Visualizing the Schwarzian Derivative

Noah Luntzlara, Piriyakorn Piriyatamwong, Mengxi Wang, Sharon Ye
Patrick Haggerty, Didac Martinez-Granado, Maxime Scott, Dylan Thurston
Laboratory of Geometry at Michigan

Introduction

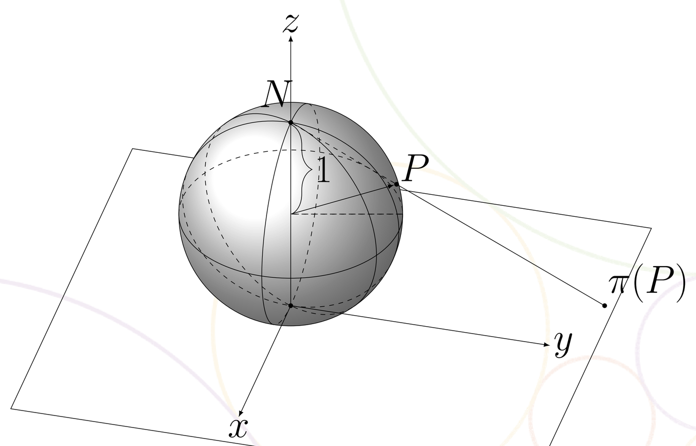
Goal

Draw pictures to better understand the Schwarzian Derivative.

Some Preliminaries

We need to know about the stereographic projection and Möbius transformations.

Definition. *A(n extended) stereographic projection is a function $\pi : S^2 \to \widehat{\mathbb{C}}$ that maps S^2, i.e. the surface of the unit sphere in $\mathbb{R}^3$, to the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (we identify $\mathbb{C}$ with the xy-plane in $\mathbb{R}^3$).*



Let N in S^2 denote the north pole. Each point P in $S^2 - \{N\}$ is mapped to the intersection of the unique line through N and P with the xy-plane, denoted as $\pi(P)$. However, when $P = N$, this procedure fails, because there is no unique line through N and P. Notice that the points near N are mapped to points that have large absolute value, so it makes sense to map N to ∞. So we set $\pi(N) = \infty$.

This produces a bijection from the sphere S^2 to the extended complex numbers $\widehat{\mathbb{C}}$, thereby equipping S^2 with the algebraic structure of $\widehat{\mathbb{C}}$; we call this the Riemann Sphere.

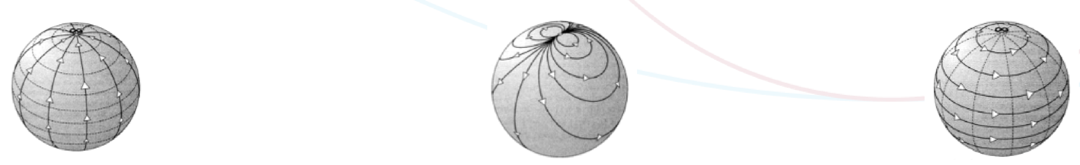
Definition. *A Möbius transformation is a complex function of the form*

$$M(z) = \frac{az+b}{cz+d}$$

where a, b, c, d are complex numbers and $ad - bc \neq 0$. Being careful, we can extend the domain and range of M to the extended complex plane $\widehat{\mathbb{C}}$.

One of the key properties of Möbius transformations is that they preserve circles on the Riemann Sphere.

Here we give three examples of Möbius transformations. Every Möbius transformation can be decomposed as a composition of Möbius transformations similar to these three.



(a) Hyperbolic: $z \mapsto 2z$ (b) Parabolic: $z \mapsto z+1$ (c) Elliptic: $z \mapsto iz$

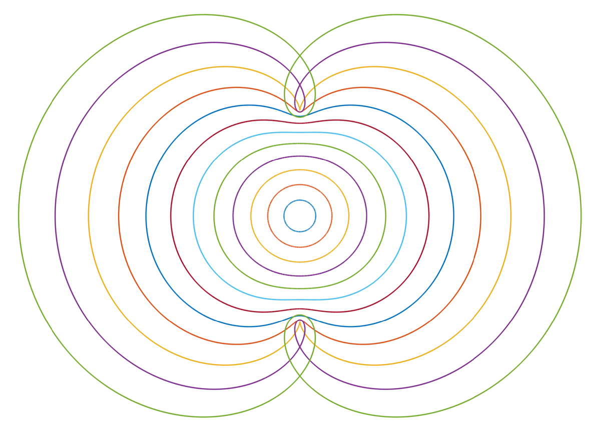
What is the Schwarzian Derivative?

