Visualizing The Unit Ball of The AGY Norm

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Motivation

- We wish to understand the Avila-Gouezel-Yoccoz (AGY) norm defined on a vector space attached to a surface built out of triangles.
- This surface built out of triangles is called a **Translation Surface**.
- The goal of this project is to visualize two-dimensional slices of the unit ball under this norm.
- Our primarily goal for the first half of the semester is build a program to derive and visualize connections between points in this translation surface, known as **Saddle Connections**, which will help us visualize the unit ball.

Defining a Translation Surface

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- A **Translation Surface** can be thought of as a collection of triangles with edges identified.
- The identifications follow some basic properties:
  - every edge is identified with exactly one other (parallel) edge
  - Every edge has a fixed orientation (with respect to the triangle, edges must be orientated counter clockwise)
  - Only edges with opposite orientation can be identified.
- Any two triangles will be regarded as the same if there is a translation between them.
- The vertices in this surface are called **singularities**.
  - *Note*: Not every vertex is a singularity, but here we treat every vertex as one. In reality, to be a singularity, the sum of the angles around the vertex must be $> 2\pi$.

Easy Identification Example

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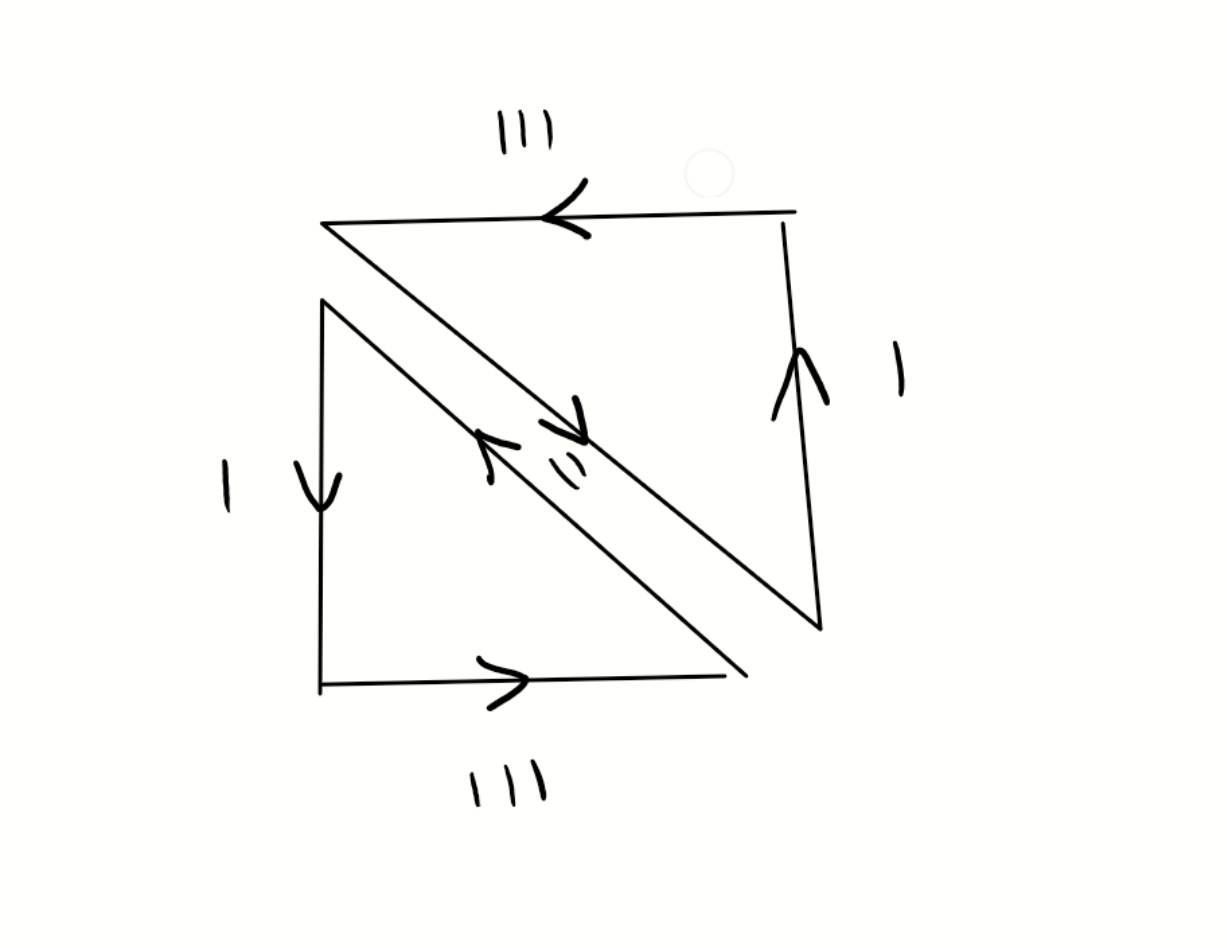

