A Visual Approach to Complex Analysis

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University of Michigan

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- Real functions $f : \mathbb{R} \to \mathbb{R}$

Graphing Functions

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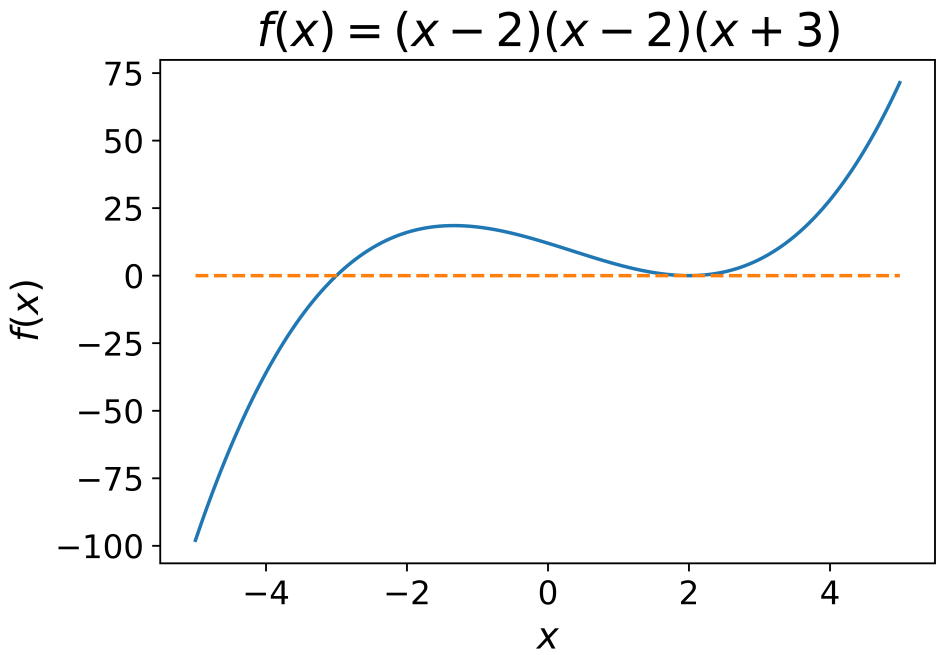

