Flag Triangulations of Low-Dimensional Manifolds

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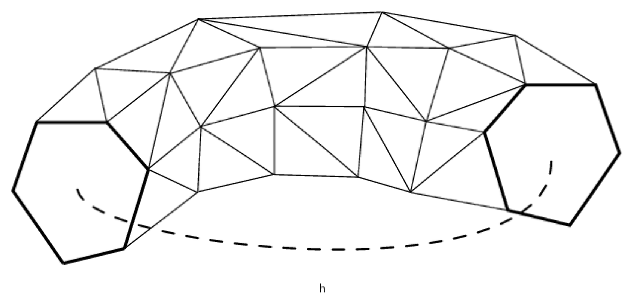
Introduction

A triangulation of a manifold M is a simplicial complex whose geometric realization is homeomorphic to M. The combinatorics of a triangulated manifold is encoded in the face numbers. Bistellar flips, edge subdivisions and contraction are useful tools to construct a new triangulated manifold from an old one while preserving the homeomorphism type of the manifold.

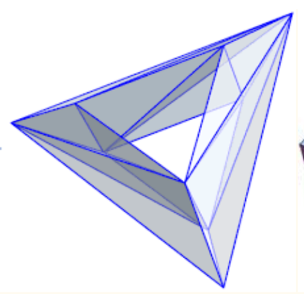


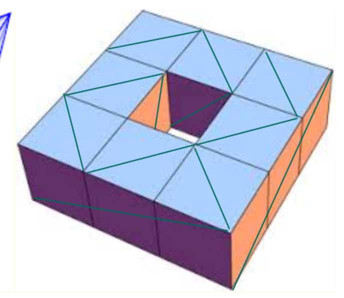
Set-up

- The f-vector of a $(d-1)$-dimensional simplicial complex Δ is $(1, f_0(\Delta), \ldots, f_{d-1}(\Delta))$, where $f_i(\Delta)$ is the number of i-dimensional faces.
- For any face $F \in \Delta$, $\mathrm{lk}(F) = \{\tau \in \Delta : \tau \cap F = \emptyset, \tau \cup F \in \Delta\}$ and $\mathrm{st}(F) = \{\tau \in \Delta : \tau \cup F \in \Delta\}$. In particular, if Δ is a triangulated surface, then vertex links are cycles.
- The *edge subdivision* of a simplicial complex Δ along the edge $\{u, v\}$ is obtained by replacing $\mathrm{st}(e)$ with the cone over $\partial\mathrm{st}(e)$.
- The *edge contraction* of a simplicial complex Δ along its edge $\{u, v\}$ is obtained by identifying u with v in Δ.
- A simplicial complex is said to be *flag* if it has no empty simplices.
- An *induced* subcomplex of Δ in W is the subcomplex $\Delta[W] = \{\tau \in \Delta : \tau \subseteq W.\}$ In particular, the face links are induced in flag complexes.

For example, the image on the left is not flag, since it has empty triangle. The image on the right is flag.

Goal: Find minimal flag triangulations for the torus, Klein bottle, $\mathbb{R}P^2$ and other surfaces.





Minimization Algorithm

Although we know a fair amount of information about minimal triangulation of manifolds (for example, the minimal triangulation of torus has 7 vertices), we have less knowledge about flag triangulation of manifolds. For 2-manifolds, their f-vectors are uniquely determined by the number of vertices.

Lemma 1. *The f-vector of a flag triangulated surface is of the following form:* $f(\Delta) = (1, f_0(\Delta), 3(f_0(\Delta) - \chi(\Delta)), 2(f_0(\Delta) - \chi(\Delta)))$, *where* $\chi(\Delta)$ *is the Euler characteristic of* Δ.

Our implementation is based on the following theorem.

Theorem 1: (Nevo, Lutz [1])

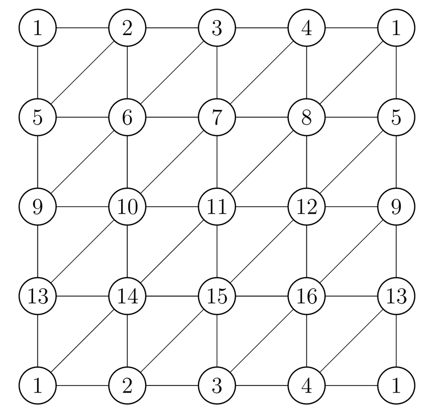
Two flag simplicial complexes are piecewise linearly homeomorphic if and only if they can be connected by a sequence of flag complexes, each obtained from the previous one by either an edge subdivision or admissable edge contraction.

Algorithm

For any given manifold, we start with a small flag triangulation and apply edge subdivision to it until it reaches a predetermined maximum (50 vertices to 70 vertices, depending on specific manifold), and the manifold would go through a sequence of admissible edge contraction, until a point that no edge can be contracted. The resulting manifold would be a local minimum in f_0.