Figure: This is how we model a torus in $\mathbb{R}^3$.

Translation Surfaces in the Wild

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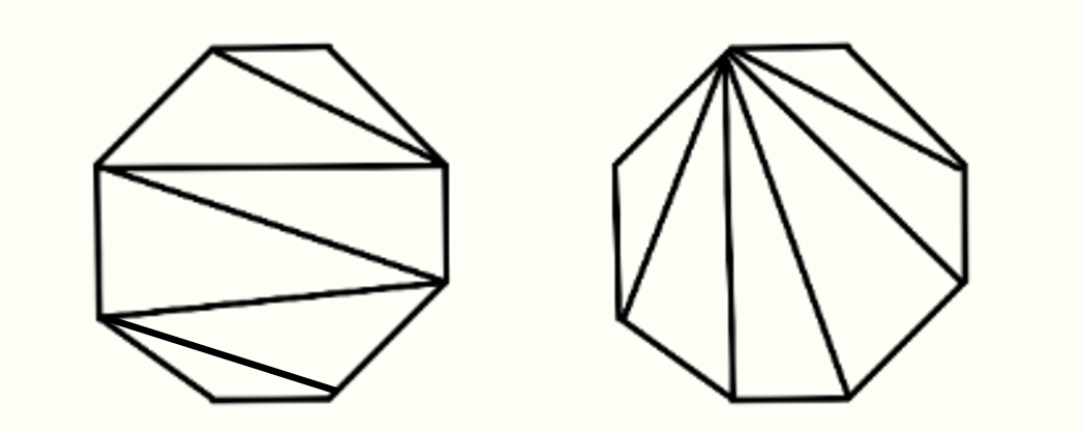

- Any regular polygon can be triangulated and turned into a translation surface.
- Once the triangulation is defined, by our restrictions on orientation, we can induce one as follows:

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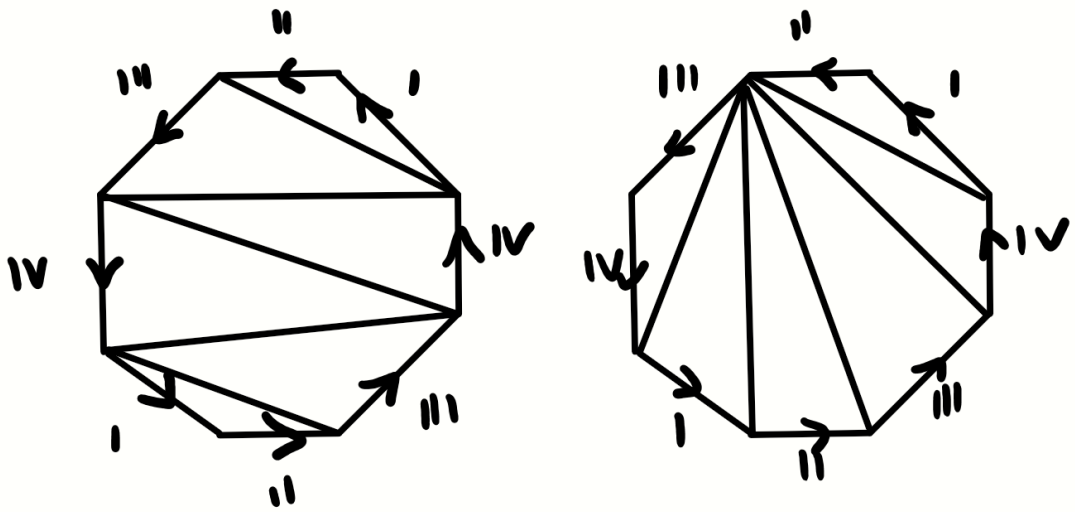

Figure: Given the natural counter clockwise orientation of the edges of the polygon, the orientation of the internal lines can be inferred.