Graphing Functions

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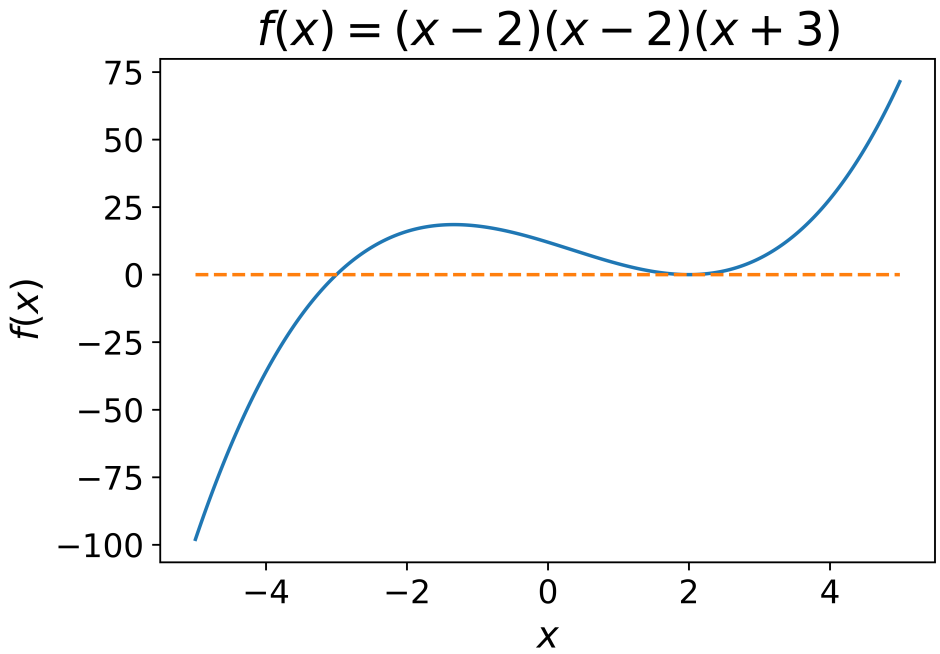
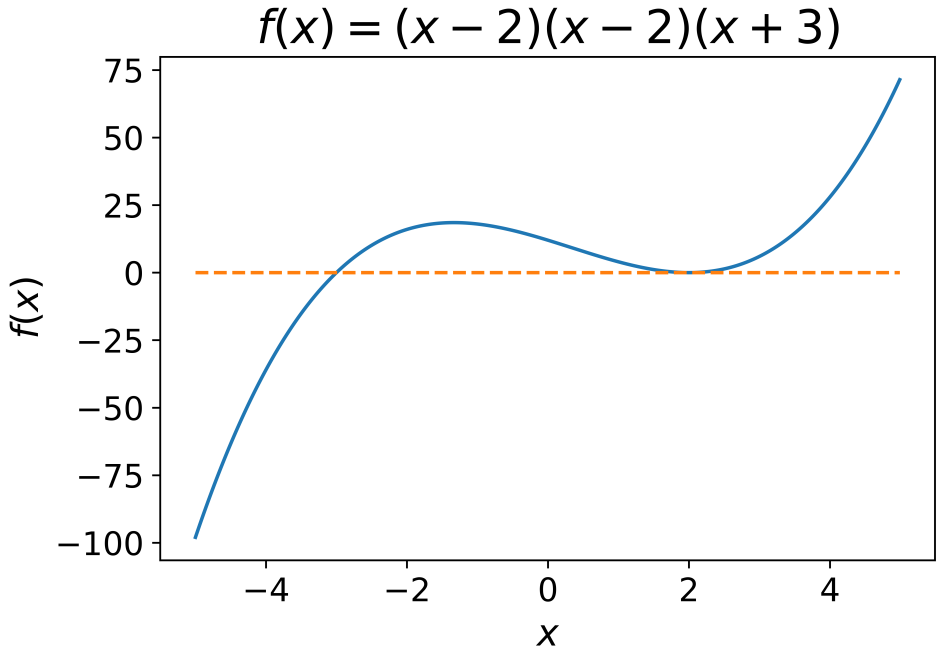


- Complex functions $g : \mathbb{C} \to \mathbb{C}$

Graphing Functions

- Real functions $f : \mathbb{R} \to \mathbb{R}$



- Complex functions $g : \mathbb{C} \to \mathbb{C}$

?

