Origami on Lattices

Anamaria Cuza, Yuqing Liu, Osama Saeed, Harry Richman, Zachary Hamaker, Ian Tobasco

Laboratory of Geometry at Michigan

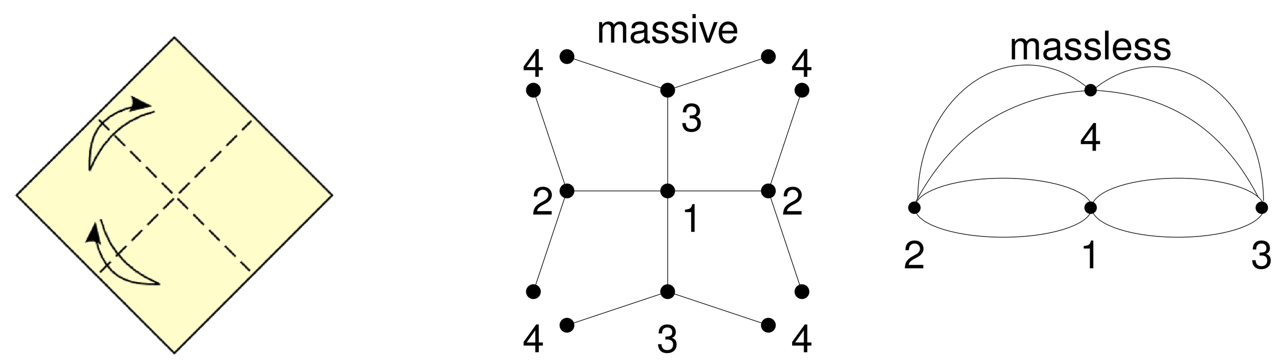
LOG(M)

Introduction

By using origami folding processes, researchers have been able to solve a wide range of engineering problems, such as fabricating different robot morphologies. Applying origami structures in this way requires an understanding of their restrictions and use of energy. In our work we have built up from easier examples to the hexagonal structure.

Goal Classify all origami configurations of a regular hexagon with 6 standard creases. Determine the degrees of freedom and the quantitative relationships between fold angles.

Definition. A sheet is **massless** if allowed to pass through itself; Otherwise, it is **massive**.
Example 1. Configuration space for sheet with four perpendicular creases.



The **degrees of freedom** is the number of independent variables that affect the space of possible configurations. The above example has one degree of freedom.

The Perplexing Case of the Massless Hexagon

Global Structure of the Hexagon

We characterize a hexagon configuration using six vectors $v_1, \ldots, v_6 \in \mathbb{R}^3$ with constraints $|v_i| = 1$, $|v_{i+1} - v_i| = 1$. This gives a heuristic for degrees of freedom for the hexagon:

$$\text{degrees of freedom} = (\text{total dim.}) - (\#\text{ constraints}) - (\text{dim. of symmetries}) = 6 \cdot 3 - 12 - 3 = 3$$

Here the space of symmetries is rotations of $\mathbb{R}^3$, which has dimension 3. Kapovich and Millson [3] prove that the singular points of hexagon configuration space are exactly the **flat configurations** in Figure 1.

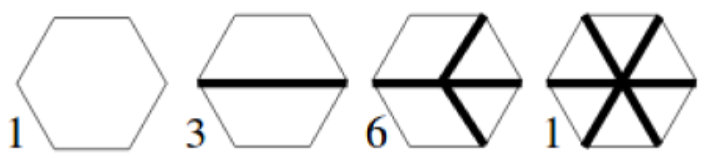


Figure 1: *Flat configurations of the massless hexagon*

Local Structure of the Hexagon

The origami configuration space near a singular point is described by the null cone of a quadratic form [2, 3]. A symmetric matrix Q defines a **quadratic form** f_Q by

$$f_Q : (u, v) \mapsto u^T Q v$$

and the **unit null cone** of this quadratic form is defined by

$$\hat{Z}(Q) = \{x \in \mathbb{R}^n : x^T Q x = 0 \text{ and } |x|^2 = 1\}.$$

Theorem. *The unit null cone of a form of signature (p, q) is a product of spheres $S^{p-1} \times S^{q-1}$*

For our hexagon, the signature is either $(3, 1)$ or $(2, 2)$. For a signature of $(3, 1)$ our unit null cone can be described geometrically as two disjoint spheres, whereas for a signature of $(2, 2)$ our unit null cone can be described as a torus.
For now, we will consider the more interesting case where our unit null cone is two disjoint spheres, and examine one of these spheres in greater detail.

