinates. These tilings allow us to visualize the metric induced on the pair of pants (see Figure 4). Because this space has 8 parameters, convex projective structures can vary in complicated ways (see Figure 5), offering many opportunities for further investigation.

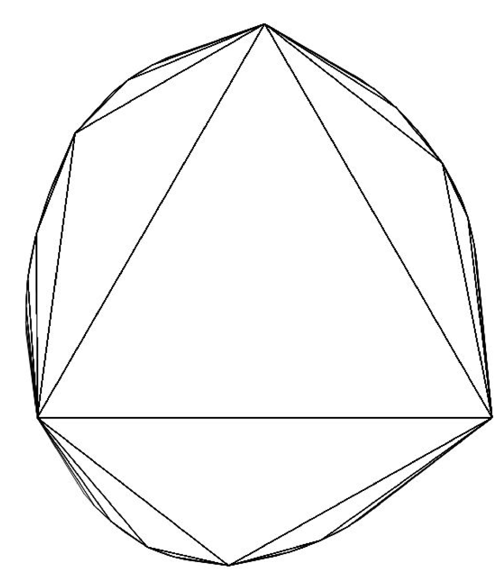

Figure 4: *A convex subset of the projective plane can be tiled by triangles parameterized by Goldman's coordinates.*

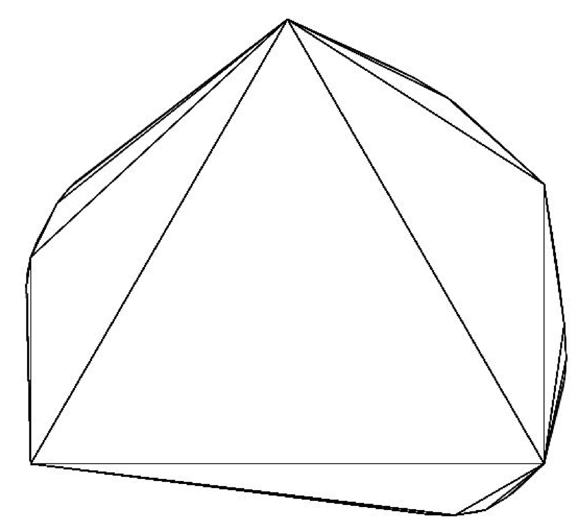

Figure 5: *Choosing different values for the parameters can result in significant distortions of the tiling.*

References

[1] William M. Goldman. Convex Real Projective Structures On Compact Surfaces. *J. Differential Geometry*, Volume 31, pp 791-845, (1990).

[2] Anton Zorich. Flat Surfaces. *Frontiers in Number Theory, Physics, and Geometry*, Springer Verlag, Volume 1, pp 439-586, (2006).