Phase Plots

- $z = re^{i\theta} \in \mathbb{C}$

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Figure: Phase Color Wheel

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Figure: Phase Color Wheel

Figure: $f(z) = z$

Phase Plot Examples

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Figure: $f(z) = z$

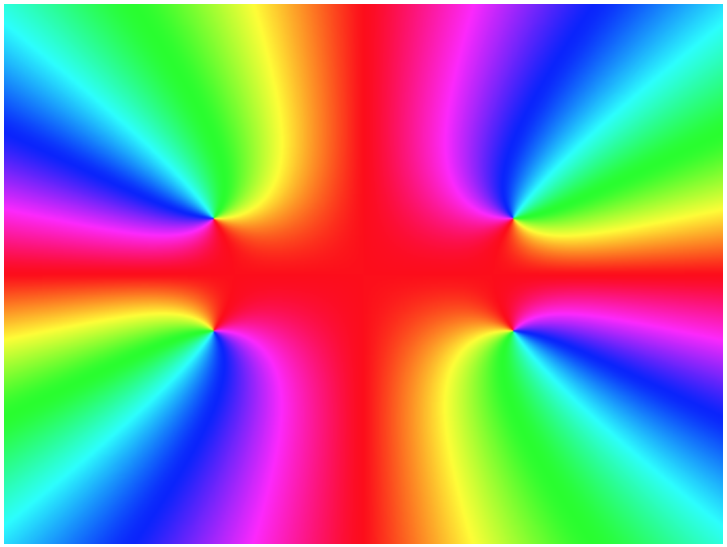


Figure: $f(z) = (z-2-i)(z-2+i)(z+2-i)(z+2+i)$

Phase Plot Examples

Figure: $f(z) = z$

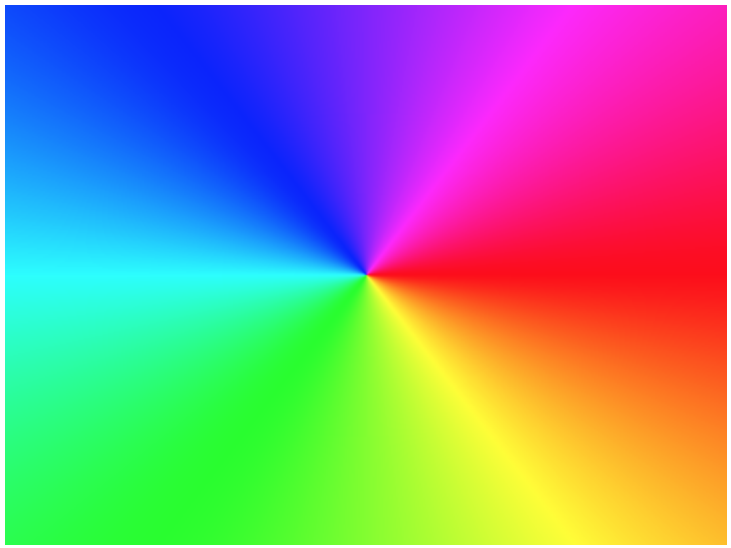
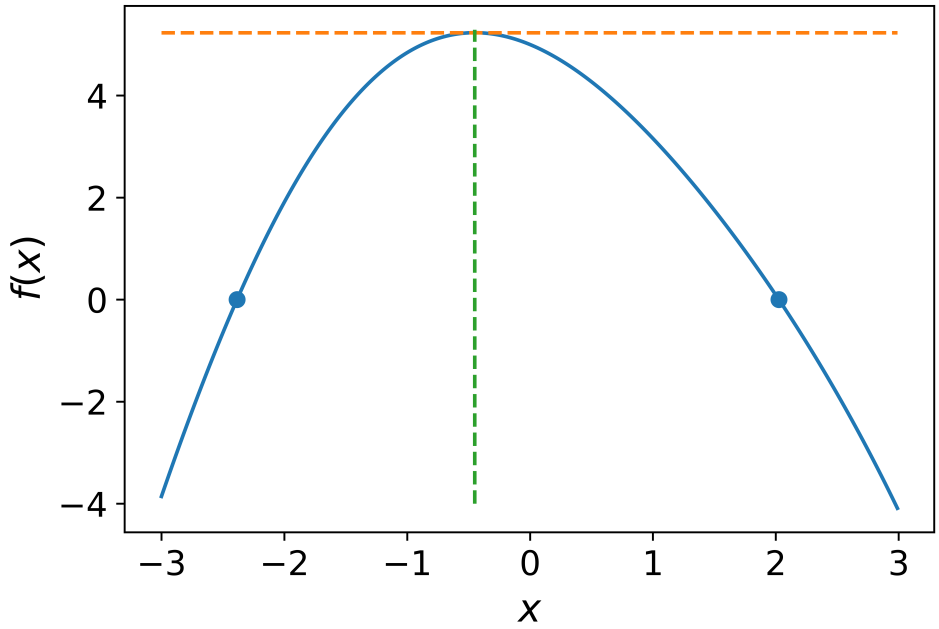


Figure: $f(z) = 1/z$

Roots of f, f'

- Research focus: relationship between roots of polynomial f and roots of f'
- Calculus I: Rolle's Theorem



