Piercing \( d \)-intervals

C. Puritz, B. Sakelaris, W. Warner, Y. Chen, S. Zerbib
Laboratory of Geometry at Michigan

Introduction

The following is a well-known theorem:

**Theorem 1.** Let \( \mathcal{F} \) be a finite family of intervals in \( \mathbb{R} \) such that every two intervals in \( \mathcal{F} \) intersect. Then there exists a point \( p \in \mathbb{R} \) that intersects every interval of \( \mathcal{F} \). Such a point is said to pierce every element in \( \mathcal{F} \).

**Example 1.** A family of four intervals, two of them intersect. Note that the leftmost end point in the family intersects all the intervals in the family.

Our project focuses on a natural generalization of this problem: instead of looking at families of intervals, we consider now families of \( \mathcal{F} \)-intervals, and given such a family with some local intersection properties, we want to bound from above the minimal number of points needed to pierce (that is, to intersect) every element in the family.

Previous Results

- In 1997, Tardos [6] and Kaiser [2] proved that if a family \( \mathcal{F} \) of \( d \)-intervals satisfies the \((p, p)\) property then \( \tau(\mathcal{F}) = (d^2 - d + 1)p \).
- In 2001, Matoušek [5] constructed an example showing that Kaiser result is not far from being tight: for any \( d \geq 1 \) there exists a family of \( d \)-intervals satisfying the \((d, d)\) property with \( \tau(\mathcal{F}) \geq \frac{d}{d-1} \) for some constant \( c \).
- In 1999, Kaiser and Rabinovich [3] proved that if a family \( \mathcal{F} \) of \( d \)-intervals satisfies the \((p, p)\) property for \( p = (\log(1 + d))^2 \) then \( \tau(\mathcal{F}) \leq 1 \).
- Recently, Zerbib [7] proved that if a family \( \mathcal{F} \) of \( d \)-intervals satisfies the \((p, p)\) property then \( \tau(\mathcal{F}) \leq \frac{d^2}{d+1} \).

Our Results

We generalize the method of Kaiser and Rabinovich [3], to slightly improve the known bound on the piercing number of families of \( \mathcal{F} \)-intervals which satisfy the \((p, p)\) property. We prove:

**Theorem 2.** Every family \( \mathcal{F} \) of \( d \)-intervals that satisfy the \((p, p)\)-property has \( \tau(\mathcal{F}) \leq \frac{d^2}{d+1} \).

Future Research Directions

Many questions in this area are still open.

- Find a tight upper bound on the piercing numbers of families of \( \mathcal{F} \)-intervals satisfying the \((p, p)\) property.

This question is open even in the \( d=2 \) case: there are no known examples with piercing number larger than \( \frac{d^2}{d} \), but the best known upper bound is \( \frac{d^2}{d+1} \). For every \( p \), find infinitely many examples of families of \( d \)-intervals satisfying the \((p, p)\) property with piercing numbers at least \( \ell \) (or \( d-1 \)). Characterize such examples.

- Prove (or disprove) Conjecture 1 and Conjecture 2.

References