A Missing Entry in Sullivan's Dictionary?

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Introduction

Goal

To study degenerate Julia sets and limit sets that share similar behaviors and to use computer programs to generate images of these objects.

Sullivan's Dictionary

A compilation of relations of the form (1-dimensional complex dynamics : Kleinian groups and 3-dimensional hyperbolic geometry).

Dictionary Entries or Translation Examples

"Language"	Complex Dynamics	Hyperbolic Geometry
Object acting on Riemann sphere	Rational maps	Kleinian groups
Parameter space	Mandelbrot set	Bers slice
Objects of interest	Julia sets	Limit sets

A possible missing entry is the correspondence between degeneracy of Julia sets and degeneracy of limit sets of quasi-Fuchsian groups.

Background

Complex Dynamics

Complex dynamics is the study of iterations of rational maps $f : \hat{\mathbb{C}} \mapsto \hat{\mathbb{C}}$.
We focus on complex dynamics of rational maps acting on the Riemann sphere ($\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$), specifically polynomials over $\mathbb{C}$ of the form $z^2 + c$.

Definition. *A* Julia set *of a rational map f on the sphere is the smallest closed set J such that $f^{-1}(J) = J$, with $|J| > 2$.*

The Mandelbrot set and Julia set examples

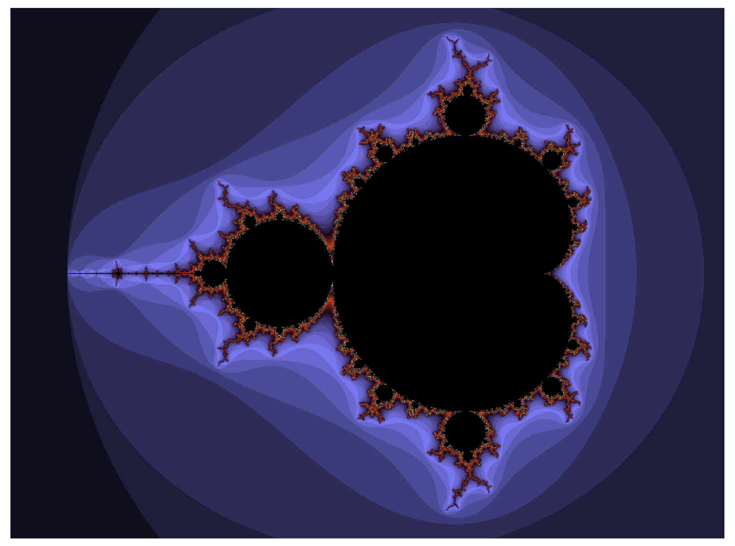

Figure 1: *The Mandelbrot set: $\{c \mid$ Julia set of $z^2 + c$ is connected$\}$*

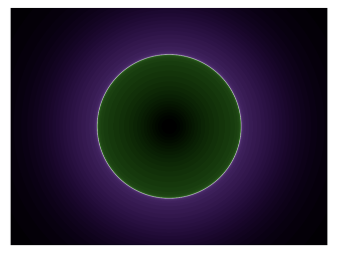

Figure 2: *Julia Set of $f(z) = z^2$*

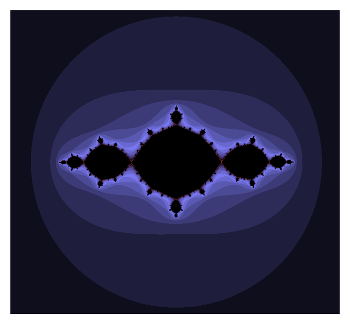

Figure 3: *Julia Set of $f(z) = z^2 - 1$*

Möbius Transformations

Möbius transformations are rational functions of the form $f(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$.

Remarks:

- The set of Möbius transformations form a group under composition.
- A Möbius transformation f is uniquely determined by its image of three points. (Note: this implies that there are some Möbius transformations that fix a circle.)

Theorem. *Let G be the group of Möbius transformations. Then G is isomorphic to $PSL(2, \mathbb{C})$.*

Hyperbolic Geometry

Kleinian Groups

- Kleinian groups are discrete subgroup of $PSL(2, \mathbb{C})$.
- Fuchsian groups are Kleinian groups preserving circles in $\hat{\mathbb{C}}$, discrete subgroups of $PSL(2, \mathbb{R})$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \frac{az+b}{cz+d}$$

$$a, b, c, d \in \mathbb{R}, ad - bc = 1$$

- Quasi-Fuchsian groups preserve quasi-circles in $\hat{\mathbb{C}}$.

Definition. *$\Lambda(\Gamma)$, the* Limit Set *of a* Kleinian Group *$\Gamma \subset PSL(2, \mathbb{C})$ is the smallest closed Γ-invariant subset of $\hat{\mathbb{C}}$, or similarly the set of accumulation points of Γx for $x \in B^3$, the unit ball.*

A Kleinian Group acting on B^3 has accumulation points on the boundary of B^3, the Riemann sphere.

Examples of limit sets

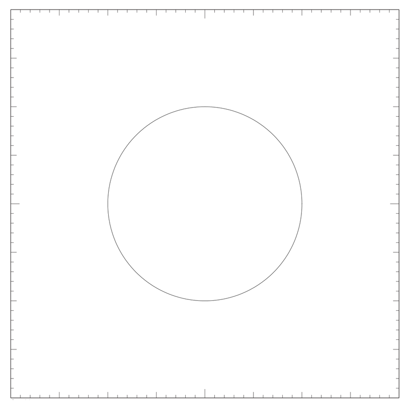

Figure 4: *Circle limit set of a Fuchsian group.*

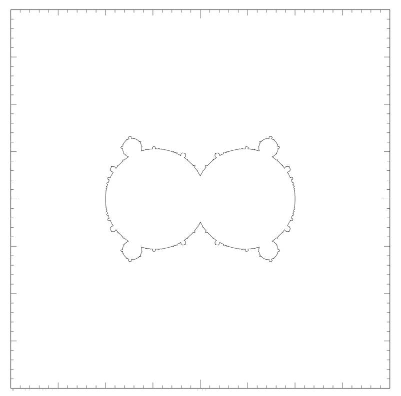

Figure 5: *Quasi-circle limit set of a quasi-Fuchsian group.*

A Missing Entry?