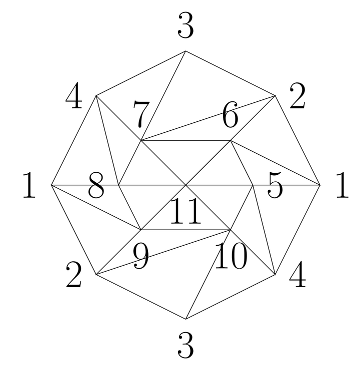
Lemma 2. *A triangulation of a manifold is a local minimal triangulation if every edge is in an induced 4-cycle.*

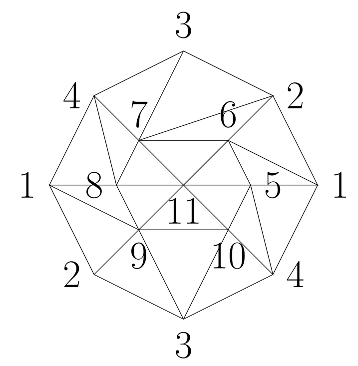
Note: 1) Contracting an edge not in any induced 4-cycle of a flag complex preserves flagness. 2) Although the algorithm does not always produce minimal triangulations, it provides us with a set of candidates and a plausible lower bound on f_0. This algorithm is not susceptible to the start manifold, as all homeomorphic flag manifold are linked by a series of edge contraction and admissible edge contraction. However, it is probably easier to reach the optimal result if we start a smaller triangulation. Thus, we manually searched for a relatively small triangulation and used it in the algorithm.



Real Projective Plane

Example 1 (Triangulation of $\mathbb{R}P^2$)**.** *We started with an 11-vertices version of* $\mathbb{R}P^2$ *and verified that it is indeed the minimal flag triangulation. Moreover, we found another non-isomorphic instance of* $\mathbb{R}P^2$ *with 11 vertices which can be transformed to the existing one by a single bistellar flip.*





Theorem 2. *Let* Δ *be a minimal flag triangulation of a surface* M, $M \neq \mathbb{S}^2$. *Then* $f_0(\Delta) \geq 11$.

Klein Bottle

Example 2 (Triangulation of Klein's bottle)**.** *We started with a conjectured minimum flag triangulation of Klein's bottle with 16 vertices. Prima facie evidence suggests that the minimum flag triangulation of Klein's bottle has 14 vertices, and has at least 28 non-isomorphic triangulations.*

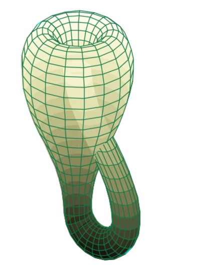


Figure 2: *The Klein bottle is a non-orientable surface.*

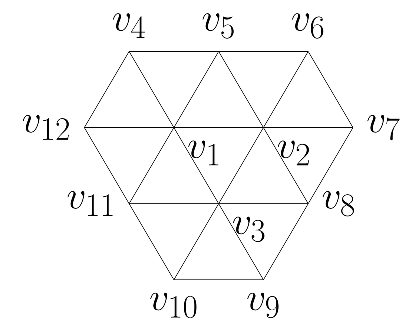
Torus

Theorem 2

There is a unique flag triangulation of twelve vertices of the torus up to isomorphism. It is the minimal flag triangulation of the torus.

Sketch of proof: First we show that every vertex link must be of degree 6 in Δ. Then gives the lower bound $f_0 \geq 12$. We then characterize Δ with $f_0 = 12$ using flagness and symmetry.

Example 3 (Triangulation of Torus)**.** *The conjectured minimum flag triangulation of torus has 16 vertices(the image on the right). After the program, we found the unique flag triangulation of flag torus to have 12 vertices, and we have proved that it is indeed unique and minimal.*

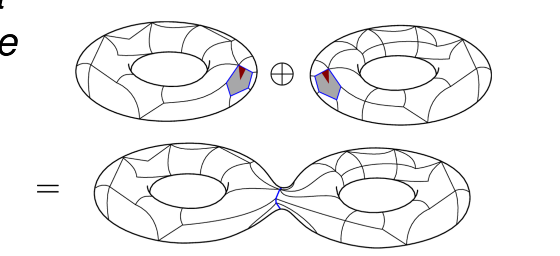


Remark 4. *Torus an orientable surface which the top edge is identified with the bottom edge, the left edge identified with the right edge.*

Future Directions

Theorem (Classification Theorem [2])**.** *Any triangulation of a 2-manifold must be homeomorphic to some member of the following family:*

- *a sphere,*
- *a connected sum of tori,*
- *a connected sum of copies of* $\mathbb{R}P^2$.



By the classification theorem, we can generate all surfaces by sphere, torus, and $\mathbb{R}P^2$. So the triangulation we found can also be used to generate flag triangulation of all 2-manifolds.

3 dimensional manifolds

The same algorithm can be developed for 3-dimensional manifolds. As the dimension increases, the types of non-isomorphic manifold also increase. By evaluating the triangulation of 3-sphere, 3-torus, $\mathbb{R}P^2$ and other manifolds, we can get a better understanding of the 3-dimensional space.

References

[1] F. Lutz and E. Nevo: Stellar theory for flag complexes, *Math. Scand.* **118**(2016), 70â82.

[2] S, Herbert; T, William *A textbook of topology* Pure and Applied Mathematics, 89, Academic Press, ISBN 0126348502, English translation of 1934 classic German textbook

[3] Frank H. Lutz, Thom Sulanke, and Ed Swartz. f -vectors of 3-manifolds. Electron. J. Combin., 16(2, Special volume in honor of Anders Bjo Ìrner):Research Paper 13, 33, 2009.