- Complex Analysis: Gauss-Lucas Theorem

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Gauss-Lucas Theorem

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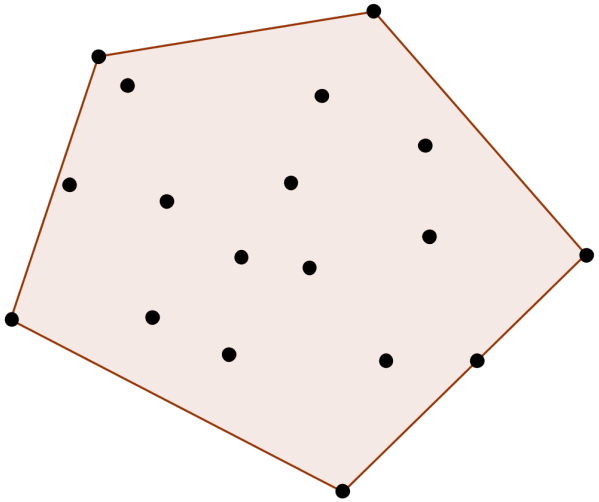
Definition (Convex Hull)

Let X be a bounded subset of the plane, the convex hull can be visualized as the shape enclosed by a rubber band stretched around X.



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Theorem (Gauss-Lucas Theorem)

Let P be a polynomial, the roots of P' all lie within the convex hull of the roots of P.

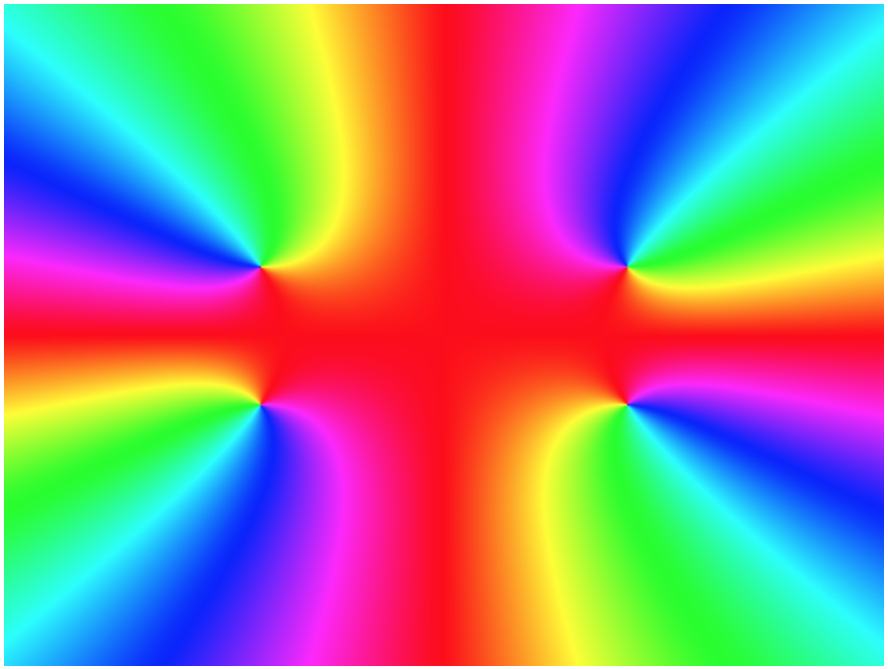

Figure: $f(z) = (z-2-i)(z-2+i)(z+2-i)(z+2+i)$

Marden's Theorem

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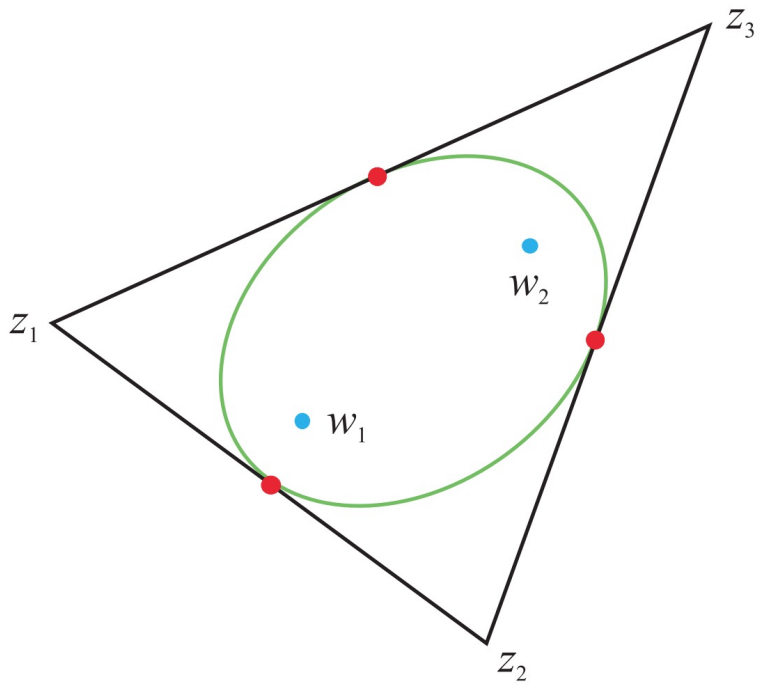
Theorem (Marden's Theorem)

