Gluing Tetrahedra

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Introduction

Goal & Set Up

Gluing tetrahedra along their faces, one can build various kinds of spaces which locally look like the ordinary 3-space. These are called 3-manifolds, which we aim to explore through this project. Our goals are:

- Understanding 3-dimensional manifolds (spaces) in terms of simpler finite combinatorial data obtained from Regina, a software package for low-dimensional topology.
- Using computational tools to derive certain topological invariants.

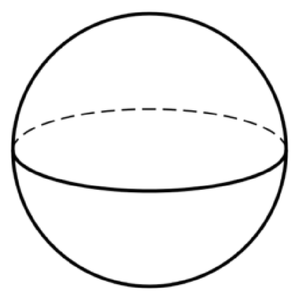
1. Gluing Tetrahedra

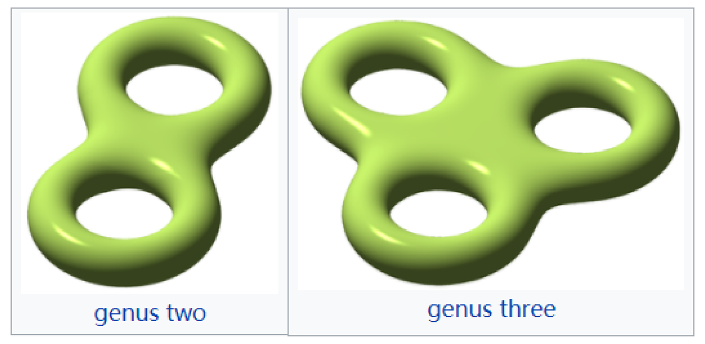
Manifolds

Definition. *An n-manifold is a space that looks locally like $\mathbb{R}^n$.*

Sphere and tori are simple examples of two manifolds. See the picture below.

Example 1 (Two-dimensional manifolds)**.** *Image credit to Wikipedia.*





n-simplex

Definition. *An n-simplex is the convex hull of a set of (n+1) affinely independent points in some Euclidean space of dimension n or higher.*

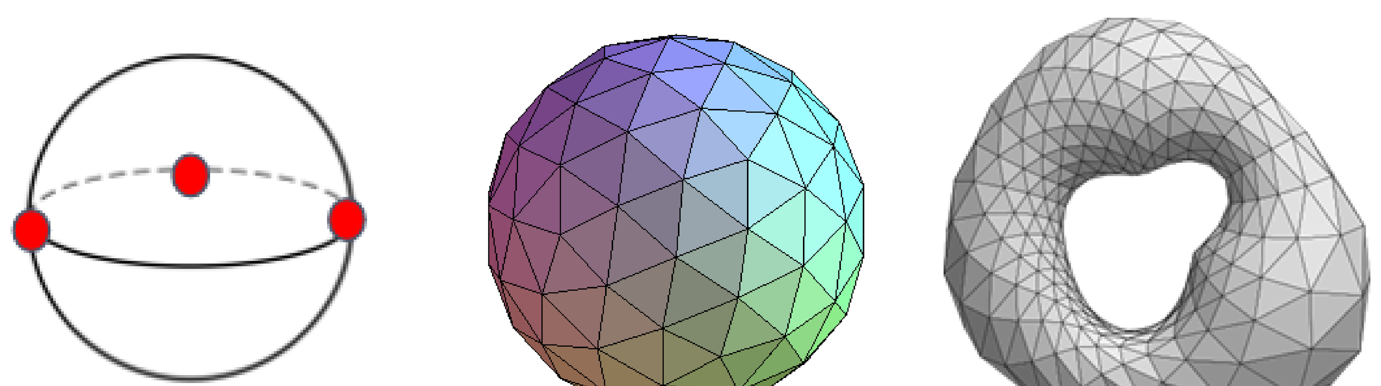
For example, a 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a triangle, a 3-simplex is a tetrahedron.