Definition. *Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. The Schwarzian derivative of f is the function*

$$\mathcal{S}(f) = \frac{f'''}{f'} - \frac{3}{2}\Big(\frac{f''}{f'}\Big)^2$$

The Schwarzian measures how much a function deviates from a Möbius transformation; it vanishes for Möbius transformations but gives useful information about other maps. We noted in the previous section that Möbius transformations take circles in the Riemann sphere to other circles; but other functions may distort circles.

Example 1. *Images of concentric circles under the polynomial $f(z) = z + z^3/3$; note the self-intersection.*



Definition. *Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function, and let $s \in \mathbb{C}$. The osculating Möbius transformation of f at s is the unique Möbius transformation $M(f,s)(z)$ that matches f in value, first, and second derivative at s:*

$$M(f,s)(s) = f(s),\ M'(f,s)(s) = f'(s),\ M''(f,s)(s) = f''(s).$$

The Schwarzian derivative is determined by the rate of change of the osculating Möbius transformation as we change s. After some renormalization, $\frac{\partial}{\partial w}M(f,w)$ is a quadratic in $z - w$. The leading coefficient of this quadratic is the Schwarzian derivative.

Why is the Schwarzian derivative important?

The Schwarzian is the unique operator with the following properties.

- Vanishes precisely for Möbius transformations: $\mathcal{S}(f) = 0 \iff f$ is a Möbius transformation
- Invariant under composition with Möbius transformations; if μ is a Möbius transformation, $\mathcal{S}(\mu \circ f) = \mathcal{S}(f)$

It appears in many branches of mathematics, including hypergeometric functions and Teichmüller theory.

Relation to Curvature

Motivation.

While Möbius transformations are the right framework for doing computations, they can be difficult to visualize. On the other hand, it is relatively easy to see curvature–how "bent" one-dimensional curves are. Luckily, curvature is related to the Schwarzian derivative!

Definitions. *Let $\gamma : \mathbb{R} \to \mathbb{C}$ be a smooth parameterized curve. Let T be the unit tangent vector $T(t) = \gamma'(t)/||\gamma'(t)||$.*

- *The curvature of γ at t is defined to be*

$$\kappa(t) = \frac{d}{dt}T(t).$$

(Note that this is invariant under different parameterizations for the curve $\operatorname{im}(\gamma)$.)

- *The osculating circle of γ at t is the unique circle tangent to γ at t with the same curvature; it has radius $r = 1/\kappa$.*

The name osculating circle might remind you of osculating Möbius transformations. The two are related by the following theorem.

Theorem 1. *Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function, let $\gamma : \mathbb{R} \to \mathbb{C}$ be a parameterized circle in $\mathbb{C}$, and let $z_0 = \gamma(t_0)$ be a point on this circle.*
If $\mu = M(f, z_0)$ is the osculating Möbius transformation of f at z_0, then μ sends the circle γ to the osculating circle of the curve $f \circ \gamma$ at the point $f(z_0)$.

To prove this, we use the following lemma relating curvature before and after applying a holomorphic map.

Lemma. *Let $\gamma : \mathbb{R} \to \mathbb{C}$ be a smooth parameterized curve, T be the unit tangent vector $T(t) = \gamma'(t)/||\gamma'(t)||$. Let $p = \gamma(s_0)$ be a point on this curve. Let κ be the curvature of γ at p. Then the curvature of $f \circ \gamma$ at $f(p)$ is given by*

$$\tilde{\kappa} = \frac{1}{|f'(p)|}\Big(\Im\Big[\frac{f''(p)T(s_0)}{f'(p)}\Big] + \kappa\Big).$$

The proof is a computation. □

Proof of Theorem 1. *Note that since $\mu = M(f, z_0)$, we have $\mu(z_0) = f(z_0)$, $\mu'(z_0) = f'(z_0)$, $\mu''(z_0) = f''(z_0)$.*
Since the osculating circle is unique, it suffices to show that $\mu \circ \gamma$ is tangent to $f \circ \gamma$ at $f(z_0)$ and has the same curvature.