Suppose the roots of a third-degree polynomial f are z_1, z_2 and z_3 and they form a triangle. There is a unique ellipse inscribed in the triangle and tangent to the sides at their midpoints. The foci of that ellipse are the zeroes of the derivative f'.



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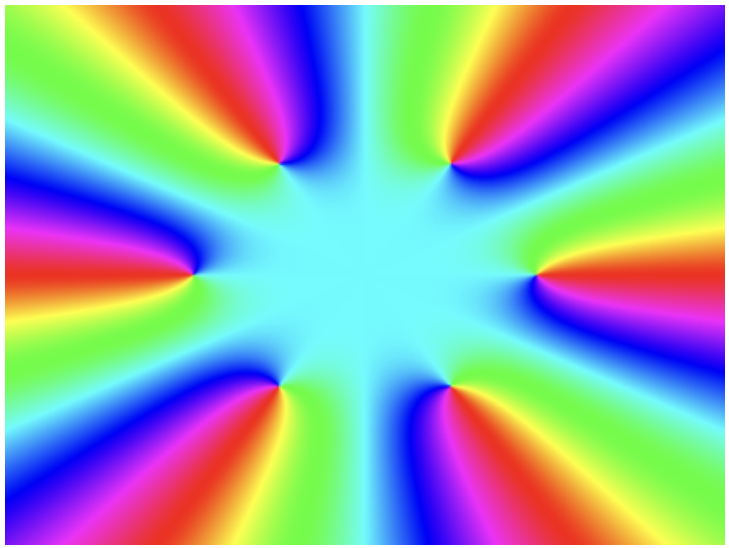
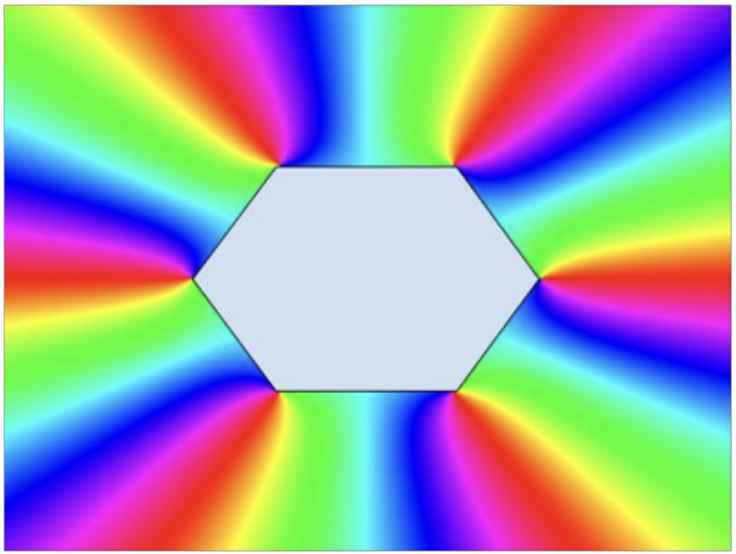
Extension to Higher Degrees





