A Visual Approach to Complex Analysis

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Graphing Functions

- Real functions $f : \mathbb{R} \rightarrow \mathbb{R}$
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$$f(x) = (x - 2)(x - 2)(x + 3)$$
Graphing Functions

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  \[ f(x) = (x - 2)(x - 2)(x + 3) \]

  ![Graph of a cubic function](image)

- Complex functions $g : \mathbb{C} \rightarrow \mathbb{C}$
Graphing Functions

- Real functions $f : \mathbb{R} \rightarrow \mathbb{R}$

\[ f(x) = (x - 2)(x - 2)(x + 3) \]

- Complex functions $g : \mathbb{C} \rightarrow \mathbb{C}$

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Phase Plots

- $z = re^{i\theta} \in \mathbb{C}$
Phase Plots

- $z = re^{i\theta} \in \mathbb{C}$
- Modulus = $r$, Phase = $e^{i\theta}$
Phase Plots

- \( z = re^{i\theta} \in \mathbb{C} \)
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**Figure:** Phase Color Wheel
Phase Plots

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**Figure:** Phase Color Wheel

**Figure:** \( f(z) = z \)
Phase Plot Examples

Figure: $f(z) = z$

Figure: $f(z) = (z - 2 - i)(z - 2 + i)(z + 2 - i)(z + 2 + i)$
Phase Plot Examples

**Figure:** $f(z) = z$

**Figure:** $f(z) = 1/z$
Roots of $f, f'$

- Research focus: relationship between roots of polynomial $f$ and roots of $f'$
- Calculus I: Rolle’s Theorem

Complex Analysis: Gauss-Lucas Theorem
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Definition (Convex Hull)

Let $X$ be a bounded subset of the plane, the convex hull can be visualized as the shape enclosed by a rubber band stretched around $X$. 
**Gauss-Lucas Theorem**

**Theorem (Gauss-Lucas Theorem)**

Let $P$ be a polynomial, the roots of $P'$ all lie within the convex hull of the roots of $P$.

**Figure:** $f(z) = (z - 2 - i)(z - 2 + i)(z + 2 - i)(z + 2 + i)$
Marden’s Theorem

**Theorem (Marden’s Theorem)**

Suppose the roots of a third-degree polynomial $f$ are $z_1, z_2$ and $z_3$ and they form a triangle. There is a unique ellipse inscribed in the triangle and tangent to the sides at their midpoints. The foci of that ellipse are the zeroes of the derivative $f'$.
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References


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