Not a Translation Surface

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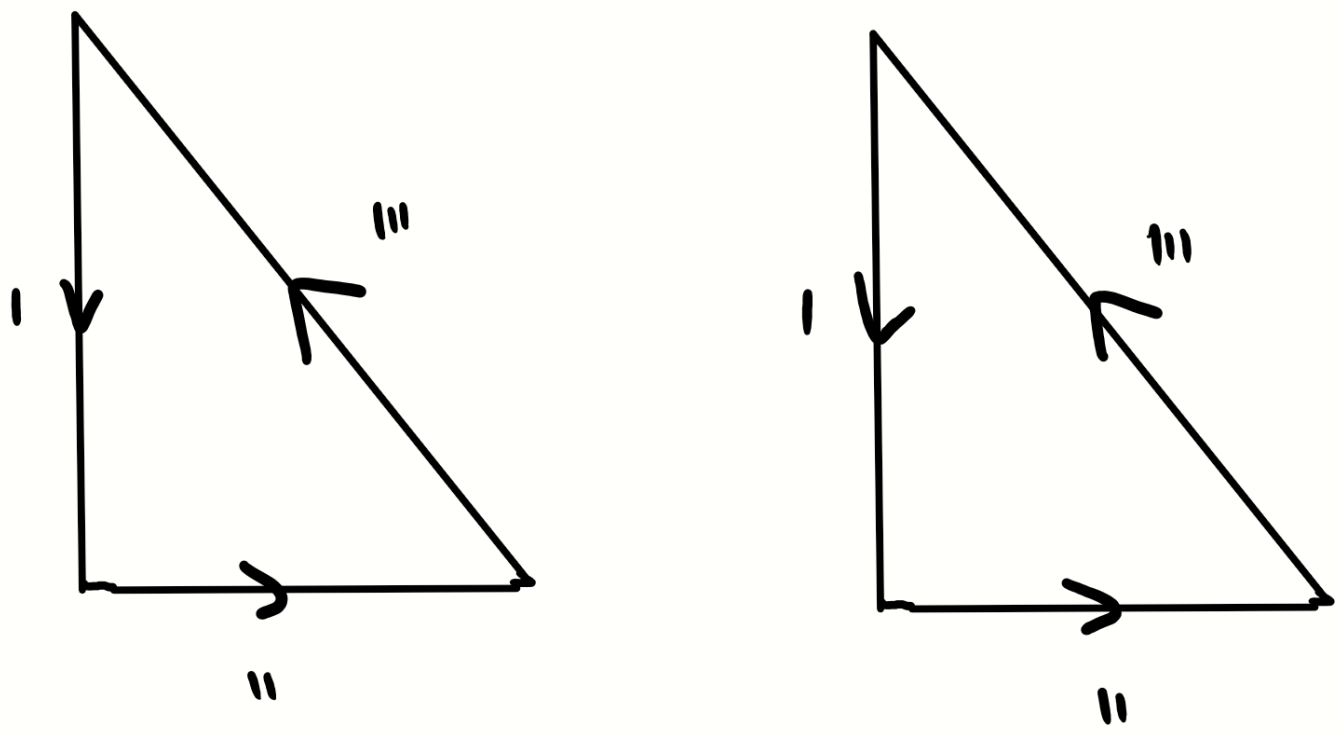

Figure: This triangulation corresponds to S^2. While a surface, it is not a translation surface, since the identified edges have the same orientation.