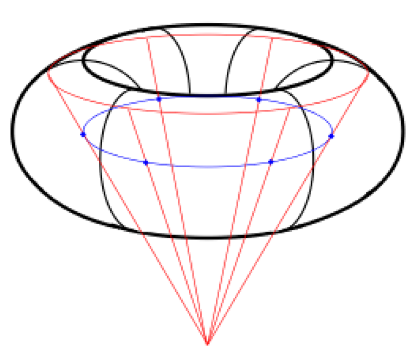


Figure 2: Null cone for signature $(2, 2)$

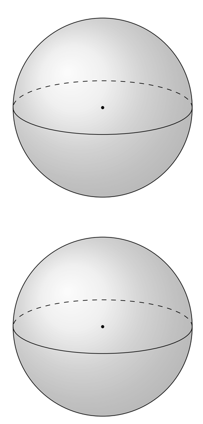


Figure 3: *Null cone for signature* $(3, 1)$

Dancing Near the Unfolded State

To understand a topological space it often helps to cut it up into smaller, more manageable pieces. For a fold configuration which is near state, we say a crease is a **mountain fold** if it is higher than a secant line between its two adjacent flat regions, and a **valley fold** if it is lower.

Dividing the space according to the mountain-valley labellings, we get nine 0-dimensional, twenty-four 1-dimensional configurations, and seventeen 2-dimensional configurations. Note that we are defining dimension here as (# degrees of freedom) $-$ 1.
If we are to consider the 0-dimensional ones as vertices, the 1-dimensional ones as edges, and the 2-dimensional ones as faces, then we can form a polyhedron which satisfies Euler's formula: V $-$ E $+$ F $= 2$.

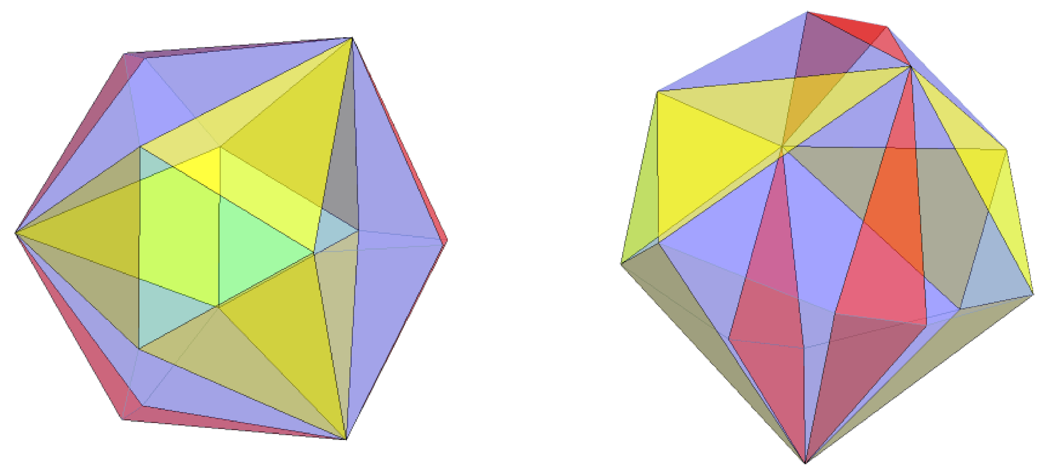


Figure 4: *Polyhedron visualization of hexagon configurations*

We know that every 2-dimensional configuration has three 1-dimensional configurations as neighbours (except for one 2-dimensional configurations that only have two 1-dimensional neighbours), and each one of those has two 0-dimensional configurations as neighbours.