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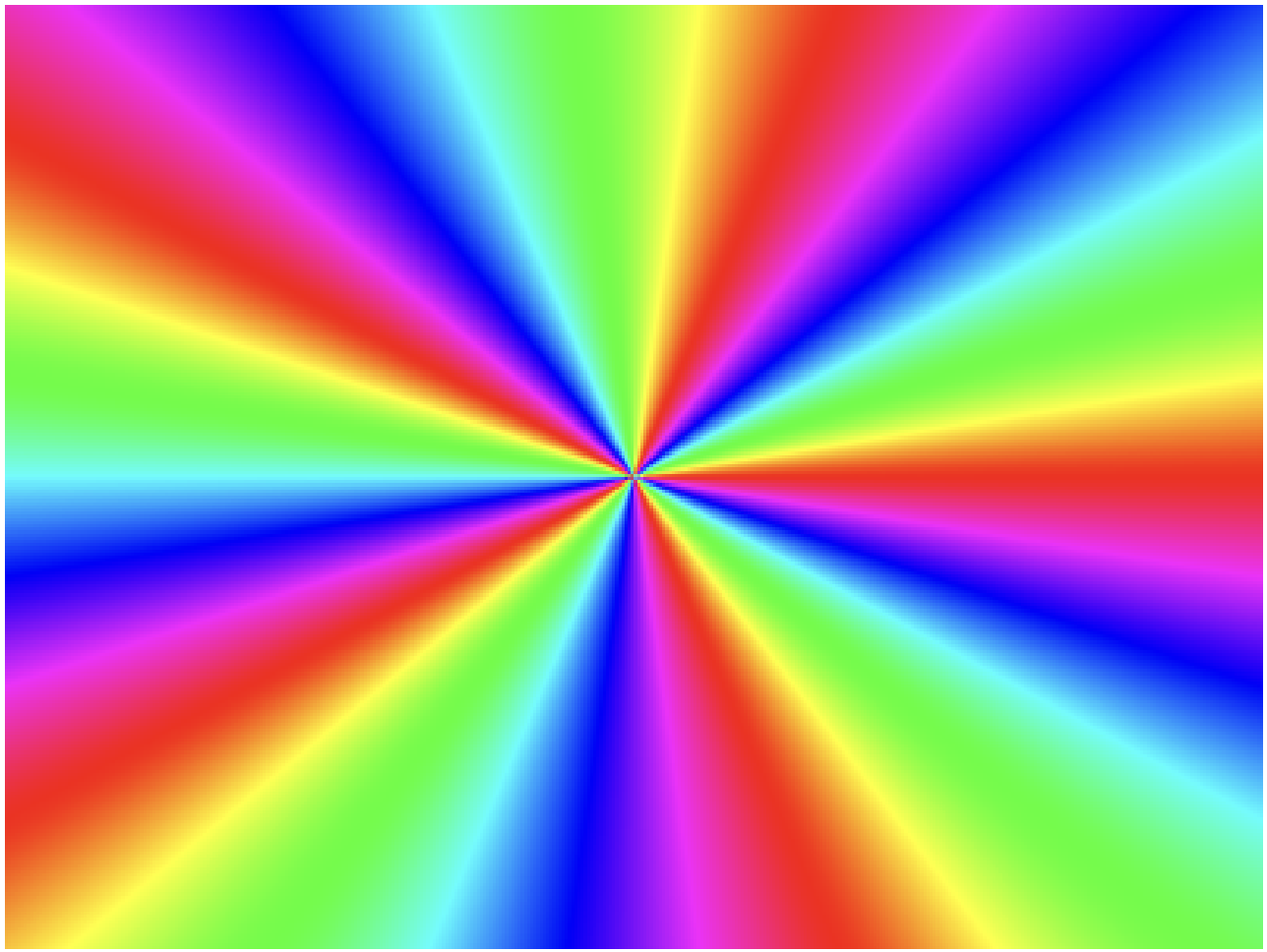