Degenerate behavior of the cauliflower Julia set

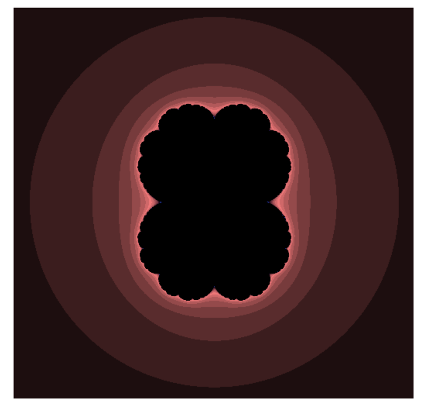

Figure 6: *Julia Set of $f(z) = z^2 + \frac{1}{4}$*

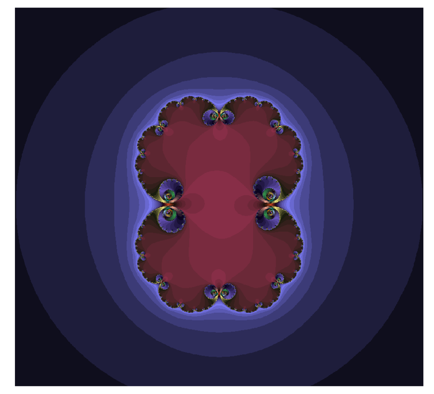

Figure 7: *Julia Set of $f(z) = z^2 + \frac{1}{4} + \varepsilon$*

Degenerate behavior of limit sets

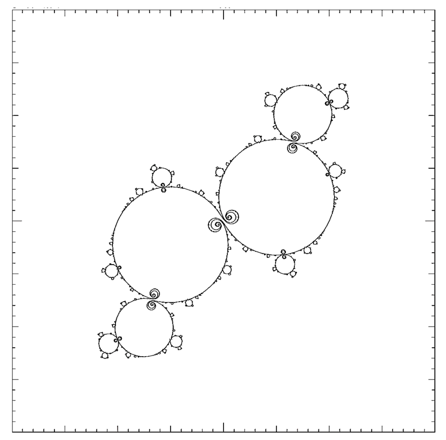

Figure 8: *Quasi-circle limit set of a quasi-Fuchsian group near a cusp of the Bers slice*

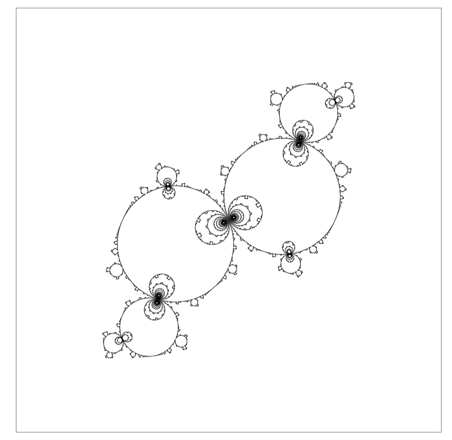

Figure 9: *Quasi-circle limit set of a quasi-Fuchsian group even nearer a cusp of the Bers slice*

Similar degenerate behaviors of limit sets and Julia sets

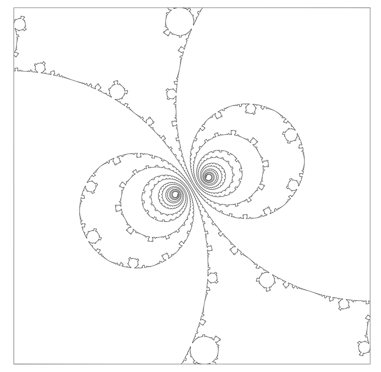

Figure 10: *Part of the quasi-circle limit set in figure 9*

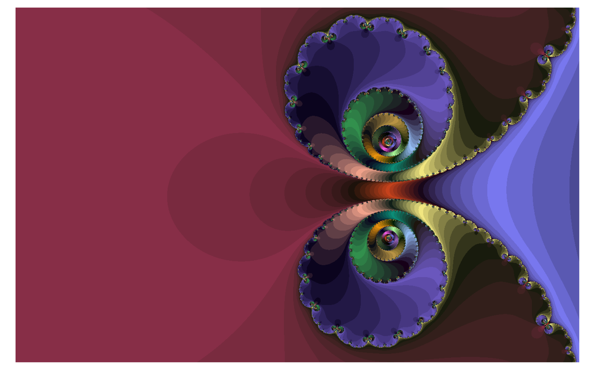

Figure 11: *East 'cusp' of the Julia Set of $f(z) = z^2 + \frac{1}{4} + \varepsilon$*

Future Directions

We shall write a program that generates images of limit sets as we approach the boundary of the Bers slice on various paths. Our focus is on behavior near cusps of the Bers slice compared to behavior of Julia sets near cusps of the Mandelbrot set.

References

[1] John Hubbard. *Teichmüller Theory and Applications to Geometry, Topology, and Dynamics Volume 2: Surface Homeomorphisms and Rational Functions.*

[2] David Mumford, Caroline Series and David Wright. *Indra's Pearl: The Vision of Felix Klein.*

[3] Albert Marden. *Hyperbolic Manifolds.*