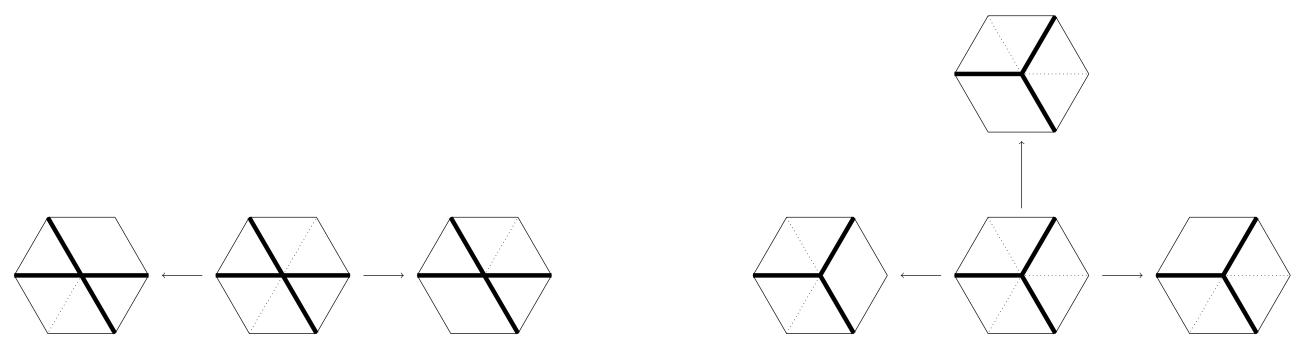


Figure 5: *Examples of two 2-dimensional configurations, with arrows pointing towards their 1-dimensional neighbours.(bolded lines equivalent to mountain folds, dotted lines equivalent to valley folds)*

Thus, by knowing the neighbours we can assign the mountain/valley configurations to our polyhedron's vertices, edges, and faces. Flattening our polyhedron gives us a better view of the positioning of the configurations:

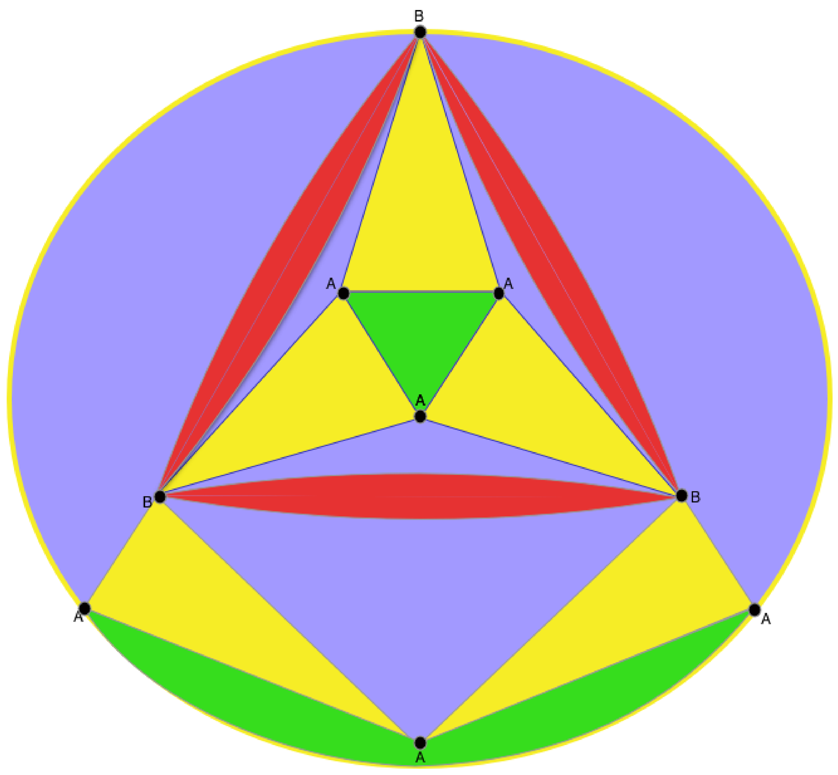


Figure 6: *Flattened polyhedron*

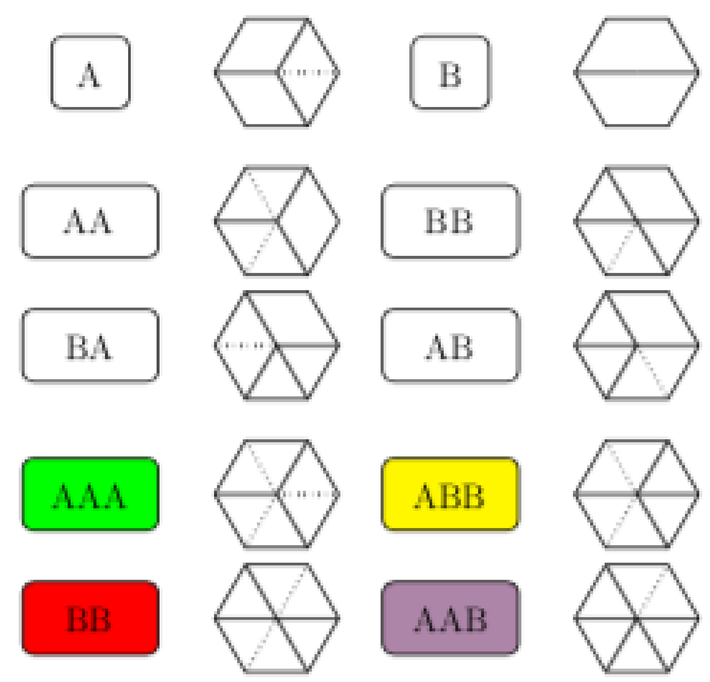


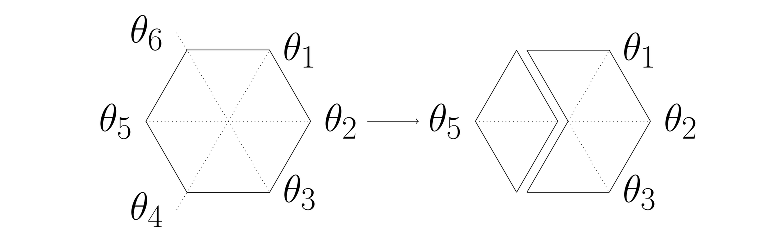
Figure 7: *Associated mountain-valley configurations*

Working Out the Messy Math

Each of our fold angles is in the interval $[-\pi, \pi]$ so the configuration space of a single hexagon is a bounded space in $[-\pi, \pi]^6$ hypercube. We can project this space to $[-\pi, \pi]^3$ by only considering three of the six angles, based on the fact that our hexagon has 3 degrees of freedom.

Divide and Bound Our goal is to generate a boundary restriction relationship by taking three consecutive fold angles as variables.

Divide We divide the hexagon into two parts: a rhombus consisting of two neighbouring equilateral triangles, and a 6-crease concave polygon.



Bound Given the fold angle θ_5, we can calculate the bounds of the distance between vertices 4 and 6 (where each vertex corresponds to the appropriately numbered fold angle). To have a valid configuration, the distance must be within the interval $[0, \sqrt{3}]$.

$$d_{4,6}(\theta_5) = \sqrt{\frac{3}{2} - \frac{3\cos\theta_5}{2}} \in [0, \sqrt{3}] \Rightarrow d_{4,6}(\theta_1, \theta_2, \theta_3) \in [0, \sqrt{3}] \Leftrightarrow \cos(\angle(\mathbf{v_4}, \mathbf{v_6})) = \mathbf{v_4} \cdot \mathbf{v_6} \geq -\frac{1}{2}$$

$$\begin{aligned} -\frac{1}{2} \leq \frac{1}{16}(1 &- 3\cos\theta_1 - 3\cos\theta_2 - 3\cos\theta_3 - 3\cos\theta_1\cos\theta_2 - 3\cos\theta_2\cos\theta_3 \\ &+ 9\cos\theta_1\cos\theta_3 - 3\cos\theta_1\cos\theta_2\cos\theta_3 + 6\sin\theta_1\sin\theta_2 + 6\sin\theta_2\sin\theta_3 \\ &+ 6\cos\theta_1\sin\theta_2\sin\theta_3 + 6\sin\theta_1\sin\theta_2\cos\theta_3 + 12\sin\theta_1\cos\theta_2\sin\theta_3) \end{aligned}$$

Visualization and Interpretation The configuration space defined by the above equation is represented in Figure 8. On its surface, θ_5 is 0. Figure 9 is a representation of the configuration space when θ_1, θ_2, θ_3 are near 0. The corresponding contours consist of four 1-dimensional configurations.

