Visualizing The Unit Ball of The AGY Norm

Vignesh Jagathese    Yichen Liu    Jacob Shulkin

University of Michigan

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Motivation

- We wish to understand the Avila-Gouezel-Yoccoz (AGY) norm defined on a vector space attached to a surface built out of triangles.
- This surface built out of triangles is called a Translation Surface.
- The goal of this project is to visualize two-dimensional slices of the unit ball under this norm.
- Our primarily goal for the first half of the semester is build a program to derive and visualize connections between points in this translation surface, known as Saddle Connections, which will help us visualize the unit ball.
Defining a Translation Surface

- A Translation Surface can be thought of as a collection of triangles with edges identified.
- The identifications follow some basic properties:
  - every edge is identified with exactly one other (parallel) edge
  - Every edge has a fixed orientation (with respect to the triangle, edges must be orientated counter clockwise)
  - Only edges with opposite orientation can be identified.
- Any two triangles will be regarded as the same if there is a translation between them.
- The vertices in this surface are called singularities.
  - Note: Not every vertex is a singularity, but here we treat every vertex as one. In reality, to be a singularity, the sum of the angles around the vertex must be $> 2\pi$. 


Easy Identification Example

Figure: This is how we model a torus in $\mathbb{R}^3$. 
Any regular polygon can be triangulated and turned into a translation surface.

Once the triangulation is defined, by our restrictions on orientation, we can induce one as follows:
**Figure:** Given the natural counter clockwise orientation of the edges of the polygon, the orientation of the internal lines can be inferred.
Not a Translation Surface

**Figure:** This triangulation corresponds to $S^2$. While a surface, it is not a translation surface, since the identified edges have the same orientation.
(Also) Not a Translation surface

**Figure:** While at first glance this may seem similar to the torus case from earlier, this is not a translation surface because the identified edges are not parallel.
Saddle Connections

- A **Saddle Connection** is a straight line joining one singularity of a triangle to another.
- While a saddle connection can go through as many triangles as it wants, it cannot go through other singularities.
- Each saddle connection has some fixed length $L$. 

(Not) Saddle Connections

- The connection in the first figure fails to be a saddle connection because it does not terminate at a singularity.
  - If the line continued on, it could possibly be a saddle connection. However, this has a fixed length $L$ such that it terminates at the edge, so it cannot be a saddle connection.

- The connection in the second figure fails to be a saddle connection because it is not a straight line connection between two singularities.
Saddle Connections

- Not all saddle connections are like this:

- While this is in fact a saddle connection, the magic comes from the edge identifications.
Saddle Connections

- How many saddle connections are there on the torus?
  - *Hint:* The length can be as long as you like.
Saddle Connections

- There are actually infinitely many saddle connections!
- Followup question: For a fixed length $L$, how many saddle connections are there with length at most $L$?
- This is a more nuanced question, and one that we are writing code to solve for.
Given $L \in \mathbb{R}_{>0}$ and $S$ a surface built out of triangles, we want to find all the saddle connections with the length smaller than $L$.

A saddle connection can be represented as a vector emanating from a singularity and ending at a singularity.

We first find the saddle connections originating from a singularity $p \in S$ going through some triangle of which $p$ is a vertex.

We loop this through all such possible triangles and apply this approach to each singularity.

We then plot each of these saddle connections as vectors in $\mathbb{R}^2$. 

Computing Saddle Connections within a Given Length: The Algorithm

- Starting at a singularity \( p \in S \) of a triangle \( T \) we initialize a ”window” as the two edges of \( T \) adjacent to \( p \) as vectors emanating away from \( p \).
- The window gives bounds for where a possible saddle connection may be.
- We move into the next triangle extending the window vectors until they hit an edge or vertex of the triangle.
- If the vectors of the window hit the same edge there is no saddle connection found yet and we go to the next triangle.
- If the vectors of the window hit different edges we have found a saddle connection, there are now two different windows as we continue forward through the surface.
- Repeat until all saddle connections shorter than \( L \) have been found.
References


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