The Chromatic Number of Flip Graphs

John Paul Koenig, Sanjana Kolisetty, Zihui Qi, Francesca Gandini, Nicholas Vlamis

Laboratory of Geometry at Michigan

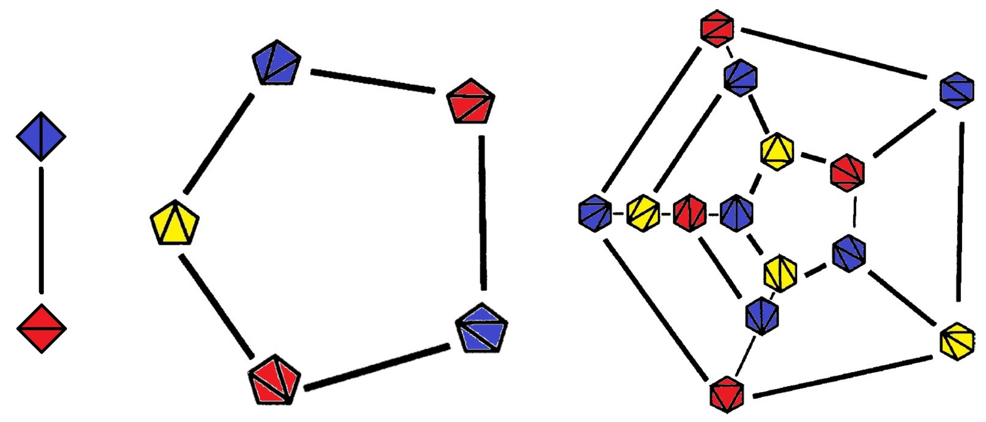
LOG(M)

Introduction

Goal

To investigate the chromatic number of the flip graph associated with a convex n-sided polygon. We want to understand how the chromatic number changes as the number of sides of the n-gon increases. As the number of sides goes to infinity does the chromatic number go to infinity?

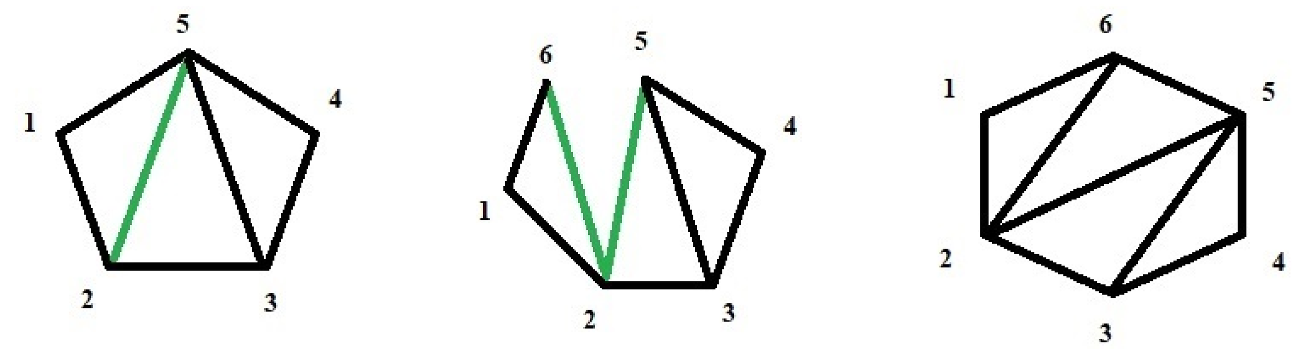
Definition. *A **flip graph** F_n is associated to a finite-sided convex n-gon P living in the plane. Given the polygon P, the vertices of the flip graph associated to P correspond to a triangulation of P (i.e. a choice of disjoint diagonals that decompose P into triangles). Two vertices are connected by an edge if the associated triangulations differ by a **flip** (i.e. if one triangulation can be obtained from the other by changing a single diagonal).*