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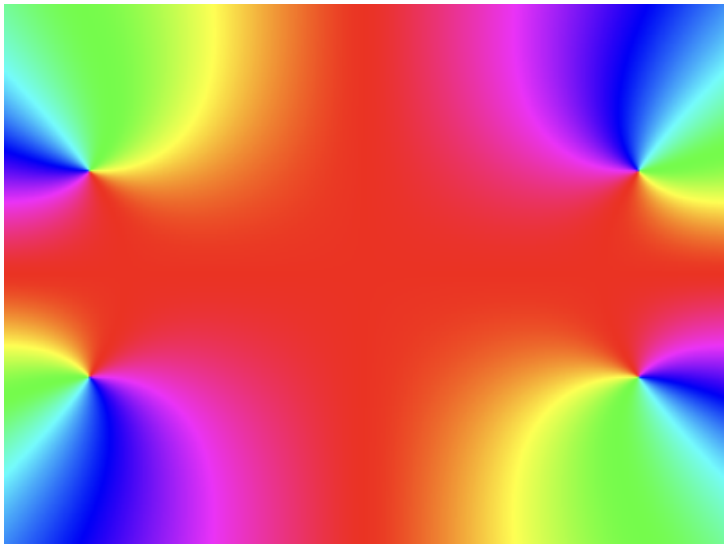

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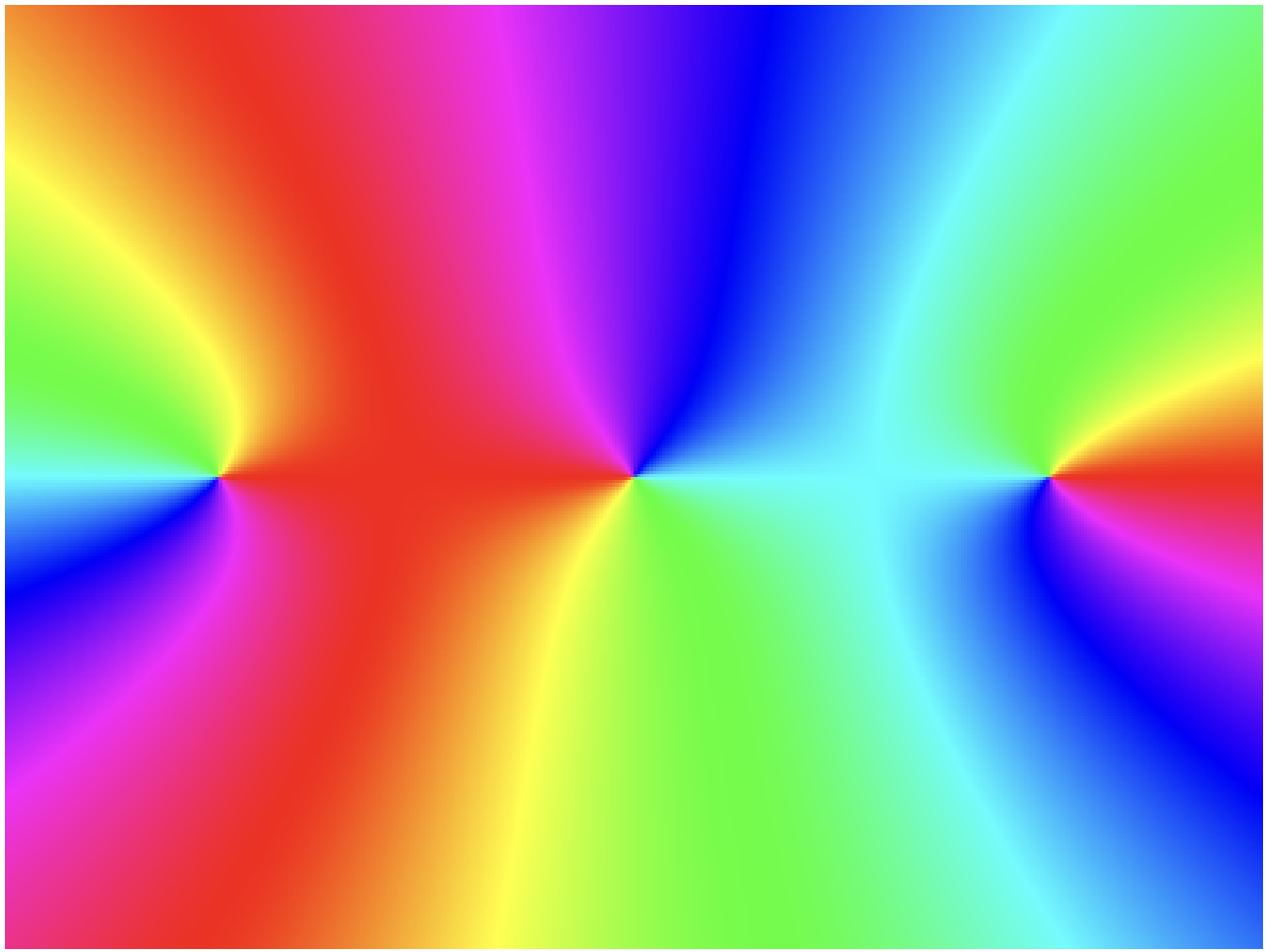