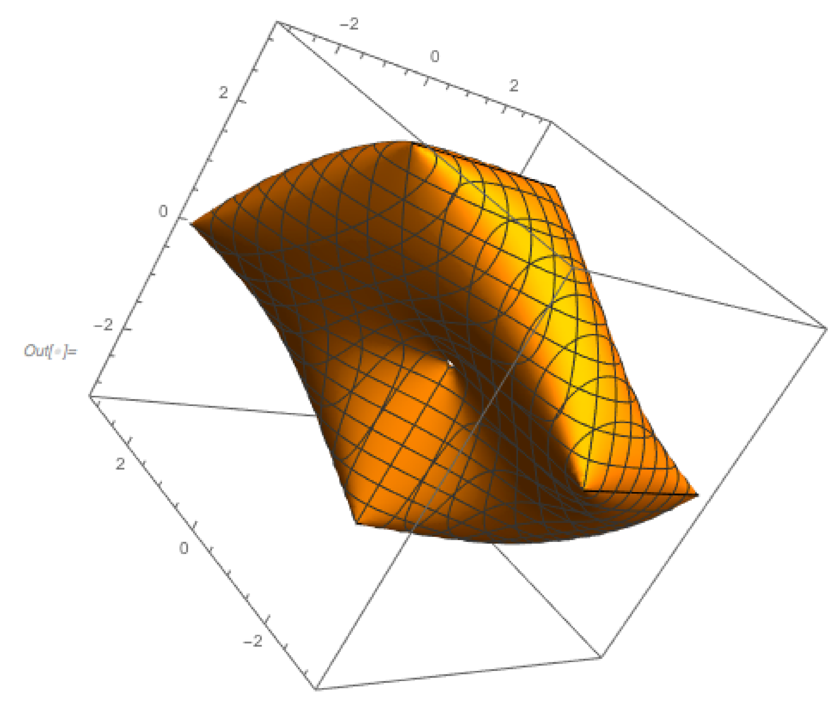


Figure 8: *Configuration space boundary of hexagon*

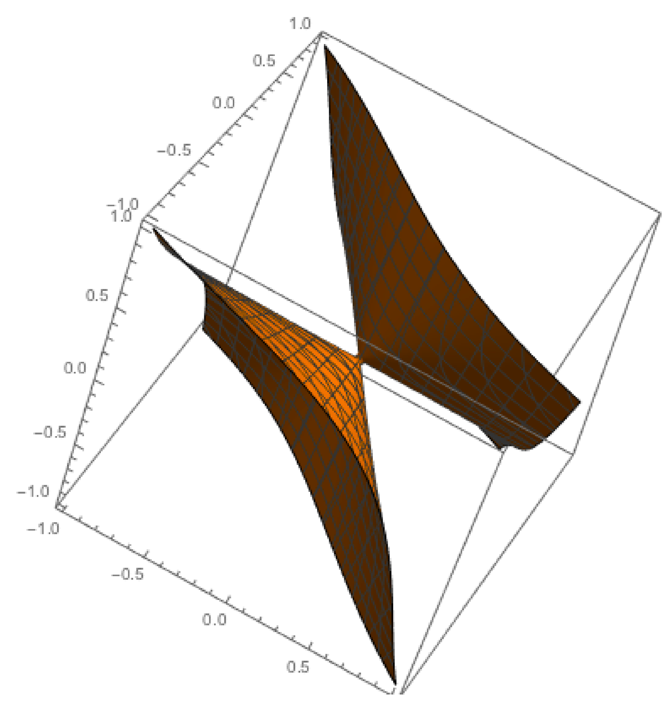


Figure 9: *Boundary near unfolded configuration*

Future Direction: Energy Propagation

Using the energy function defined by $E(\Phi) = (\sum |\theta_i|^2)^{1/2}$ and our understanding of the configuration space for the hexagon, we would like to explore how energy propagates in a triangular lattice. Imagine a hexagonal region embedded within this lattice that contains a smaller hexagonal region within itself. There is an energy well that each hexagon must escape to no longer be considered a flat configuration. We would like to show that the energy well of the outer hexagon is at least as big as the one for the inner hexagon.

References

[1] M. J. Boswick, P. Di Francesco, O. Gollinelli and E. Guitter. *Discrete Folding.* preprint 1996.
[2] B. G. Chen and C. D. Santangelo. Branches of Triangulated Origami Near the Unfolded State. *Physical Review X*, **8** 011034, 2018.
[3] M. Kapovich and J. J. Millson. Hodge Theory and the Art of Paper Folding. *Publ. RIMS, Kyoto Univ.* **33**, pp. 1–33, 1997.
[4] Illinois Geometry Lab. IGL Poster Template. *University of Illinois at Urbana-Champaign Department of Mathematics*, 2017.