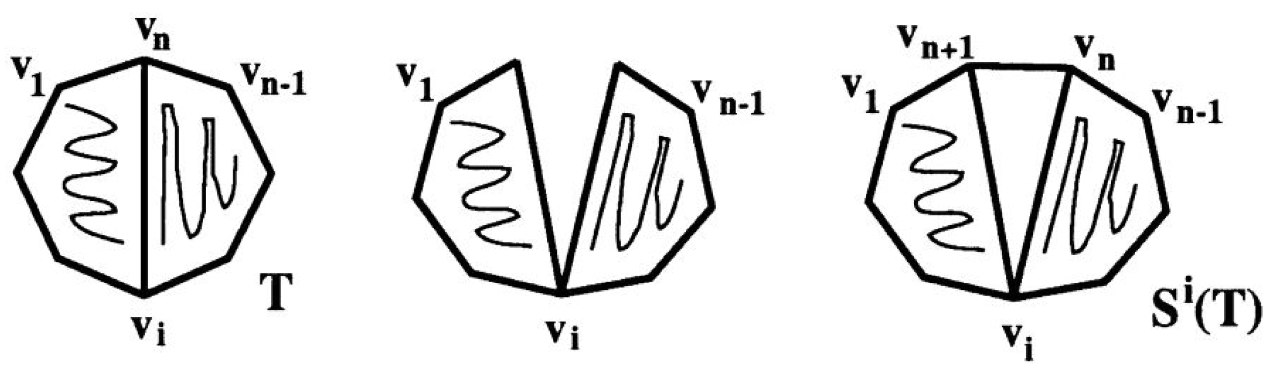
Colored F_4,F_5,F_6 [1]

Constructing Flip Graphs

In the first figure we construct a triangulation of the hexagon from a triangulation of the pentagon. The general procedure for the n-gon is shown in the second figure.



Constructing a hexagon child from a pentagon parent



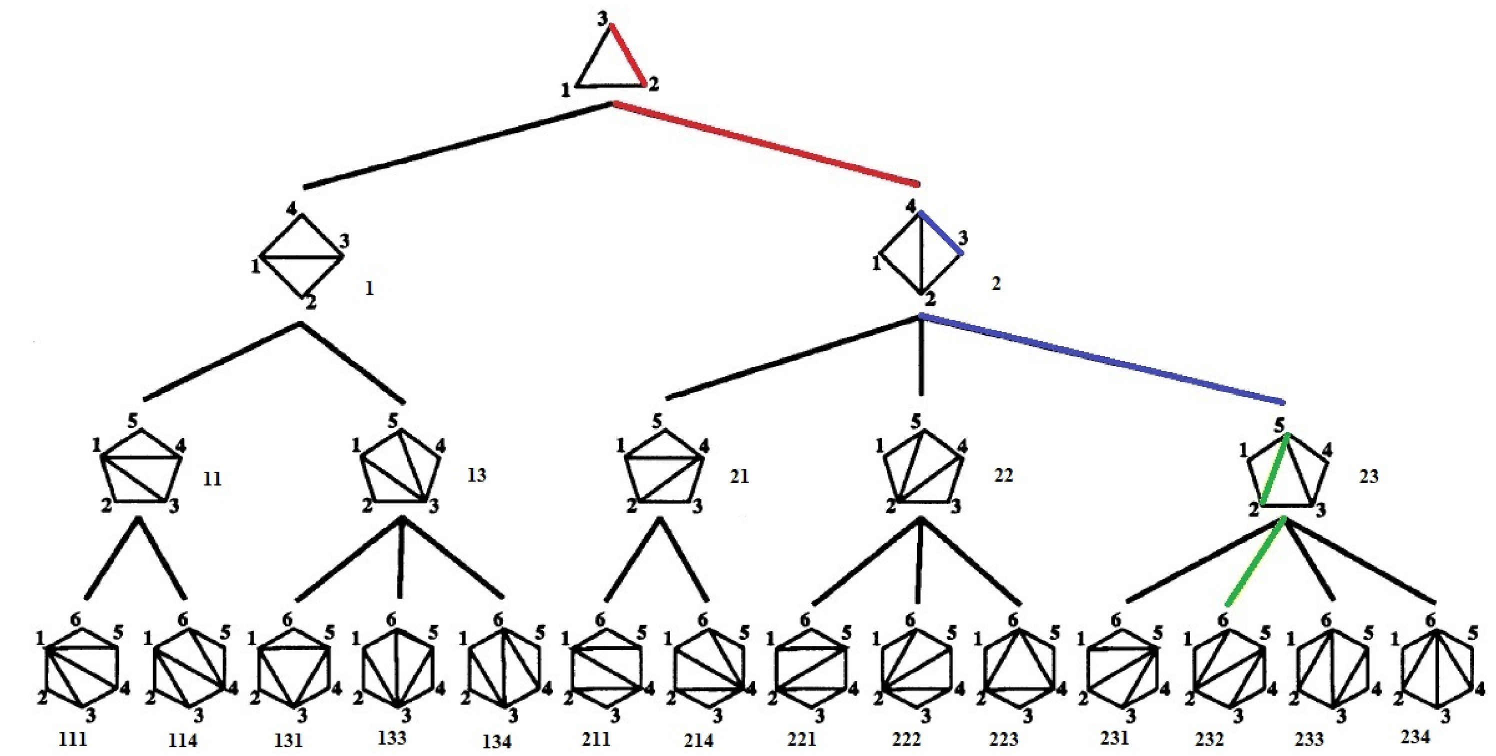
Construction of the child $S^i(T)$ of T [1]

