Piercing d-intervals

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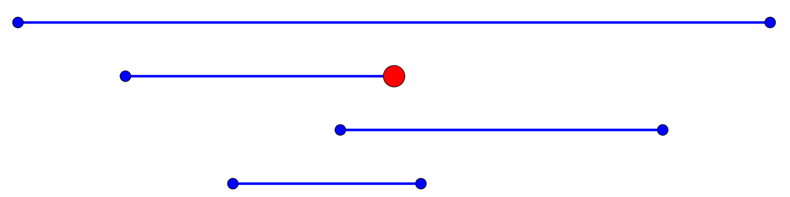
Introduction

The following is a well-known theorem:

Theorem 1. Let $\mathcal{F}$ be a finite family of intervals in $\mathbb{R}$ such that every two intervals in $\mathcal{F}$ intersect. Then there exists a point $p \in \mathbb{R}$ that intersect every interval of $\mathcal{F}$. Such a point is said to pierce every element in $\mathcal{F}$.

Example 1. A family of four intervals, every two of them intersect. Note that the leftmost right end point in the family intersects all the intervals in the family.

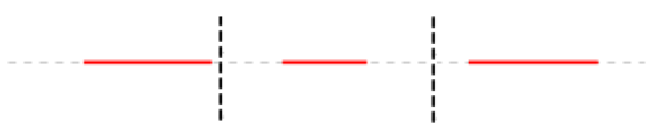


Our project focuses on a natural generalization of this problem: instead of looking at families of intervals, we consider now families of d-intervals, and given such a family with some local intersection properties, we want to bound from above the minimal number of points needed to pierce (that is, to intersect) every element in the family.

Definition

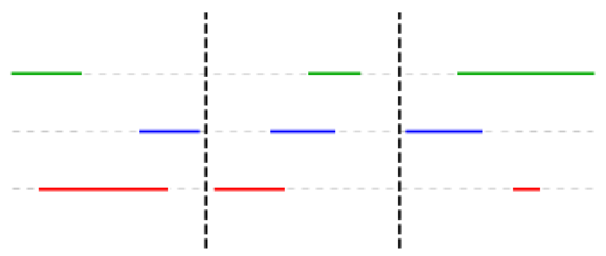
Definition 1. A d-interval is a union of d intervals, one on each of d disjoint underlying segments of $\mathbb{R}$. Namely, it is a set $J = \bigcup_{i=1}^{d} I_i$, where I_i is a closed non-empty interval in the segment $(i-1, i)$.

Example 2. A 3-interval.



Definition 2. Given integers $p \geq q \geq 2$, a family of sets $\mathcal{F}$ is said to satisfy the (p, q) property if in every p members of $\mathcal{F}$ some q intersect. In particular, a family of d-intervals satisfies the (p, p) property if every p d-intervals in the family intersect at a point.

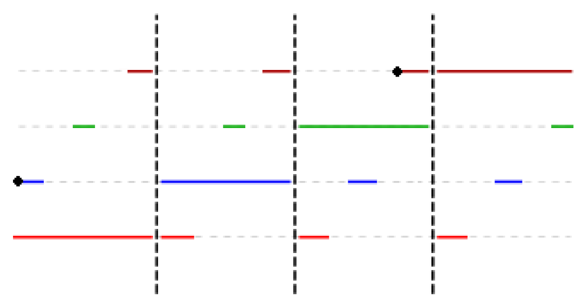
Example 3. A family of 3-intervals which satisfies the $(2, 2)$ property but does not satisfy the $(3, 3)$ property.



Definition 3. The piercing number of a family $\mathcal{F}$ of sets in $\mathbb{R}$, denoted by $\tau(\mathcal{F})$, is the minimum number of points in $\mathbb{R}$ needed to intersect every set in $\mathcal{F}$.

Example 4. A family of p d-intervals that satisfies the (p, p) property has $\tau(\mathcal{F}) = 1$.

Example 5. A family of 3-intervals with $\tau(\mathcal{F}) = 2$.



The conjecture that inspired our project was raised by Björner, Matoušek, and Ziegler [1] in 2014:

Conjecture 1 (Björner, Matoušek, and Ziegler [1]). *There exists an absolute constant p, such that any family of d-invervals $\mathcal{F}$ satisfying the (p, p)-property has $\tau(F) \leq d$.*

Previous Results

- In 1997, Tardos [6] and Kaiser [2] proved that if a family $\mathcal{F}$ of d-intervals satisfies the $(p, 2)$ property then $\tau(\mathcal{F}) \leq (d^2 - d + 1)(p - 1)$.
- In 2001, Matoušek [5] constructed an example showing that Kaiser result is not far from being tight: for any $d \geq 1$ there exists a family of d-intervals satisfying the $(2, 2)$-property with $\tau(F) \geq c\frac{d^2}{\log(d)}$ for some constant c.
- In 1999, Kaiser and Rabinovich [3] proved that if a family $\mathcal{F}$ of d-intervals satisfies the (p, p) for $p = \lceil (log(d + 2)) \rceil$ then $\tau(\mathcal{F}) \leq d$.
- Recently Zerbib [7] proved that if a family $\mathcal{F}$ of d-intervals satisfies the (p, p)-property then $\tau(\mathcal{F}) < p^{\frac{1}{p-1}} d^{\frac{p}{p-1}} + d$.