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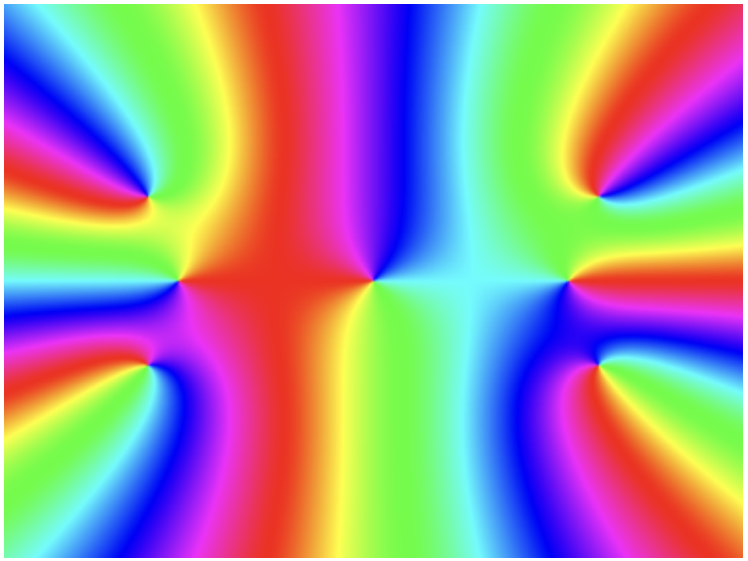

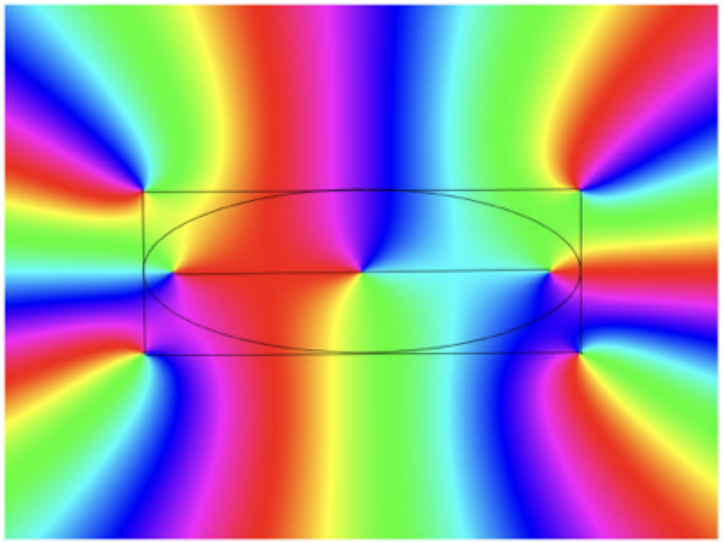

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[1] Elias Wegert. *Visual Complex Functions*. 2012.

[2] Lab of Geometry at Michigan. LoG(M) Beamer Template. *University of Michigan Department of Mathematics*. 2018.

Special thanks to our mentors, Professor Luke Edholm and Rachel Webb, and to the LoG(M) Leadership.