Theorem 1: (Constructing edges of a Flip Graph[1])

Let T and T' be triangulations of an n-gon. Let $S^i(T)$ define the *child* operation on a triangulation T at vertex i. Then, $S^i(T) \sim S^j(T') \iff$

1. $T = T'$ and $\delta_{k,n} \notin T \; \forall \; i < k < j$ *or*
2. $T \sim T'$ and $i = j$ and $\delta_{i,n} \in T \cap T'$

Next, we illustrate a choice of labeling for the vertices of the flip graph that records the child operations needed to obtain them. The labeling is given by diagonal $\delta_{i,n}$ that we are opening. Every time we open a diagonal $\delta_{i,n}$ to get a child, we add i to the labeling. E.g. for the second line in the graph, we get labeling 1 or 2 depending on whether we open $\delta_{1,3}$ or $\delta_{2,3}$.



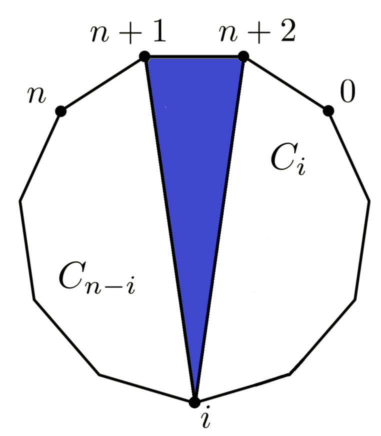
Levels three to six of the tree of triangulations [1]

Properties of Flip Graphs

Let F_n be a flip graph generated from an n-length cycle. Then,

- The number of vertices in F_{n+2} is given by the n-th Catalan Number C_n. The Catalan numbers for $n \leq 9$ are: 1, 2, 5, 14, 42, 132, 429, 1430, 4862.

  Let's see a proof of this: Assume that F_{n+1} has C_{n-1} vertices; the base case of $n = 2$ is true as F_3 has $1 = C_1$ vertex. Let P be an $(n+2)$-gon and fix a cyclic labeling of the vertices. If $1 \leq i \leq n$, let δ_i be the triangle with vertices $n+2, n+1$, and i. Any triangulation containing δ_i gives a triangulation of the two polygons in the complement of δ_i. By the inductive hypothesis, there are $C_{i-1} * C_{n-i+1}$ such triangulations. Now every triangulation of P contains a unique triangle of the form δ_i. Hence, there are $\sum_{i=1}^{n} C_{i-1} * C_{n-i}$ triangulations of P, which is exactly C_n by the recursive formula for the $(n)^{th}$-Catalan number.