Our Results

We generalize the method of Kaiser and Rabinovich [3], to slightly improve the known bound on the piercing number of families of d-intervals which satisfy the (p, p) property. We prove:

Theorem 2. Every family $\mathcal{F}$ of d-intervals that satisfy the (p, p)-property has $\tau(\mathcal{F}) \leq d^{\frac{p}{p-1}}$.

To prove this theorem, we first establish a lemma regarding balanced d-partite hypergraphs.

Definition 4. A d-partite hypergraph H is of a pair of sets (V, E), where V is called the vertex set of H, and it is the disjoint union of d vertex sets $V = V_i \cup \cdots \cup V_d$. The set E is called the edge set of H, and each element of E is a subset e of size d of V, where $|e \cap V_i| = 1$ for all i. For every subset $S \subset V$, denote by χ_S the characteristic vector of S. The hypergraph H is said to be balanced if there exists non-negative weights α_e, such that $\sum_{e \in E} \alpha_e \chi_e = \chi_V$.

Lemma 1. Let $H = (V, E)$ be a balanced d-partite hypergraph on vertex set $V = V_i \cup \cdots \cup V_d$, where $|V_i| = t$ for all i. If every $\ell + 1$ edges of H have at least m vertices in common, then every ℓ edges of H have at least $tm + 1$ vertices in common.

Proof. Assume that there exists a set A of ℓ edges in H such that $|\bigcap A| \leq tm$. Write $C = \bigcap A$, and let $I \subset [d]$ be the set of indices i such that $C \cap V_i \neq \emptyset$. Write $\bar{C} = \left(\bigcup_{i \in I} V_i\right) \setminus C$. Since H is d-partite and $|V_i| = t$ for all i, we have $|\bar{C}| = (t-1)|C|$. On the other hand, for every edge $e \notin A$, the set $A \cup \{e\}$ have at least m vertices in common. Since H is balanced we have

$$|C| \geq |C| \sum_{e \in A} \alpha_e + m \sum_{e \notin A} \alpha_e > m \sum_{e \notin A} \alpha_e.$$

Therefore,

$$|\bar{C}| \leq (|C| - m) \sum_{e \notin A} \alpha_e \leq (t-1)m \sum_{e \notin A} \alpha_e < (t-1)|C|,$$

a contradiction. □

We are now ready to prove Theorem 2.

Proof of Theorem 2. Suppose $\tau(\mathcal{F}) > d^{\frac{p}{p-1}}$. Then any set consisting of $d \cdot \lfloor d^{\frac{1}{p-1}} \rfloor$ points, $\lfloor d^{\frac{1}{p-1}} \rfloor$ of them on each one of the d underlying segments of $\mathcal{F}$, do not pierce $\mathcal{F}$. By a topological theorem due to Komiya [4], this implies that there exists a balanced d-partite hypergraph H on vertex set $V = V_i \cup \cdots \cup V_d$, such that $|V_i| = \lfloor d^{\frac{1}{p-1}} \rfloor + 1 > d^{\frac{1}{p-1}}$ for all i, and H satisfies the (p, p) property.

Write $t = \lfloor d^{\frac{1}{p-1}} \rfloor + 1$. By the (p, p) property of H, every p edges of H intersect at at least one vertex. Applying the lemma above $p - 1$ times, we obtain that every edge of H intersects itself at more than $t^{p-1} > d^{\frac{p-1}{p-1}} = d$ vertices. But this contradicts the fact that every edge of H contains exactly d vertices. □

Numerical Experiments

Inspired by Conjecture 1, we wanted to use computational power in order to generate a large database of examples of families of d-intervals satisfying the (p, p) property. Our motivation was either to find an example disproving Conjecture 1, or to characterize the structure of such families of 3-intervals with large piercing number.

To this end, we developed a computer program in C++, which randomly generates families of separated d-intervals with some (p, p)-property, for specified positive integers p and d, and then calculates the piercing number of the family. The programs generates random d-intervals across a specified partition of the unit interval. This continues on, with a newly generated d-interval being added only if the family still satisfied the (p, p) property with the new d-interval. The piercing number is calculated by finding the smallest possible subset of endpoints of all d-intervals of a family such that at least one endpoint pierced each d-interval. The program was run on the Flux high-performance computing cluster, which is maintained by the ARC-TS department at the University of Michigan.

Results: For computational limitation reasons, we focused mainly on generating data of families of d-intervals satisfying the $(3, 3)$ property where $d < 30$. Our goal was to find a family with $\tau \geq d$. Examples with $\tau = d$ would have provided helpful stepping stones for proving Conjecture 1, whereas an example with $\tau > d$ would have disproved this conjecture.

Our program failed to generate an example with $\tau > 2$. We ran several million tests with the family size ranging from 10 up to 500. All the results we have got so far generated a family with $\tau \leq 2$. This might suggests that the following stronger version of Conjecture 1, for the the case $d < 30$:

Conjecture 2. *A family of d-invervals $\mathcal{F}$ (where $d < 30$) satisfying the $(3, 3)$-property has $\tau(F) \leq cd$ for some $c < 1$.*

Future Research Directions

Many questions in this area are still open.

- Find a tight upper bound on the piercing numbers of families of d-intervals satisfying the (p, p) property.
  This question is open even in the $p = 2$ case: there are no known examples with piercing number larger than $O(d^2/\log(d))$, but the best known upper bound is $d^2 - d$.
- For every p, find infinitely many examples of families of d-intervals satisfying the (p, p) with piercing numbers at least d (or $d - 1$). Characterize such examples.
- Prove (or disprove) Conjecture 1 and Conjecture 2.

References

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