- *Tangent: We have $\mu(z_0) = f(z_0)$, so the two curves intersect. Moreover, $\mu'(z_0) = f'(z_0)$, so by the chain rule*

$$(\mu \circ \gamma)'(t_0) = \mu'(z_0)\gamma'(t_0) = f'(z_0)\gamma'(t_0) = (f \circ \gamma)'(t_0).$$

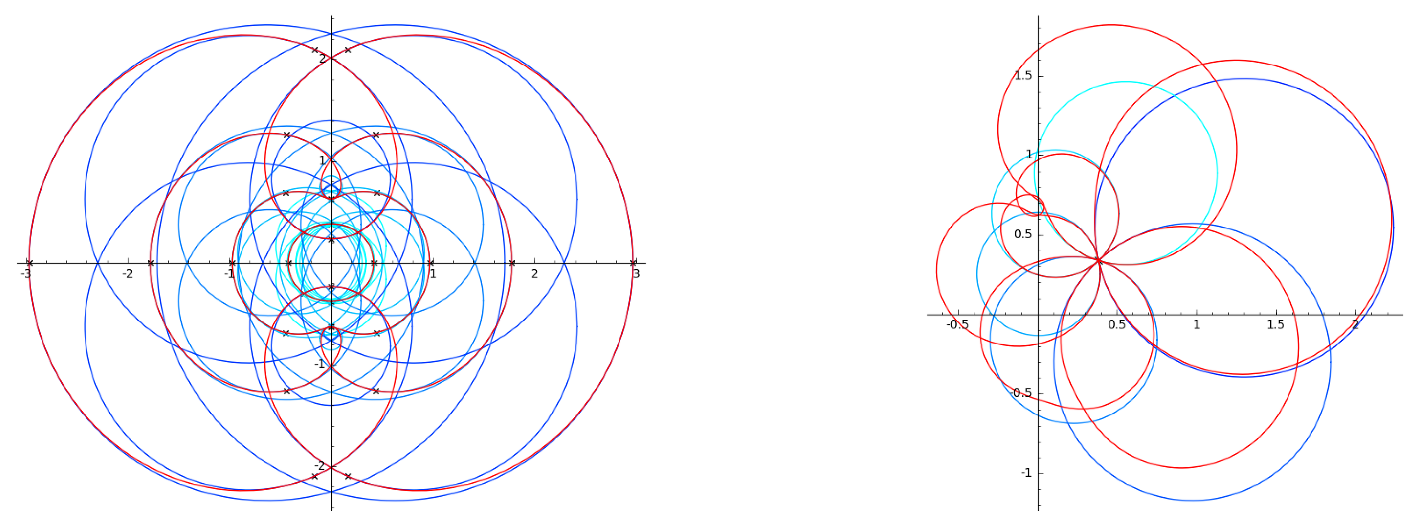
Thus the two curves are tangent.

- *Equal curvature: by the lemma, we can write an expression in terms of derivatives for the curvature of $\mu \circ \gamma$ at $\mu(z_0) = f(z_0)$, and similarly for f. Since the first two derivatives are equal,*

$$\frac{1}{|\mu'(p)|}\Big(\Im\Big[\frac{\mu''(p)T(s_0)}{\mu'(p)}\Big] + \kappa\Big) = \frac{1}{|f'(p)|}\Big(\Im\Big[\frac{f''(p)T(s_0)}{f'(p)}\Big] + \kappa\Big).$$

Thus the circle $\operatorname{im}(\mu \circ \gamma)$ is the osculating circle of $\operatorname{im}(f \circ \gamma)$ at $f(z_0)$. □

By drawing nearby osculating circles to the images of circles under the map f, we can see how the osculating Möbius transformation changes and glean information about the shape of the Schwarzian derivative.



Example 2. *Left: the images of concentric circles under $f(z) = z + z^3/3$ (red), and osculating circles at equidistant points around them (blue). Right: the images under the same map of circles with equal radii through a point (red), and osculating circles at that point (blue).*

Reference

[1] Needham, Tristan. *Visual Complex Analysis*, Oxford University Press, (1997).
[2] Osgood, Brad. (1998). *Old and New on the Schwarzian Derivative.*
[3] Haggerty Patrick, Martínez Granado Dídac, Scott Maxime. *LOG(M): Visualizing the Schwarzian derivative*, online notes `https://www.overleaf.com/read/tvfpfcmqytdp#/41298056/`