Triangulation

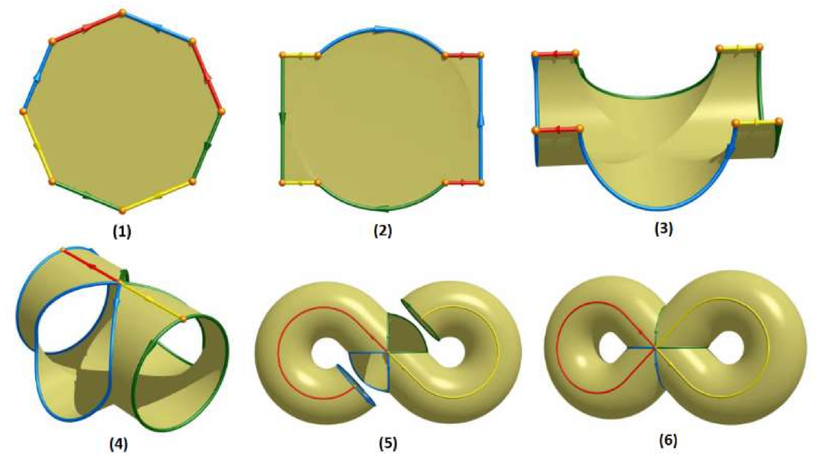
Definition. *Triangulation is a subdivision of a planar object into triangles, and by extension the subdivision of a higher-dimension geometric object into simplices. Triangulations of a three-dimensional manifold would involve subdividing it into tetrahedra ("pyramids" of various shapes and sizes) packed together.*

Note: For a given manifold, we may have many different triangulations. But the topological invariants are determined by the underlying manifolds.

Example 2 (Triangulated two-dimensional manifolds)**.** *Image credit to Fracademic and Wikipedia.*



Example 3 (Gluing a two-holed torus)**.** *We can get a two-holed torus by gluing six triangles. Image credit to the Youtube video "From an octagon to a genus 2 surface - Mathlapse"*

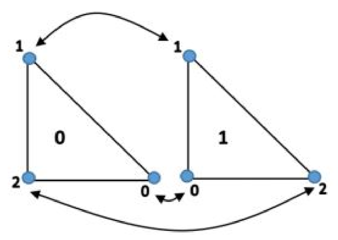


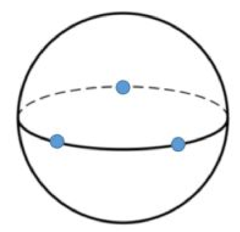
Euler Characteristic

Definition. *The Euler characteristic of a triangulated manifolds is an invariant of the underlying manifold. It is defined by $\chi = \sum_i (-1)^i \alpha_i$, where α_i is the number of i-simplices used to triangulate this space.*

Example 4 (Euler characteristic of S^2)**.** *(Table credit to Regina) With the gluing table generated from Regina, we know the Euler characteristic for S^2 is $\chi = 3 - 3 + 2 = 2$.*

Triangle	Edge 01	Edge 02	Edge 12
0	1(01)	1(02)	1(12)
1	0(01)	0(02)	0(12)





2. Betti Numbers

We wanted to upgrade the Euler characteristic to the Betti numbers, which are topological invariants that provide more information about the manifold. In fact, we can compute Betti numbers from triangulations and then recover the Euler characteristic from the Betti numbers.

Simplicial Complex

Definition. *Suppose M is a d-manifold with triangulation $\mathcal{K}$. We say $\mathcal{K}$ is a simplicial complex if:*
1. The vertices of a fixed simplex are not glued together.
2. Every k-simplex is determined by its vertices. In other words, if two k-simplices have the same set of vertices, then they are in fact the same.

The simplest example of a simplicial complex is a single tetrahedron.

Cochain Complex

Definition. *Given a simplicial complex $\mathcal{K}$, let Y_i be the set of all i-simplices and correspondingly let $C^i(\mathcal{K})$ be the set of all $\mathbb{Q}$-valued functions defined on Y_i.*

For each $y \in Y$, we define its characteristic function $[y] \in C^\bullet(\mathcal{K})$ by

$$[y](y') = \begin{cases} 1, & \text{if } y = y' \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

$C^i(\mathcal{K})$ is a vector space with the basis $[y]$.
Let $r_j(z)$ denote the subset of z obtained by removing the $(j+1)^{th}$ vertex of the simplex.