(Also) Not a Translation surface

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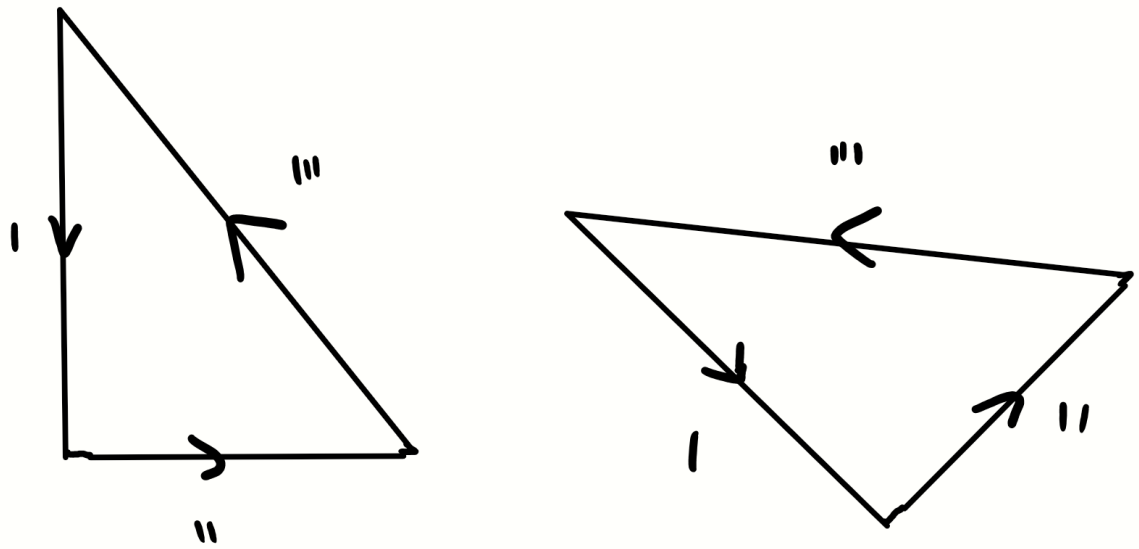

Figure: While at first glance this may seem similar to the torus case from earlier, this is not a translation surface because the identified edges are not parallel.

Saddle Connections

- A **Saddle Connection** is a straight line joining one singularity of a triangle to another.
- While a saddle connection can go through as many triangles as it wants, it cannot go through other singularities.
- Each saddle connection has some fixed length L.

(Not) Saddle Connections

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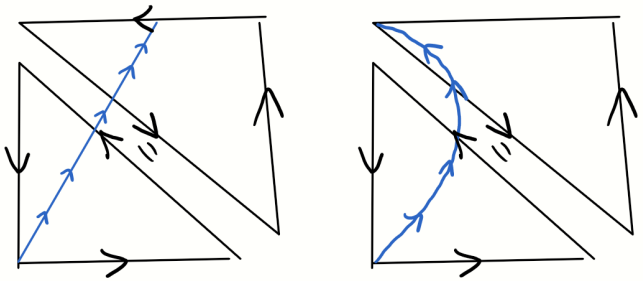

- The connection in the first figure fails to be a saddle connection because it does not terminate at a singularity.
  - If the line continued on, it could possibly be a saddle connection. However, this has a fixed length L such that it terminates at the edge, so it cannot be a saddle connection.
- The connection in the second figure fails to be a saddle connection because it is not a straight line connection between two singularities.

Saddle Connections

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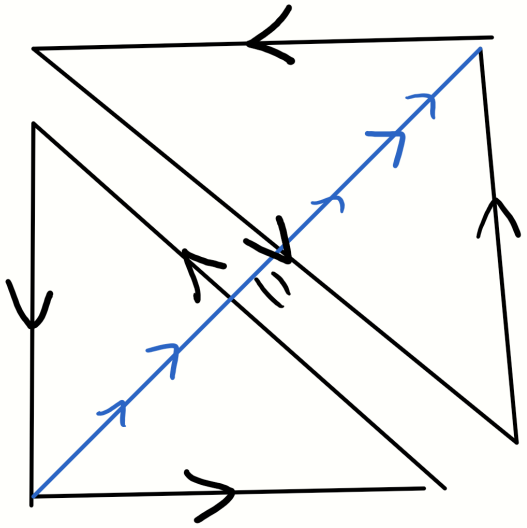
- Not all saddle connections are like this:



- While this is in fact a saddle connection, the magic comes from the edge identifications.