A unique δ_i partitioning P into two polygons

- Every vertex in F_n has degree $n-3$.
  This follows from the fact that a triangulation of an n-gon consists of $n-3$ triangles.
- Planarity of F_n
  - F_6 is planar because it can be embedded into the plane as shown in the first figure.
  - F_7 is non-planar by using Euler's Formula: $|V|$ - $|E|$ + $|F|$ = 2.
  - F_n is non-planar for $n \geq 7$: F_7 fails Euler's Formula, so is it non-planar. By containment relations, every F_n contains a copy of F_7 for $n > 7$; hence, it is not planar.
- F_n is connected. By induction, for any arbitrary triangulation we can always perform a series of flips to reach the fan at vertex 1.
- F_n is triangle free. A triangle in the flip graph would imply the existence of a triangulation with crossing diagonals.

Current Work

Owing to the rapid growth of the Catalan numbers, higher order flip graphs are impractical to compute by hand. Therefore, we have recently been designing an algorithm to generate all the labels (i.e. vertices) of the flip graph. This is achieved through two algorithms: algorithm 1 takes in the adjacency matrix of a triangulation and generates the labels of its children, while algorithm 2 takes in the label of a triangulation and generates its adjacency matrix. Recursively calling these algorithms generates the vertices of F_n.

The Algorithm

Algorithm 1 starts with the adjacency matrix of a triangle. We use this as input to our algorithm to generate the labels of the triangle's children, which are labels 1 and 2 in the tree diagram. These labels are converted into their adjacency matrices by algorithm 2, which are then used to generate their children's labels. This procedure propagates through the tree until the desired level is reached.
We define a matrix A_i to correspond to the triangulation with label i.

For example, in the 4-by-4 matrix A_2, we notice that 4 is connected to 1, 2 and 3. If we put A_2 in algorithm 1, since we have edges $\delta_{1,4}$, $\delta_{2,4}$, and $\delta_{3,4}$, we can open each edge to get the children's labeling 21, 22 and 23.

A_2

	1	2	3	4
1	0	1	0	1
2	1	0	1	1
3	0	1	0	1
4	1	1	1	0

A_{23}

	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	0	1
3	0	1	0	1	1
4	0	0	1	0	1
5	1	1	1	1	0

Without loss of generality we take labeling 23 into the algorithm 2, and it will generate the 5-by-5 matrix A_{23}. Again, we put this A_{23} into the algorithm 1 so we can get possible labeling of 23's children, which are 231, 232, 233 and 234 since we can tell from A_{23} that vertex 5 is connected to 1, 2, 3 and 4.

To go to the adjacency matrix representing label 233, we apply algorithm 2. This algorithm begins with the last value of the label and connects it to the highest, valid vertex in the n-gon. 0s are inserted in the matrix wherever diagonals are invalidated. This process repeats for each value in the label. Thus, the size of the label corresponds to the number of diagonals in the triangulation.

A_{233}

	1	2	3	4	5	6
1	0	1	0	0	0	1
2	1	0	1	0	0	1
3	0	1	0	1	1	1
4	0	0	1	0	1	0
5	0	0	1	1	0	1
6	1	1	1	0	1	0

Further Research and Questions

Once we are able to generate and store the vertices of higher order flip graphs efficiently, it will not be difficult to generate the edges of F_n by applying **Theorem 1**. The next step will be exploring proper colorings of these graphs and their chromatic numbers. We hope to design a branch-and-bound algorithm to generate colorings of the flip graphs, in order for us to investigate upper bounds on $\chi(F_n)$ and make conjectures about whether $\chi(F_n)$ tends to infinity as n tends to infinity.

References

[1] F. Hurtado, M. Noy. *Graph of triangulations of a convex polygon and tree of triangulations, Computational Geometry, Volume 13, Issue 3, 1999, Pages 179-188.*

[2] *White, Dennis. The Catalan Numbers. PDF. Minneapolis: University of Minnesota, November 3, 2010.*