Definition. *The exterior differentials from $C^i(\mathcal{K})$ to $C^{i+1}(\mathcal{K})$ are defined by*

$$(d[y])(z) = \begin{cases} 1, & \text{if } y = r_j(z) \text{ and j is even} \\ -1, & \text{if } y = r_j(z) \text{ and j is odd} \\ 0, & \text{otherwise} \end{cases} \tag{2.2}$$

Definition. *(Betti numbers) If we denote the differentials from $C^i(\mathcal{K})$ to $C^{i+1}(\mathcal{K})$ by d_i, then the i-th Betti number of a manifold is h^i= nullity(Δ_i)= dim(Ker($d_i^T * d_i + d_{i-1} * d_{i-1}^T$)), where $i = 0, 1, 2, .., n$.*

Since we deal with 3-manifold, we only have Δ_0 through Δ_3. When $i = 0$, it degenerates to h^0= dim(Ker($d_0^T * d_0$)) and when $i = 3$, it degenerates to h^3= dim(Ker($d_2 * d_2^T$)).

Methods and Results

Computing Euler Characteristic from Regina

The first thing we deal with is Euler characteristic. Given source code from Regina, we wrote a program to find the equivalence classes of vertices, edges, faces, tetrahedra, and used the formula to get the Euler characteristic.

```
General statistics of the triangulation:
            vertices        edges        faces        tetrahedra
Count              1            3            4                 2
The Euler Characteristic is 0.

Lists of equivalent classes(separated by --):
Equivalent classes of vertices:
0(0) | 0(1) | 0(2) | 0(3) | 1(0) | 1(1) | 1(2) | 1(3) |
------------------------------------------------------------------------------

Equivalent classes of edges:
0(01) | 0(02) | 0(03) | 0(12) | 0(13) | 1(01) | 1(02) | 1(03) | 1(12) | 1(13) |
------------------------------------------------------------------------------
0(23) |
------------------------------------------------------------------------------
1(23) |
------------------------------------------------------------------------------

Equivalent classes of faces:
0(012) | 1(012) |
------------------------------------------------------------------------------
0(013) | 1(013) |
------------------------------------------------------------------------------
0(023) | 0(123) |
------------------------------------------------------------------------------
1(023) | 1(123) |
------------------------------------------------------------------------------
```

Besides, we also verified that Euler characteristic is a topological invariant. If we use two different triangulations, we still get the same Euler characteristic. In other words, Euler characteristic is independent of the triangulation. The example below is outputs for two different triangulations of S^3. The first triangulation is "3-sphere (dual to Bing's house)" in Regina and the second one is "3-sphere (minimal)" after one barycentric subdivision.

```
General statistics of the triangulation:
            vertices        edges        faces        tetrahedra
Count              1            3            4                 2
The Euler Characteristic is 0.

General statistics of the triangulation:
            vertices        edges        faces        tetrahedra
Count              6           30           48                24
The Euler Characteristic is 0.
```

Computing the Matrix of the Differential

d is actually a linear map from $C^i(\mathcal{K})$ to $C^{i+1}(\mathcal{K})$. So for a fixed basis, we can represent d by a matrix. However, life is not always good. Some triangulations may not us give a simplicial complex. To make it a simplicial complex, we can do barycentric subdivision twice. After one subdivision, the number of tetrahedra is 24 times as many as that of the original triangulation. So the matrix we compute here is quite big. For example, d_1 corresponding to L(11,1) is a 9216*5444 matrix.

Computing Betti Numbers

We started to compute Betti numbers by the definition in the above section, but it took too long to get the result. So we turned to another formula:h^i=nullity(d_i)-rank(d_{i-1}). With this method, we successfully computed the Betti numbers of some manifolds in a relatively short time. Here are some results we got from our code.

Manifold	#tetrahedra (before barycentric subdivision)	#tetrahedra (in simplicial complex)	Betti number	Run time
S^3	1	576	1001	10 s
S^2 x S^1	2	1152	1111	1m 57s
Poincaré homology sphere	5	2880	1001	25m 44s
L(10,1)	7	4032	1001	1h 10m 19s
T x I /[2,1|1,1]	7	4032	1111	1h 10m 07s
T x I /[3,2|1,1]	8	4608	1111	1h 44m 32s

Recovering Euler Characteristic from Betti Numbers

In fact, the Euler characteristic and Betti numbers can be linked in the following formula:

$$\chi = \Delta_0 - \Delta_1 + \Delta_2 - \Delta_3 + ... = \sum_{i=0}^{n} (-1)^i \Delta_i$$

So from the table above, we can recover Euler characteristic of these manifolds by taking the alternating sum of Betti numbers we got. For example, the Euler characteristic of S^3 is $\chi = 1 - 0 + 0 - 1 = 0$. This result coincides with what we got in the previous part!

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