Saddle Connections

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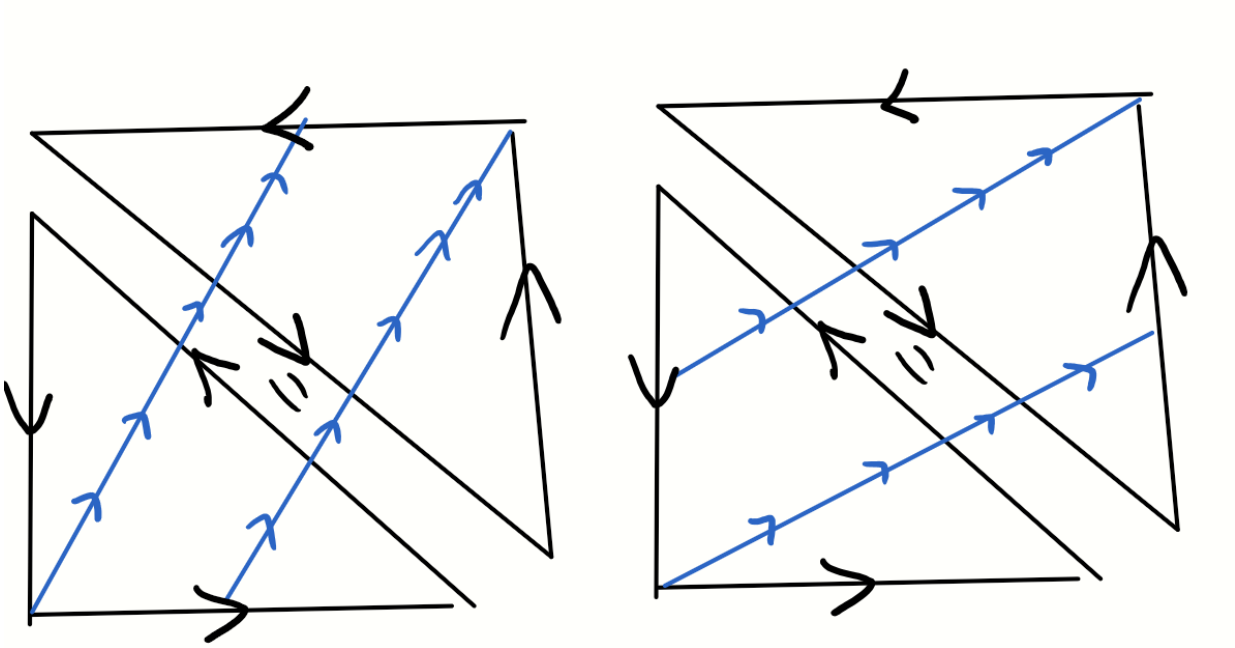

- How many saddle connections are there on the torus?
  - *Hint:* The length can be as long as you like.

Saddle Connections

- There are actually infinitely many saddle connections!
- Followup question: For a fixed length L, how many saddle connections are there with length at most L?
- This is a more nuanced question, and one that we are writing code to solve for.

Computing Saddle Connections within a Given Length: The Approach

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- Given $L \in \mathbb{R}_{>0}$ and S a surface built out of triangles, we want to find all the saddle connections with the length smaller than L.
- A saddle connection can be represented as a vector emanating from a singularity and ending at a singularity.
- We first find the saddle connections originating from a singularity $p \in S$ going through some triangle of which p is a vertex.
- We loop this through all such possible triangles and apply this approach to each singularity.
- We then plot each of these saddle connections as vectors in $\mathbb{R}^2$.

Computing Saddle Connections within a Given Length: The Algorithm

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- Starting at a singularity $p \in S$ of a triangle T we initialize a "window" as the two edges of T adjacent to p as vectors emanating away from p.
- The window gives bounds for where a possible saddle connection may be.
- We move into the next triangle extending the window vectors until they hit an edge or vertex of the triangle.
- If the vectors of the window hit the same edge there is no saddle connection found yet and we go to the next triangle.
- If the vectors of the window hit different edges we have found a saddle connection, there are now two different windows as we continue forward through the surface.
- Repeat until all saddle connections shorter than L have been found.

References

[1] Alex Wright. *Visualizing the Unit Ball of the AGY Norm*. 2018.

[2] Lab of Geometry at Michigan. LoG(M) Beamer Template. *University of Michigan Department of Mathematics*. 2018.

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